Carlos S. Kubrusly

The Elements of Operator Theory

Second Edition

Birkhäuser

Carlos S. Kubrusly

The Elements of Operator Theory

Second Edition

Birkhäuser

Carlos S. Kubrusly
Electrical Engineering Department
Catholic University of Rio de Janeiro
R. Marques de S. Vicente 225
22453-900, Rio de Janeiro, RJ, Brazil
carlos@ele.puc-rio.br

ISBN 978-0-8176-4997-5 e-ISBN 978-0-8176-4998-2
DOI 10.1007/978-0-8176-4998-2
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2011922537

Mathematics Subject Classification (2010): 47-01, 47A-xx, 47B-xx, 47C-xx, 47D-xx, 47L-xx

© Springer Science+Business Media, LLC 2011
All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, LLC, 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.
The use in this publication of trade names, trademarks, service marks, and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Printed on acid-free paper

www.birkhauser-science.com

To the memory of my father

The truth, he thought, has never been of any real value to any human being — it is a symbol for mathematicians and philosophers to pursue. In human relations kindness and lies are worth a thousand truths. He involved himself in what he always knew was a vain struggle to retain the lies.

Graham Greene

Preface to the Second Edition

This is a revised, corrected, enlarged, updated, and thoroughly rewritten version for a second edition of *Elements of Operator Theory* (Birkhäuser, Boston, 2001). Although a considerable amount of new material has been added to this new edition, it was not altered in a significant way: the original focus and organization were preserved. In particular, the numbering system of the former edition (concerning chapters, sections, definitions, propositions, lemmas, theorems, corollaries, examples, and problems) has been kept. New material was either embedded in the text without changing those numbers (to catch up with the previous edition for reference purposes so that citations made to the previous edition still hold for the new edition) or included at the end of each chapter with a subsequent numbering. All problems and references of the first edition have also been kept, and 33 new problems and 24 new references (22 books and 2 papers) were added to the present edition.

The logical dependence of the various sections (and chapters) is roughly linear and reflects approximately the minimum amount of material needed to proceed further. A few parts might be compressed or even skipped in a first reading. Chapter 1 may be taken for self-study (and an important one at that), and a formal course of lectures might begin with Chapter 2. Sections 3.8, 3.9, and 4.7 may be postponed to a second reading, as well as Section 6.8 if the readers are still to acquire their first contact with measure theory.

The first edition was written about ten years ago. During this period an extensive Errata was posted on the Web. All corrections listed in it were, of course, also incorporated in the present edition. I thank Torrey Adams, Patricia T. Bandeira, Renato A. A. da Costa, Moacyr V. Dutra, Jorge S. Garcia, Jessica Q. Kubrusly, Nhan Levan, José Luis C. Lyra Jr, Adrian H. Pizzinga, Regina Posternak, André L. Pulcherio, James M. Snyder, Guilherme P. Temporão, Fernando Torres-Torija, Augusto C. Gadelha Vieira, and João Zanni, who helped in compiling that Errata.

Rio de Janeiro, November 2010 Carlos S. Kubrusly

Preface

"Elements" in the title of this book has its standard meaning, namely, basic principles and elementary theory. The main focus is operator theory, and the topics range from sets to the spectral theorem. Chapter 1 (*Set-Theoretic Structures*) introduces the reader to ordering, lattices, and cardinality. Linear spaces are presented in Chapter 2 (*Algebraic Structures*), and metric (and topological) spaces are studied in Chapter 3 (*Topological Structures*). The purpose of Chapter 4 (*Banach Spaces*) is to put algebra and topology to work together. Continuity plays a central role in the theory of topological spaces, and linear transformation plays a central role in the theory of linear spaces. When algebraic and topological structures are compatibly laid on the same underlying set, leading to the notion of topological vector spaces, then we may consider the concept of continuous linear transformations. By an operator we mean a continuous linear transformation of a normed space into itself. Chapter 5 (*Hilbert Spaces*) is central. There a geometric structure is properly added to the algebraic and topological structures. The spectral theorem is a cornerstone in the theory of operators on Hilbert spaces. It gives a full statement on the nature and structure of normal operators, and is considered in Chapter 6 (*The Spectral Theorem*).

The book is addressed to graduate students, both in mathematics or in one of the sciences, and also to working mathematicians exploring operator theory and scientists willing to apply operator theory to their own subject. In the former case it actually is a first course. In the latter case it may serve as a basic reference on the so-called elementary theory of single operator theory. Its primary intention is to introduce operator theory to a new generation of students and provide the necessary background for it. Technically, the prerequisite for this book is some mathematical maturity that a first-year graduate student in mathematics, engineering, or in one of the formal sciences is supposed to have already acquired. The book is largely self-contained. Of course,

a formal introduction to analysis will be helpful, as well as an introductory course on functions of a complex variable. Measure and integration are not required up to the very last section of the last chapter. Each section of each chapter has a short and concise (sometimes a compound) title. They were selected in such a way that, when put together in the contents, they give a brief outline of the book to the right audience.

The focus of this book is on concepts and ideas as an alternative to the computational approach. The proofs avoid computation whenever possible or convenient. Instead, I try to unfold the structural properties behind the statements of theorems, stressing mathematical ideas rather than long calculations. Tedious and ugly (all right, "ugly" is subjective) calculations were avoided when a more conceptual way to explain the stream of ideas was possible. Clearly, this is not new. In any event, every single proof in this book was specially tailored to meet this requirement, but they (at least the majority of them) are standard proofs, perhaps with a touch of what may reflect some of the author's minor idiosyncrasies. In writing this book I kept my mind focused on the reader. Sometimes I am talking to my students and sometimes to my colleagues (they surely will identify in each case to whom I am talking). For my students, the objective is to teach mathematics (ideas, structures, and problems).

There are 300 problems throughout [the first edition of] the book, many of them with multiple parts. These problems, at the end of each chapter, comprise complements and extensions of the theory, further examples and counterexamples, or auxiliary results that may be useful in the sequel. They are an integral part of the main text, which makes them different from traditional classroom exercises. Many of these problems are accompanied by hints, which may be a single word or a sketch, sometimes long, of a proof. The idea behind providing these long and detailed hints is that just talking to students is not enough. One has to motivate them too. In my view, motivation (in this context) is to reveal the beauty of pure mathematics, and to challenge students with a real chance to reconstruct a proof for a theorem that is "new" to them. Such a real chance can be offered by a suitable, sometimes rather detailed, hint.

At the end of each chapter, just before the problems, the reader will find a list of suggested readings that contains only books. Some of them had a strong influence in preparing this book, and many of them are suggested as a second or third reading. The reference section comprises a list of all those books and just a few research papers (82 books and 11 papers — for the first edition), all of them quoted in the text. Research papers are only mentioned to complement occasional historical remarks so that the few articles cited there are, in fact, classical breakthroughs. For a glance at current research in operator theory the reader is referred to recent research monographs suggested in Chapters 5 and 6.

I started writing this book after lecturing on its subject at Catholic University of Rio de Janeiro for over 20 years. In general, the material is covered in two one-semester beginning graduate courses, where the audience comprises mathematics, engineering, economics, and physics students. Quite often senior undergraduate students joined the courses. The dividing line between these two one-semester courses depends a bit on the pace of lectures but is usually somewhere at the beginning of Chapter 5. Questions asked by generations of students and colleagues have been collected. When the collection was big enough, some former students, as well as current students, insisted upon a new book but urged that it should not be a mere collection of lecture notes and exercises bound together. I hope not to disappoint them too much.

At this point, where a preface is coming to an end, one has the duty and pleasure to acknowledge the participation of those people who somehow effectively contributed in connection with writing the book. Certainly, the students in those courses were a big help and a source of motivation. Some friends among students and colleagues have collaborated by discussing the subject of this book for a long time on many occasions. They are: Gilberto O. Corrêa, Oswaldo L. V. Costa, Giselle M. S. Ferreira, Marcelo D. Fragoso, Ricardo S. Kubrusly, Abilio P. Lucena, Helios Malebranche, Carlos E. Pedreira, Denise O. Pinto, Marcos A. da Silveira, and Paulo César M. Vieira. Special thanks are due to my friend and colleague Augusto C. Gadelha Vieira, who read part of the manuscript and made many valuable suggestions. I am also grateful to Ruth F. Curtain, who, back in the early 1970s, introduced me to functional analysis. I wish to thank Catholic University of Rio de Janeiro for providing the release time that made this project possible. Let me also thank the staff of Birkhäuser Boston and Elizabeth Loew of TEXniques for their ever-efficient and friendly partnership. Finally, it is just fair to mention that this project was supported in part by CNPq (Brazilian National Research Council) and FAPERJ (Rio de Janeiro State Research Council).

Rio de Janeiro, November 2000 Carlos S. Kubrusly

Contents

Preface to the Second Edition vii

Preface ix

1 Set-Theoretic Structures 1

1.1 Background ... 1
1.2 Sets and Relations ... 3
1.3 Functions ... 4
1.4 Equivalence Relations ... 7
1.5 Ordering ... 8
1.6 Lattices ... 10
1.7 Indexing ... 12
1.8 Cardinality ... 14
1.9 Remarks ... 21
Problems ... 26

2 Algebraic Structures 37

2.1 Linear Spaces ... 37
2.2 Linear Manifolds ... 43
2.3 Linear Independence ... 45
2.4 Hamel Basis ... 48
2.5 Linear Transformations ... 55
2.6 Isomorphisms ... 58
2.7 Isomorphic Equivalence ... 63
2.8 Direct Sum ... 66
2.9 Projections ... 70
Problems ... 75

3 Topological Structures **87**

3.1 Metric Spaces 87
3.2 Convergence and Continuity 95
3.3 Open Sets and Topology 102
3.4 Equivalent Metrics and Homeomorphisms 108
3.5 Closed Sets and Closure 114
3.6 Dense Sets and Separable Spaces 121
3.7 Complete Spaces 128
3.8 Continuous Extension and Completion 135
3.9 The Baire Category Theorem 144
3.10 Compact Sets 149
3.11 Sequential Compactness 156
Problems 165

4 Banach Spaces **199**

4.1 Normed Spaces 199
4.2 Examples 204
4.3 Subspaces and Quotient Spaces 210
4.4 Bounded Linear Transformations 217
4.5 The Open Mapping Theorem and Continuous Inverse 225
4.6 Equivalence and Finite-Dimensional Spaces 232
4.7 Continuous Linear Extension and Completion 239
4.8 The Banach–Steinhaus Theorem and Operator Convergence .. 244
4.9 Compact Operators 252
4.10 The Hahn–Banach Theorem and Dual Spaces 259
Problems 270

5 Hilbert Spaces **309**

5.1 Inner Product Spaces 309
5.2 Examples 315
5.3 Orthogonality 321
5.4 Orthogonal Complement 326
5.5 Orthogonal Structure 332
5.6 Unitary Equivalence 336
5.7 Summability 340
5.8 Orthonormal Basis 349
5.9 The Fourier Series Theorem 356
5.10 Orthogonal Projection 364
5.11 The Riesz Representation Theorem and Weak Convergence ... 374
5.12 The Adjoint Operator 384
5.13 Self-Adjoint Operators 393
5.14 Square Root and Polar Decomposition 398
Problems 405

6 The Spectral Theorem **443**

6.1 Normal Operators 443
6.2 The Spectrum of an Operator 450
6.3 Spectral Radius 458
6.4 Numerical Radius 465
6.5 Examples of Spectra 468
6.6 The Spectrum of a Compact Operator 478
6.7 The Compact Normal Case 484
6.8 A Glimpse at the General Case 492
Problems 499

References **521**

Index **529**

1

Set-Theoretic Structures

The purpose of this chapter is to present a brief review of some basic set-theoretic concepts that will be needed in the sequel. By basic concepts we mean standard notation and terminology, and a few essential results that will be required in later chapters. We assume the reader is familiar with the notion of *set* and *elements* (or *members*, or *points*) of a set, as well as with the basic set operations. It is convenient to reserve certain symbols for certain sets, especially for the basic *number systems*. The set of all *nonnegative integers* will be denoted by $\mathbb{N}_0$, the set of all *positive integers* (i.e., the set of all *natural numbers*) by $\mathbb{N}$, and the set of all *integers* by $\mathbb{Z}$. The set of all *rational numbers* will be denoted by $\mathbb{Q}$, the set of all *real numbers* (or the *real line*) by $\mathbb{R}$, and the set of all *complex numbers* by $\mathbb{C}$.

1.1 Background

We shall also assume that the reader is familiar with the basic rules of elementary (classical) logic, but acquaintance with formal logic is not necessary. The foundations of mathematics will not be reviewed in this book. However, before starting our brief review on set-theoretic concepts, we shall introduce some preliminary notation, terminology, and logical principles as a background for our discourse.

If a *predicate* $P(\ )$ is meaningful for a subject x, then $P(x)$ (or simply P) will denote a *proposition*. The terms *statement* and *assertion* will be used as synonyms for proposition. A statement on statements is sometimes called a *formula* (or a *secondary proposition*). Statements may be *true* or *false* (not true). A *tautology* is a formula that is true regardless of the truth of the statements in it. A *contradiction* is a formula that is false regardless of the truth of the statements in it. The symbol $\Rightarrow$ denotes *implies* and the formula $P \Rightarrow Q$ (whose logical definition is "either P is false or Q is true") means "the statement P implies the statement Q". That is, "*if* P is true, *then* Q is true", or "P is a *sufficient condition* for Q". We shall also use the symbol $\not\Rightarrow$ for the denial of $\Rightarrow$, so that $\not\Rightarrow$ denotes *does not imply* and the formula $P \not\Rightarrow Q$

C.S. Kubrusly, *The Elements of Operator Theory*, DOI 10.1007/978-0-8176-4998-2_1,

© Springer Science+Business Media, LLC 2011

means "the statement P does not imply the statement Q". Accordingly, let $\not P$ stand for the denial of P (read: *not* P). If P is a statement, then $\not P$ is its *contradictory*.

Let us first recall one of the basic rules of deduction called *modus ponens*: "if a statement P is true and if P implies Q, then the statement Q is true" — "anything implied by a true statement is true". Symbolically, $\{P \text{ true and } P \Rightarrow Q\} \Longrightarrow \{Q \text{ true}\}$. A *direct proof* is essentially a chain of modus ponens. For instance, if P is true, then the string of implications $P \Rightarrow Q \Rightarrow R$ ensures that R is true. Indeed, if we can establish that P holds, and that P implies Q, then (modus ponens) Q holds. Moreover, if we can also establish that Q implies R, then (modus ponens again) R holds.

However, modus ponens alone is not enough to ensure that such a reasoning may be extended to an arbitrary (endless) string of implications. In certain cases the *Principle of Mathematical Induction* provides an alternative reasoning. Let $\mathbb{N}$ be the set of all natural numbers. A set S of natural numbers is called *inductive* if $n+1$ is an element of S whenever n is. The Principle of Mathematical Induction states that "if 1 is an element of an inductive set S, then $S = \mathbb{N}$". This leads to a second scheme of proof, called *proof by induction*. For instance, for each natural number n let P_n be a proposition. If P_1 holds true and if $P_n \Rightarrow P_{n+1}$ for each n, then P_n holds true for every natural number n. The scheme of proof by induction works for $\mathbb{N}$ replaced with $\mathbb{N}_0$. There is nothing magical about the number 1 as far as a proof by induction is concerned. All that is needed is a "beginning" and the notion of "induction". Example: Let i be an arbitrary integer and let $\mathbb{Z}_i$ be the set made up of all integers greater than or equal to i. For each integer k in $\mathbb{Z}_i$ let P_k be a proposition. If P_i holds true and if $P_k \Rightarrow P_{k+1}$ for each k, then P_k holds true for every integer k in $\mathbb{Z}_i$ (particular cases: $\mathbb{Z}_0 = \mathbb{N}_0$ and $\mathbb{Z}_1 = \mathbb{N}$).

"If a statement leads to a contradiction, then this statement is false". This is the rule of a *proof by contradiction — reductio ad absurdum*. It relies on the *Principle of Contradiction*, which states that "P and $\not P$ are impossible". In other words, the Principle of Contradiction says that the formula "P and $\not P$" is a contradiction. But this alone does not ensure that any of P or $\not P$ must hold. The *Law of the Excluded Middle* (or *Law of the Excluded Third — tertium non datur*) does: "either P or $\not P$ holds". That is, the Law of the Excluded Middle simply says that the formula "P or $\not P$" is a tautology. Therefore, the formula $\not Q \Rightarrow \not P$ means "P holds *only if* Q holds", or "Q is a *necessary condition* for P". If $P \Rightarrow Q$ and $Q \Rightarrow P$, then we write $P \Leftrightarrow Q$ which means "P *if and only if* Q", or "P is a *necessary and sufficient condition* for Q", or still "P and Q are *equivalent*" (and vice versa). Indeed, the formulas $P \Rightarrow Q$ and $\not Q \Rightarrow \not P$ are equivalent: $\{P \Rightarrow Q\} \Longleftrightarrow \{\not Q \Rightarrow \not P\}$. This equivalence is the basic idea behind a *contrapositive proof*: "to verify that a proposition P implies a proposition Q, prove, instead, that the denial of Q implies the denial of P".

We conclude this introductory section by pointing out another usual but slightly different meaning for the term "proposition". We shall often say "prove the following proposition" instead of "prove that the following proposition holds true". Here the term *proposition* is being used as a synonym for *theorem* (a true statement for which we demand a proof of its truth), and not as a synonym for assertion or statement (that may be either true or false). A *conjecture* is a statement that has not been proved yet — it may turn out to be either true or false once a proof of its truth or falsehood is supplied. If a conjecture is proved to be true, then it becomes a theorem. Note that there is no "false theorem" — if it is false, it is not a theorem. Another synonym for theorem is *lemma*. There is no logical difference among the terms "theorem", "lemma", and "proposition", but it is usual to endow them with a psychological hierarchy. Generally, a theorem is supposed to bear a greater importance (which is subjective) and a lemma is often viewed as an intermediate theorem (which may be very important indeed) that will be applied to prove a further theorem. Propositions are sometimes placed a step below, either as an isolated theorem or as an auxiliary result. A *corollary* is, of course, a theorem that comes out as a consequence of a previously proved theorem (i.e., whose proof is mainly based on an application of that previous theorem). Unlike "conjecture", "proposition", "lemma", "theorem", and "corollary", the term *axiom* (or *postulate*) is applied to a fundamental statement (or assumption, or hypothesis) upon which a theory (i.e., a set of theorems) is built. Clearly, a set of axioms (or, more appropriately, a system of axioms) should be *consistent* (i.e., they should not lead to a contradiction), and they are said to be *independent* if none of them is a theorem (i.e., if none of them can be proved by the remaining axioms).

1.2 Sets and Relations

If x is an element of a set X, then we shall write $x \in X$ (meaning that x *belongs* to X, or x is *contained* in X). Otherwise (i.e., if x is not an element of X), $x \notin X$. We also write $A \subseteq B$ to mean that a set A is a *subset* of a set B ($A \subseteq B \iff \{x \in A \Rightarrow x \in B\}$). In such a case A is said to be *included* in B. The *empty set*, which is a subset of every set, will be denoted by $\varnothing$. Two sets A and B are *equal* (notation: $A = B$) if $A \subseteq B$ and $B \subseteq A$. If A is a subset of B but not equal to B, then we say that A is a *proper subset* of B and write $A \subset B$. In such a case A is said to be *properly included* in B. A *nontrivial subset* of a set X is a nonempty proper subset of it. If $P(\ )$ is a predicate which is meaningful for every element x of a set X (so that $P(x)$ is a proposition for each x in X), then $\{x \in X\colon P(x)\}$ will denote the subset of X consisting of all those elements x of X for which the proposition $P(x)$ is true. The *complement* of a subset A of a set X, denoted by $X \backslash A$, is the subset $\{x \in X\colon x \notin A\}$. If A and B are sets, the *difference* between A and B, or the *relative complement* of B in A, is the set

$$A\backslash B \;=\; \big\{x \in A\colon\; x \notin B\big\}.$$

We shall also use the standard notation $\cup$ and $\cap$ for *union* and *intersection*, respectively ($x \in A \cup B \iff \{x \in A \text{ or } x \in B\}$ and $x \in A \cap B \iff \{x \in A \text{ and } x \in B\}$). The sets A and B are *disjoint* if $A \cap B = \varnothing$ (i.e., if they have an empty intersection). The *symmetric difference* (or *Boolean sum*) of two sets A and B is the set

$$A \triangledown B \;=\; (A\backslash B) \cup (B\backslash A) \;=\; (A \cup B)\backslash(A \cap B).$$

The terms *class*, *family*, and *collection* (as their related terms prefixed with "sub") will be used as synonyms for set (usually applied for sets of sets, but not necessarily) without imposing any hierarchy among them. If $\mathcal{X}$ is a collection of subsets of a given set X, then $\bigcup \mathcal{X}$ will denote the union of all sets in $\mathcal{X}$. Similarly, $\bigcap \mathcal{X}$ will denote the intersection of all sets in $\mathcal{X}$ (alternative notation: $\bigcup_{A\in\mathcal{X}} A$ and $\bigcap_{A\in\mathcal{X}} A$). An important statement about complements that exhibits the duality between union and intersection are the *De Morgan laws*:

$$X\backslash\Big(\bigcup_{A\in\mathcal{X}} A\Big) = \bigcap_{A\in\mathcal{X}} (X\backslash A) \quad \text{and} \quad X\backslash\Big(\bigcap_{A\in\mathcal{X}} A\Big) = \bigcup_{A\in\mathcal{X}} (X\backslash A).$$

The *power set* of any set X, denoted by $\wp(X)$, is the collection of all subsets of X. Note that $\bigcup \wp(X) = X \in \wp(X)$ and $\bigcap \wp(X) = \varnothing \in \wp(X)$.

A *singleton* in a set X is a subset of X containing one and only one point of X (notation: $\{x\} \subseteq X$ is a singleton on $x \in X$). A *pair* (or a *doubleton*) is a set containing just two points, say $\{x, y\}$, where x is an element of a set X and y is an element of a set Y. A pair of points $x \in X$ and $y \in Y$ is an *ordered pair*, denoted by (x, y), if x is regarded as the first member of the pair and y is regarded as the second. The *Cartesian product* of two sets X and Y, denoted by $X\times Y$, is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. A *relation* R between two sets X and Y is any subset of the Cartesian product $X\times Y$. If R is a relation between X and Y and (x, y) is a pair in $R \subseteq X\times Y$, then we say that x is *related* to y under R (or x and y are related by R), and write xRy (instead of $(x, y) \in R$). Tautologically, for any ordered pair (x, y) in $X\times Y$, either $(x, y) \in R$ or $(x, y) \notin R$ (i.e., either xRy or $x\not Ry$). A relation between a set X and itself is called a *relation on* X. If X and Y are sets and if R is a relation between X and Y, then the *graph* of the relation R is the subset of $X\times Y$

$$G_R \;=\; \big\{(x, y) \in X\times Y\colon\; xRy\big\}.$$

A relation R clearly coincides with its graph G_R.

1.3 Functions

Let x be an arbitrary element of a set X and let y and z be arbitrary elements of a set Y. A relation F between the sets X and Y is a *function* if xFy and

xFz imply $y = z$. In other words, a relation F between a set X and a set Y is called a *function from X to Y* (or a *mapping of X into Y*) if for each $x \in X$ there exists a *unique* $y \in Y$ such that xFy. The terms *map* and *transformation* are often used as synonyms for function and mapping. (Sometimes the terms *correspondence* and *operator* are also used but we shall keep them for special kinds of functions.) It is usual to write $F\colon X \to Y$ to indicate that F is a mapping of X into Y, and $y = F(x)$ (or $y = Fx$) instead of xFy. If $y = F(x)$, we say that F *maps* x to y, so that $F(x) \in Y$ is the *value* of the function F at $x \in X$. Equivalently, $F(x)$, which is a point in Y, is the *image* of the point x in X under F. It is also customary to use the abbreviation "the function $X \to Y$ defined by $x \mapsto F(x)$" for a function from X to Y that assigns to each x in X the value $F(x)$ in Y. A *Y-valued function on X* is precisely a function from X to Y. If Y is a subset of the set $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{Z}$, then *complex-valued function*, *real-valued function*, or *integer-valued function*, respectively, are usual terminologies. An X-valued function on X (i.e., a function $F\colon X \to X$ from X to itself) is referred to as a *function on X*. The collection of all functions from a set X to a set Y will be denoted by Y^X. Indeed, $Y^X \subseteq \wp(X \times Y)$.

Consider a function $F\colon X \to Y$. The set X is called the *domain* of F and the set Y is called the *codomain* of F. If A is a subset of X, then the *image of A under F*, denoted by $F(A)$, is the subset of Y consisting of all points y of Y such that $y = F(x)$ for some $x \in A$:

$$F(A) \;=\; \big\{y \in Y\colon\ y = F(x) \text{ for some } x \in A \subseteq X\big\}.$$

On the other hand, if B is a subset of Y, then the *inverse image of B under F* (or the *pre-image of B under F*), denoted by $F^{-1}(B)$, is the subset of X made up of all points x in X such that $F(x)$ lies in B:

$$F^{-1}(B) \;=\; \big\{x \in X\colon\ F(x) \in B \subseteq Y\big\}.$$

The *range* of F, denoted by $\mathcal{R}(F)$, is the image of X under F. Thus

$$\mathcal{R}(F) \;=\; F(X) \;=\; \big\{y \in Y\colon\ y = F(x) \text{ for some } x \in X\big\}.$$

If $\mathcal{R}(F)$ is a singleton, then F is said to be a *constant* function. If the range of F coincides with the codomain (i.e., if $F(X) = Y$), then F is a *surjective* function. In this case F is said to map X *onto* Y. A function F is *injective* (or F is a *one-to-one* mapping) if its domain X does not contain two elements with the same image. In other words, a function $F\colon X \to Y$ is injective if $F(x) = F(x')$ implies $x = x'$ for every x and x' in X. A *one-to-one correspondence* between a set X and a set Y is a one-to-one mapping of X onto Y; that is, a surjective and injective function (also called a *bijective* function).

If A is an arbitrary subset of X and F is a mapping of X into Y, then the function $G\colon A \to Y$ such that $G(x) = F(x)$ for each $x \in A$ is *the restriction of F to A*. Conversely, if $G\colon A \to Y$ is the restriction of $F\colon X \to Y$ to some subset

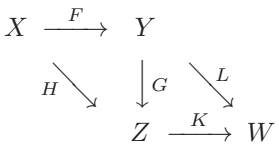
A of X, then F is *an extension of* G *over* X. It is usual to write $G = F|_A$. Note that $\mathcal{R}(F|_A) = F(A)$.

Let A be a subset of X and consider a function $F\colon A \to X$. An element x of A is a *fixed point* of F (or F *leaves* x *fixed*) if $F(x) = x$. The function $J\colon A \to X$ defined by $J(x) = x$ for every $x \in A$ is the *inclusion map* (or the *embedding*, or the *injection*) of A into X. In other words, the inclusion map of A into X is the function $J\colon A \to X$ that leaves each point of A fixed. The inclusion map of X into X is called the *identity map on* X and denoted by I, or by I_X when necessary (i.e., the identity on X is the function $I\colon X \to X$ such that $I(x) = x$ for every $x \in X$). Thus the inclusion map of a subset of X is the restriction to that subset of the identity map on X. Now consider a function on X; that is, a mapping $F\colon X \to X$ of X into itself. A subset of X, say A, is *invariant for* F (or *invariant under* F, or F*-invariant*) if $F(A) \subseteq A$. In this case the restriction of F to A, $F|_A\colon A \to X$, has its range included in A: $\mathcal{R}(F|_A) = F(A) \subseteq A \subseteq X$. Therefore, we shall often think of the restriction of $F\colon X \to X$ to an invariant subset $A \subseteq X$ as a mapping of A into itself: $F|_A\colon A \to A$. It is in this sense that the inclusion map of a subset of X can be thought of as the identity map on that subset: they differ only in that one has a larger codomain than the other.

Let $F\colon X \to Y$ be a function from a set X to a set Y, and let $G\colon Y \to Z$ be a function from the set Y to a set Z. Since the range of F is included in the domain of G, $\mathcal{R}(F) \subseteq Y$, consider the restriction of G to the range of F, $G|_{\mathcal{R}(F)}\colon \mathcal{R}(F) \to Z$. The *composition* of G and F, denoted by $G \circ F$ (or simply by GF), is the function from X to Z defined by $(G \circ F)(x) = G|_{\mathcal{R}(F)}(F(x)) = G(F(x))$ for every $x \in X$. It is usual to say that the diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ & \searrow^{H} & \downarrow{\scriptstyle G} \\ & & Z \end{array}$$

commutes if $H = G \circ F$. Although the above diagram is said to be *commutative* whenever H is the composition of G and F, the composition itself is not a commutative operation even when such a commutation makes sense. For instance, if $X = Y = Z$ and F is a constant function on X, say $F(x) = a \in X$ for every $x \in X$, then $G \circ F$ and $F \circ G$ are constant functions on X as well: $(G \circ F)(x) = G(a)$ and $(F \circ G)(x) = a$ for every $x \in X$. However $G \circ F$ and $F \circ G$ need not be the same (unless a is a fixed point of G). Composition may not be commutative but it is always associative. If F maps X into Y, G maps Y into Z, and K maps Z into W, then we can consider the compositions $K \circ (G \circ F)\colon X \to W$ and $(K \circ G) \circ F\colon X \to W$. It is readily verified that $K \circ (G \circ F) = (K \circ G) \circ F$. For this reason we may and shall drop the parentheses. In other words, the diagram



commutes (i.e., $H = G \circ F$, $L = K \circ G$, and $K \circ H = L \circ F$). If F is a function on set X, then the composition of $F\colon X \to X$ with itself, $F \circ F$, is denoted by F^2. Likewise, for any positive integer $n \in \mathbb{N}$, F^n denotes the composition of F with itself n times, $F \circ \cdots \circ F\colon X \to X$, which is called the nth *power* of F. A function $F\colon X \to X$ is *idempotent* if $F^2 = F$ (and hence $F^n = F$ for every $n \in \mathbb{N}$). It is easy to show that the range of an idempotent function is precisely the set of all its fixed points. In fact, $F = F^2$ if and only if $\mathcal{R}(F) = \{x \in X\colon\ F(x) = x\}$.

Suppose $F\colon X \to Y$ is an injective function. Thus, for an arbitrary element of $\mathcal{R}(F)$, say y, there exists a *unique* element of X, say x_y, such that $y = F(x_y)$. This defines a function from $\mathcal{R}(F)$ to X, $F^{-1}\colon \mathcal{R}(F) \to X$, such that $x_y = F^{-1}(y)$. Hence $y = F(F^{-1}(y))$. On the other hand, if x is an arbitrary element of X, then $F(x)$ lies in $\mathcal{R}(F)$ so that $F(x) = F(F^{-1}(F(x)))$. Since F is injective, $x = F^{-1}(F(x))$. Conclusion: For every injective function $F\colon X \to Y$ there exists a (unique) function $F^{-1}\colon \mathcal{R}(F) \to X$ such that $F^{-1}F\colon X \to X$ is the identity on X (and $FF^{-1}\colon \mathcal{R}(F) \to \mathcal{R}(F)$ is the identity on $\mathcal{R}(F)$). F^{-1} is called the *inverse of F on $\mathcal{R}(F)$*: *an injective function has an inverse on its range*. If F is also surjective, then $F^{-1}\colon Y \to X$ is called the *inverse* of F. Thus, an injective and surjective function is also called an *invertible* function (in addition to its other names).

1.4 Equivalence Relations

Let x, y, and z be arbitrary elements of a set X. A relation R on X is

reflexive if xRx for every $x \in X$,
transitive if xRy and yRz imply xRz, and
symmetric if xRy implies yRx.

An *equivalence relation* on a set X is a relation $\sim$ on X that is reflexive, transitive, and symmetric. If $\sim$ is an equivalence relation on a set X, then the *equivalence class* of an arbitrary element x of X (with respect to $\sim$) is the set

$$[x] = \{x' \in X\colon\ x' \sim x\}.$$

Given an equivalence relation $\sim$ on a set X, the *quotient space of X modulo* $\sim$, denoted by $X/\sim$, is the collection

$$X/\sim\ = \{[x] \subseteq X\colon\ x \in X\}$$

of the equivalence classes (with respect to $\sim$) of every $x \in X$. Set

$$\pi(x) = [x]$$

in $X/\sim$ for each x in X. This defines a surjective map $\pi\colon X \to X/\sim$ which is called the *natural mapping* of X onto $X/\sim$.

Let $\mathcal{X}$ be any collection of nonempty subsets of a set X. It *covers* X (or $\mathcal{X}$ is a *covering* of X) if $X = \bigcup \mathcal{X}$ (i.e., if every point in X belongs to some set in $\mathcal{X}$). The collection $\mathcal{X}$ is *disjoint* if the sets in $\mathcal{X}$ are pairwise disjoint (i.e., $A \cap B = \varnothing$ whenever A and B are distinct sets in $\mathcal{X}$). A *partition* of a set X is a disjoint covering of it.

Let $\approx$ be an equivalence relation on a set X, and let $X/\approx$ be the quotient space of X modulo $\approx$. It is clear that

$$X/\approx \text{ is a partition of } X.$$

Conversely, let $\mathcal{X}$ be any partition of a set X and define a relation $\sim/\mathcal{X}$ on X as follows: for every x, x' in X, x is related to x' under $\sim/\mathcal{X}$ (i.e., $x \sim/\mathcal{X} x'$) if x and x' belong to the same set in $\mathcal{X}$. In fact,

$$\sim/\mathcal{X} \text{ is an equivalence relation on } X,$$

which is called the *equivalence relation induced* by a partition $\mathcal{X}$. It is readily verified that the quotient space of X modulo the equivalence relation induced by the partition $\mathcal{X}$ coincides with $\mathcal{X}$ itself, just as the equivalence relation induced by the quotient space of X modulo the equivalence relation $\approx$ on X coincides with $\approx$. Symbolically,

$$X/(\sim/\mathcal{X}) = \mathcal{X} \quad \text{and} \quad \sim/(X/\approx) = \approx.$$

Thus an equivalence relation $\approx$ on X induces a partition $X/\approx$ of X, which in turn induces back an equivalence relation $\sim/(X/\approx)$ on X that coincides with $\approx$. On the other hand, a partition $\mathcal{X}$ of X induces an equivalence relation $\sim/\mathcal{X}$ on X, which in turn induces back a partition $X/(\sim/\mathcal{X})$ of X that coincides with $\mathcal{X}$. Conclusion: *The collection of all equivalence relations on a set X is in a one-to-one correspondence with the collection of all partitions of X.*

1.5 Ordering

Let x and y be arbitrary elements of a set X. A relation R on X is

$$\textit{antisymmetric} \text{ if } xRy \text{ and } yRx \text{ imply } x = y.$$

A relation $\leq$ on a nonempty set X is a *partial ordering* of X if it is reflexive, transitive, and antisymmetric. If $\leq$ is a partial ordering on a set X, the notation $x < y$ means $x \leq y$ and $x \neq y$. Moreover, $y > x$ and $y \geq x$ are just another

way to write $x < y$ and $x \leq y$, respectively. Thus a *partially ordered set* is a pair $(X, \leq)$ where X is a nonempty set and $\leq$ is a partial ordering of X (i.e., a nonempty set equipped with a partial ordering on it). *Warning*: It may happen that $x \not\leq y$ and $y \not\leq x$ for some $(x, y) \in X \times X$. Let $(X, \leq)$ be a partially ordered set, and let A be a subset of X. Note that $(A, \leq)$ is a partially ordered set as well. An element $x \in X$ is an *upper bound* for A if $y \leq x$ for every $y \in A$. Similarly, an element $x \in X$ is a *lower bound* for A if $x \leq y$ for every $y \in A$. A subset A of X is *bounded above* in X if it has an upper bound in X, and *bounded below* in X if it has a lower bound in X. It is *bounded* if it is bounded both above and below.

If a subset A of a partially ordered set X is bounded above in X and if some upper bound of A belongs to A, then this (*unique*) element of A is the *maximum* of A (or the *greatest* or *biggest* element of A), denoted by $\max A$. Similarly, if A is bounded below in X and if some lower bound of A belongs to A, then this (*unique*) element of A is the *minimum* of A (or the *least* or *smallest* element of A), denoted by $\min A$. An element $x \in A$ is *maximal* in A if there is no element $y \in A$ such that $x < y$ (equivalently, if $x \not< y$ for every $y \in A$). Similarly, an element $x \in A$ is *minimal* in A if there is no element $y \in A$ such that $y < x$ (equivalently, if $y \not< x$ for every $y \in A$). Note that $x \not< y$ (or $y \not< x$) does not mean that $y \leq x$ (or $x \leq y$) so that the concepts of *a* maximal (or *a* minimal) element in A and that of *the* maximum (or *the* minimum) element of A do not coincide.

Example 1.A. A collection of many (e.g., two) pairwise disjoint nonempty subsets of a set, equipped with the partial ordering defined by the inclusion relation $\subseteq$, has no maximum, no minimum, and every element in it is both maximal and minimal. On the other hand, the collection of all infinite subsets of an infinite set, whose complements are also infinite, has no maximal element in the inclusion ordering $\subseteq$. (The notion of infinite sets will be introduced later in Section 1.8 — for instance, the set all even natural numbers is an infinite subset of $\mathbb{N}$ that has an infinite complement.)

Let A be a subset of a partially ordered set X. Let $U_A \subseteq X$ be the set of all upper bounds of A, and let $V_A \subseteq X$ be the set of all lower bounds of A. If U_A is nonempty and has a minimum element, say $u = \min U_A$, then $u \in U_A$ is called the *supremum* (or the *least upper bound*) of A (notation: $u = \sup A$). Similarly, if V_A is nonempty and has a maximum, say $v = \max V_A$, then $v \in V_A$ is called the *infimum* (or the *greatest lower bound*) of A (notation: $v = \inf A$). A bounded set may not have a supremum or an infimum. However, if a set A has a maximum (or a minimum), then $\sup A = \max A$ (or $\inf A = \min A$). Moreover, if a set A has a supremum (or an infimum) *in* A, then $\sup A = \max A$ (or $\inf A = \min A$). If a pair $\{x, y\}$ of elements of a partially ordered set X has a supremum or an infimum in X, then we shall use the following notation: $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$.

Let $F\colon X \to Y$ be a function from a set X to a partially ordered set Y. Thus the range of F, $F(X) \subseteq Y$, is a partially ordered set. An *upper bound* for F is an upper bound for $F(X)$, and F is *bounded above* if it has an upper bound. Similarly, a *lower bound* for F is a lower bound for $F(X)$, and F is *bounded below* if it has a lower bound. If a function F is bounded both above and below, then it is said to be *bounded*. The *supremum* of F, $\sup_{x\in X} F(x)$, and the *infimum* of F, $\inf_{x\in X} F(x)$, are defined by $\sup_{x\in X} F(x) = \sup F(X)$ and $\inf_{x\in X} F(x) = \inf F(X)$. Now suppose X also is partially ordered and take an arbitrary pair of points x_1, x_2 in X. F is an *increasing* function if $x_1 \leq x_2$ in X implies $F(x_1) \leq F(x_2)$ in Y, and *strictly increasing* if $x_1 < x_2$ in X implies $F(x_1) < F(x_2)$ in Y. (For notational simplicity we are using the same symbol $\leq$ to denote both the partial ordering of X and the partial ordering of Y.) In a similar way we can define *decreasing* and *strictly decreasing* functions between partially ordered sets. If a function is either decreasing or increasing, then it is said to be *monotone*.

1.6 Lattices

Let X be a partially ordered set. If every pair $\{x, y\}$ of elements of X is bounded above, then X is a *directed set* (or the set X is said to be *directed upward*). If every pair $\{x, y\}$ is bounded below, then X is said to be *directed downward*. X is a *lattice* if every pair of elements of X has a supremum and a infimum in X (i.e., if there exits a unique $u \in X$ and a unique $v \in X$ such that $u = x \vee y$ and $v = x \wedge y$ for every pair $x \in X$ and $y \in X$). A nonempty subset A of a lattice X that contains $x \vee y$ and $x \wedge y$ for every x and y in A is a *sublattice* of X (and hence a lattice itself). Every lattice is directed both upward and downward. If every bounded subset of X has a supremum and an infimum, then X is a *boundedly complete lattice*. If every subset of X has a supremum and an infimum, then X is a *complete lattice*. The following chain of implications:

$$\textit{complete lattice} \Rightarrow \textit{boundedly complete lattice} \Rightarrow \textit{lattice} \Rightarrow \textit{directed set}$$

is clear enough, and neither of them can be reversed. If X is a complete lattice, then X has a supremum and an infimum in X, which actually are the maximum and the minimum of X, respectively. Since $\min X \in X$ and $\max X \in X$, this shows that a complete lattice in fact is nonempty (even if this had not been assumed when we defined a partially ordered set). Likewise, the empty set $\varnothing$ of a complete lattice X has a supremum and an infimum. Since every element of X is both an upper and a lower bound for $\varnothing$, it follows that $U_\varnothing = V_\varnothing = X$. Hence $\sup \varnothing = \min X$ and $\inf \varnothing = \max X$.

Example 1.B. The power set $\wp(X)$ of a set X is a complete lattice in the inclusion ordering $\subseteq$, where $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ for every pair $\{A, B\}$ of subsets of X. In this case, $\sup \varnothing = \min \wp(X) = \varnothing$ and $\inf \varnothing = \max \wp(X) = X$.

Example 1.C. The real line $\mathbb{R}$ with its natural ordering $\leq$ is a boundedly complete lattice but not a complete lattice (and so is its sublattice $\mathbb{Z}$ of all integers). The set $A = \{x \in \mathbb{R}\colon\ 0 \leq x \leq 1\}$, as a sublattice of $(\mathbb{R}, \leq)$, is a complete lattice where $\sup \varnothing = \min A = 0$ and $\inf \varnothing = \max A = 1$. The set of all rational numbers $\mathbb{Q}$ is a sublattice of $\mathbb{R}$ (in the natural ordering $\leq$) but not a boundedly complete lattice — e.g., the set $\{x \in \mathbb{Q}\colon\ x^2 \leq 2\}$ is bounded in $\mathbb{Q}$ but has no infimum and no supremum in $\mathbb{Q}$.

Example 1.D. The notion of connectedness needs topology and we shall define it in due course. (Connectedness will be defined in Chapter 3.) However, if the reader is already familiar with the concept of a connected subset of the plane, then he can appreciate now a rather simple example of a directed set that is not a lattice The subcollection of $\wp(\mathbb{R}^2)$ made up of all connected subsets of the Euclidean plane $\mathbb{R}^2$ is a directed set in the inclusion ordering $\subseteq$ (both upward and downward) but not a lattice.

Lemma 1.1. (Banach–Tarski). *An increasing function on a complete lattice has a fixed point.*

Proof. Let $(X, \leq)$ be a partially ordered set, consider a function $F\colon X \to X$, and set $A = \{x \in X\colon\ F(x) \leq x\}$. Suppose X is a complete lattice. Then X has a supremum in X $(\sup X = \max X)$. Since $\max X \in X$, it follows that $F(\max X) \in X$ so that $F(\max X) \leq \max X$. Conclusion: A is nonempty. Take $x \in A$ arbitrary and let a be the infimum of A $(a = \inf A \in X)$. If F is increasing, then $F(a) \leq F(x) \leq x$ since $a \leq x$ and $x \in A$. Hence $F(a)$ is a lower bound for A, and so $F(a) \leq a$. Thus $a \in A$. On the other hand, since $F(x) \leq x$ and F is increasing, $F(F(x)) \leq F(x)$. Thus $F(x) \in A$ so that $F(A) \subseteq A$, and hence $F(a) \in A$ (for $a \in A$), which implies that $a = \inf A \leq F(a)$. Therefore $a \leq F(a) \leq a$. Thus (antisymmetry) $F(a) = a$. □

The next theorem is an extremely important result that plays a central role in Section 1.8. Its proof is based on the previous lemma.

Theorem 1.2. (Cantor–Bernstein). *If there exist an injective mapping of X into Y and an injective mapping of Y into X, then there exists a one-to-one correspondence between the sets X and Y.*

Proof. First note that the theorem statement can be translated into the following problem. Given an injective function from X to Y and also an injective function from Y to X, construct a bijective function from X to Y. Thus consider two functions $F\colon X \to Y$ and $G\colon Y \to X$. Let $\wp(X)$ be the power set of X. For each $A \in \wp(X)$ set

$$\Phi(A) \ = \ X \backslash G(Y \backslash F(A)).$$

It is readily verified that $\Phi\colon \wp(X) \to \wp(X)$ is an increasing function with respect to the inclusion ordering of $\wp(X)$. Therefore, by the Banach–Tarski

Lemma, it has a fixed point in the complete lattice $\wp(X)$. That is, there is an $A_0 \in \wp(X)$ such that $\Phi(A_0) = A_0$. Hence $A_0 = X\backslash G(Y\backslash F(A_0))$ so that

$$X\backslash A_0 \;=\; G(Y\backslash F(A_0)).$$

Thus $X\backslash A_0$ is included in the range of G. If $F\colon X \to Y$ and $G\colon Y \to X$ are injective, then it is easy to show that the function $H\colon X \to Y$, defined by

$$H(x) \;=\; \begin{cases} F(x), & x \in A_0, \\ G^{-1}(x), & x \in X\backslash A_0, \end{cases}$$

is injective and surjective. □

If X is a partially ordered set such that for every pair x, y of elements of X either $x \leq y$ or $y \leq x$, then X is *simply ordered* (synonyms: *linearly ordered*, *totally ordered*). A simply ordered set is also called a *chain*. Note that, in this particular case, the concepts of maximal element and maximum element (as well as minimal element and minimum element) coincide. Also note that if F is a function from a simply ordered set X to any partially ordered set Y, then *F is strictly increasing if and only if it is increasing and injective*. It is clear that every simply ordered set is a lattice. For instance, any subset of the real line $\mathbb{R}$ (e.g., $\mathbb{R}$ itself or $\mathbb{Z}$) is simply ordered.

Example 1.E. Let $\leq$ be a simple ordering on a set X and recall that $x < y$ means $x \leq y$ and $x \neq y$. This defines a transitive relation $<$ on X that satisfies the *trichotomy law*: for every pair $\{x, y\}$ in X exactly one of the three statements $x < y$, $x = y$, or $y < x$ is true. Conversely, if $<$ is a transitive relation on a set X that satisfies the trichotomy law, and if a relation $\leq$ on X is defined by setting $x \leq y$ whenever either $x < y$ or $x = y$, then $\leq$ is a simple ordering on X. Thus, according to the above notation, $\leq$ is a simple ordering on a set X if and only if $<$ is a transitive relation on X that satisfies the trichotomy law.

If X is a partially ordered set such that every nonempty subset of it has a minimum, then X is said to be *well ordered*. Every well-ordered set is simply ordered. Example: Any subset of $\mathbb{N}_0$ (equipped with its natural ordering) is well ordered.

1.7 Indexing

Let F be a function from a set X to a set Y. Another way to look at the range of F is: for each $x \in X$ set $y_x = F(x) \in Y$ and note that $F(X) = \{y_x \in Y\colon\, x \in X\}$, which can also be written as $\{y_x\}_{x\in X}$. Thus the domain X can be thought of as an *index set*, the range $\{y_x\}_{x\in X}$ as a family of elements of Y indexed by an index set X (an *indexed family*), and the function $F\colon X \to Y$

as an *indexing*. An indexed family $\{y_x\}_{x\in X}$ may contain elements y_a and y_b, for a and b in X, such that $y_a = y_b$. If $\{y_x\}_{x\in X}$ has the property that $y_a \neq y_b$ whenever $a \neq b$, then it is said to be an indexed family of *distinct* elements. Observe that $\{y_x\}_{x\in X}$ is a family of distinct elements if and only if the function $F\colon X \to Y$ (i.e., the indexing process) is injective. The identity mapping on an arbitrary set X can be viewed as an indexing of X, the *self-indexing* of X. Thus any set X can be thought of as an indexed family (the range of the self-indexing of itself).

A mapping of the set $\mathbb{N}$ (or $\mathbb{N}_0$, but not $\mathbb{Z}$) into a set Y is called a *sequence* (or an *infinite sequence*). Notation: $\{y_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\geq 1}$, $\{y_n\}_{n=1}^{\infty}$, or simply $\{y_n\}$. Thus a *Y-valued sequence* (or a *sequence of elements in Y*, or even a *sequence in Y*) is precisely a function from $\mathbb{N}$ to Y, which is commonly thought of as an indexed family (indexed by $\mathbb{N}$) where the indexing process (i.e., the function itself) is often omitted. The elements y_n of $\{y_n\}$ are sometimes referred to as the *entries* of the sequence $\{y_n\}$. If Y is a subset of the set $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{Z}$, then *complex-valued sequence*, *real-valued sequence*, or *integer-valued sequence*, respectively, are usual terminologies.

Let $\{X_\gamma\}_{\gamma\in\Gamma}$ be an indexed family of sets. The *Cartesian product* of $\{X_\gamma\}_{\gamma\in\Gamma}$, denoted by $\prod_{\gamma\in\Gamma}X_\gamma$, is the set consisting of all indexed families $\{x_\gamma\}_{\gamma\in\Gamma}$ such that $x_\gamma \in X_\gamma$ for every $\gamma \in \Gamma$. In particular, if $X_\gamma = X$ for all $\gamma \in \Gamma$, where X is a fixed set, then $\prod_{\gamma\in\Gamma}X_\gamma$ is precisely the collection of all functions from Γ to X. That is,

$$\prod_{\gamma\in\Gamma} X \;=\; X^{\Gamma}.$$

Recall: X^Γ denotes the collection of all functions from a set Γ to a set X. Suppose $\Gamma = \mathbb{I}_n$, where $\mathbb{I}_n = \{i \in \mathbb{N}\colon i \leq n\}$ for some $n \in \mathbb{N}$ ($\mathbb{I}_n$ is called an *initial segment* of $\mathbb{N}$). The Cartesian product of $\{X_i\}_{i\in\mathbb{I}_n}$ (or $\{X_i\}_{i=1}^n$), denoted by $\prod_{i\in\mathbb{I}_n}X_i$ or $\prod_{i=1}^n X_i$, is the set $X_1\times\cdots\times X_n$ of all *ordered n-tuples* $(x_1,\ldots,x_n)$ with $x_i \in X_i$ for every $i \in \mathbb{I}_n$. Moreover, if $X_i = X$ for all $i \in \mathbb{I}_n$, then $\prod_{i\in\mathbb{I}_n}X$ is the Cartesian product of n copies of X which is denoted by X^n (instead of $X^{\mathbb{I}_n}$). The n-tuples $(x_1,\ldots,x_n)$ in X^n are also called *finite sequences* (as functions from an initial segment of $\mathbb{N}$ into X). Accordingly, $\prod_{n\in\mathbb{N}}X$ is referred to as the Cartesian product of countably infinite copies of X which coincides with $X^{\mathbb{N}}$: the set of all X-valued (infinite) sequences.

An exceptionally helpful way of defining an infinite sequence is given by the *Principle of Recursive Definition* which says that, *if F is a function from a nonempty set X into itself, and if x is an arbitrary element of X, then there exists a unique X-valued sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_1 = x$ and $x_{n+1} = F(x_n)$ for every $n \in \mathbb{N}$*. The existence of such a unique sequence is intuitively clear, and it can be easily proved by induction (i.e., by using the Principle of Mathematical Induction). A slight generalization reads as follows. *For each $n \in \mathbb{N}$ let G_n be a mapping of X^n into X, and let x be an arbitrary*

element of X. Then there exists a unique X-valued sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_1 = x$ and $x_{n+1} = G_n(x_1, \ldots, x_n)$ for every $n \in \mathbb{N}$.

Since sequences are functions of $\mathbb{N}$ (or of $\mathbb{N}_0$) to a set X, the terms associated with the notion of being *bounded* clearly apply to sequences in a partially ordered set X. In particular, if X is a partially ordered set, and if $\{x_n\}$ is an X-valued sequence, then $\sup_n x_n$ and $\inf_n x_n$ are defined as the supremum and infimum, respectively, of the partially ordered indexed family $\{x_n\}$. Since $\mathbb{N}$ and $\mathbb{N}_0$ (with their natural ordering) are partially ordered sets (well-ordered, really), the terms associated with the property of being *monotone* (such as *increasing, decreasing, strictly increasing, strictly decreasing*) also apply to sequences in a partially ordered set X. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in a set Z, and let $\{n_k\}_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive integers (i.e., a strictly increasing sequence in $\mathbb{N}$). If we think of $\{n_k\}$ and $\{z_n\}$ as functions, then the range of the former is a subset of the domain of the latter (i.e., the indexed family $\{n_k\}_{k\in\mathbb{N}}$ is a subset of $\mathbb{N}$). Thus we may consider the composition of $\{z_n\}$ with $\{n_k\}$, say $\{z_{n_k}\}$, which is again a function of $\mathbb{N}$ to Z (i.e., $\{z_{n_k}\}$ is a sequence in Z). Since $\{n_k\}$ is strictly increasing, to each element of the indexed family $\{z_{n_k}\}_{k\in\mathbb{N}}$ there corresponds a unique element of the indexed family $\{z_n\}_{n\in\mathbb{N}}$. In this case the Z-valued sequence $\{z_{n_k}\}$ is called a *subsequence* of $\{z_n\}$.

A sequence is a function whose domain is either $\mathbb{N}$ or $\mathbb{N}_0$, but a similar concept could be likewise defined for a function on any well-ordered domain. Even in this case, a function with domain $\mathbb{Z}$ (equipped with its natural ordering) would not be a sequence. Now recall the following string of (nonreversible) implications:

$$\textit{well-ordered} \Rightarrow \textit{simply ordered} \Rightarrow \textit{lattice} \Rightarrow \textit{directed set}.$$

This might suggest an extension of the concept of sequence by allowing functions whose domains are directed sets. A *net* in a set X is a family of elements of X indexed by a directed set Γ. In other words, if Γ is a directed set and X is an arbitrary set, then an indexed family $\{x_\gamma\}_{\gamma\in\Gamma}$ of elements of X indexed by Γ is called a net in X indexed by Γ. Examples: Every X-valued sequence $\{x_n\}$ is a net in X. In fact, sequences are prototypes of nets. Every X-valued function on $\mathbb{Z}$ (notation: $\{x_k\}_{k\in\mathbb{Z}}$, $\{x_k\}_{k=-\infty}^{\infty}$ or $\{x_k;\ k = 0, \pm1, \pm2, \ldots\}$) is a net (sometimes called *double sequences* or *bisequences*, although these nets are not sequences themselves).

1.8 Cardinality

Two sets, say X and Y, are said to be *equivalent* (denoted by $X \leftrightarrow Y$) if there exists a one-to-one correspondence between them. Clearly (see Problems 1.8 and 1.9), $X \leftrightarrow X$ (reflexivity), $X \leftrightarrow Y$ if and only if $Y \leftrightarrow X$ (symmetry), and

$X \leftrightarrow Z$ whenever $X \leftrightarrow Y$ and $Y \leftrightarrow Z$ for some set Z (transitivity). Thus, if there exists a set upon which $\leftrightarrow$ is a relation, then it is an equivalence relation. For instance, if the notion of equivalent sets is restricted to subsets of a given set X, then $\leftrightarrow$ is an equivalence relation on the power set $\wp(X)$. If $C = \{x_\gamma\}_{\gamma\in\Gamma}$ is an indexed family of *distinct* elements of a set X indexed by a set Γ (so that $x_\alpha \neq x_\beta$ for every $\alpha \neq \beta$ in Γ), then $C \leftrightarrow \Gamma$ (the very indexing process sets a one-to-one correspondence between Γ and C).

Let $\mathbb{N}$ be the set of all natural numbers and, for each $n \in \mathbb{N}$, consider the initial segment

$$\mathbb{I}_n \,=\, \big\{i \in \mathbb{N}\colon\ i \le n\big\}.$$

A set X is *finite* if it is either empty or equivalent to $\mathbb{I}_n$ for some $n \in \mathbb{N}$. A set is *infinite* if it is not finite. If X is finite and Y is equivalent to X, then Y is finite. Therefore, if X is infinite and Y is equivalent to X, then Y is infinite. It is easy to show by induction that, for each $n \in \mathbb{N}$, $\mathbb{I}_n$ has no proper subset equivalent to it. Thus (see Problem 1.12), every finite set has no proper subset equivalent to it. That is, *if a set has a proper equivalent subset, then it is infinite*. Moreover, such a subset must be infinite too (since it is equivalent to an infinite set).

Example 1.F. $\mathbb{N}$ *is infinite*. Indeed, it is easy to show that $\mathbb{N}_0$ is equivalent to $\mathbb{N}$ (the function $F\colon\mathbb{N}_0 \to \mathbb{N}$ such that $F(n) = n + 1$ for every $n \in \mathbb{N}_0$ will do the job). Thus $\mathbb{N}_0$ is infinite, because $\mathbb{N}$ is a proper subset of $\mathbb{N}_0$ which is equivalent to it, and so is $\mathbb{N}$.

To verify the converse (i.e., to show that *every infinite set has a proper equivalent subset*) we apply the Axiom of Choice.

Axiom of Choice. *If $\{X_\gamma\}_{\gamma\in\Gamma}$ is an indexed family of nonempty sets indexed by a nonempty index set Γ, then there exists an indexed family $\{x_\gamma\}_{\gamma\in\Gamma}$ such that $x_\gamma \in X_\gamma$ for each $\gamma \in \Gamma$.*

Theorem 1.3. *A set is infinite if and only if it has a proper equivalent subset.*

Proof. We have already seen that every set with a proper equivalent subset is infinite. To prove the converse, take an arbitrary element x_0 from an infinite set X_0, and an arbitrary k from $\mathbb{N}_0$. The Principle of Mathematical Induction allows us to construct, for each $k \in \mathbb{N}_0$, a finite family $\{X_n\}_{n=0}^{k+1}$ of infinite sets as follows. Set $X_1 = X_0\backslash\{x_0\}$ and, for every nonnegative integer $n \le k$, let X_{n+1} be recursively defined by the formula

$$X_{n+1} \,=\, X_n\backslash\{x_n\},$$

where $\{x_n\}_{n=0}^{k}$ is a finite set of pairwise distinct elements, each x_n being an arbitrary element taken from each X_n. Consider the (infinite) indexed family $\{X_n\}_{n\in\mathbb{N}_0} = \bigcup_{k\in\mathbb{N}_0}\{X_n\}_{n=0}^{k+1}$ and use the Axiom of Choice to ensure the existence of the indexed family $\{x_n\}_{n\in\mathbb{N}_0} = \bigcup_{k\in\mathbb{N}_0}\{x_n\}_{n=0}^{k}$, where each x_n is arbi-

trarily taken from each X_n. Next consider the sets $A_0 = \{x_n\}_{n\in\mathbb{N}_0} \subseteq X_0$, $A = \{x_n\}_{n\in\mathbb{N}} \subset A_0$, and

$$X = A \cup (X_0 \backslash A_0) \subset A_0 \cup (X_0 \backslash A_0) = X_0.$$

Note that $A_0 \leftrightarrow \mathbb{N}_0$ and $A \leftrightarrow \mathbb{N}$ (since the elements of A_0 are distinct). Thus $A_0 \leftrightarrow A$ (because $\mathbb{N}_0 \leftrightarrow \mathbb{N}$), and hence $X_0 \leftrightarrow X$ (see Problem 1.20). Conclusion: Any infinite set X_0 has a proper equivalent subset (i.e., there exists a proper subset X of X_0 such that $X_0 \leftrightarrow X$). □

If X is a finite set, so that it is equivalent to an initial segment $\mathbb{I}_n$ for some natural number n, then we say that its *cardinality* (or its *cardinal number*) is n. Thus the cardinality of a finite set X is just the number of elements of X (where, in this case, "numbering" means "indexing" as a finite set may be naturally indexed by an index set $\mathbb{I}_n$). We shall use the symbol $\#$ for cardinality. Thus $\#\mathbb{I}_n = n$, and so $\#X = n$ whenever $X \leftrightarrow \mathbb{I}_n$. For infinite sets the concept of cardinal number is a bit more complicated. We shall not define a cardinal number for an infinite set as we did for finite sets (which "number" should it be?) but define the following concept instead. Two sets X and Y are said to have the *same cardinality* if they are equivalent. Thus, to each set X we shall assign a symbol $\#X$, called the *cardinal number* of X (or the *cardinality* of X) according to the following rule: $\#X = \#Y \iff X \leftrightarrow Y$ — two sets have the same cardinality if and only if they are equivalent; otherwise (i.e., if they are not equivalent) we shall write $\#X \neq \#Y$. We say that the cardinality of a set X is *less than or equal to* the cardinality of a set Y (notation: $\#X \leq \#Y$) if there exists an injective mapping of X into Y (i.e., if there exists a subset Y' of Y such that $\#X = \#Y'$). Equivalently, $\#X \leq \#Y$ if there exists a surjective mapping of Y onto X (see Problem 1.6). If $\#X \leq \#Y$ and $\#X \neq \#Y$, then we shall write $\#X < \#Y$.

Theorem 1.4. (Cantor). $\#X < \#\wp(X)$ *for every set* X.

Proof. Consider the function $F\colon X \to \wp(X)$ defined by $F(x) = \{x\}$ for every $x \in X$, which is clearly injective. Thus $\#X \leq \#\wp(X)$. Hence $\#X < \#\wp(X)$ if and only if $\#X \neq \#\wp(X)$. Suppose $\#X = \#\wp(X)$ so that there is a surjective function $G\colon X \to \wp(X)$. Consider the set $A = \{x \in X\colon\ x \notin G(x)\}$ in $\wp(X)$ and take $a \in X$ such that $G(a) = A$ (recall: G is surjective). If $a \in A$, then $a \notin G(a)$ and so $a \notin A$, which is a contradiction. Conclusion 1: $a \notin A$. On the other hand, if $a \notin A$, then $a \in G(a)$ and so $a \in A$, which is another contradiction. Conclusion 2: $a \in A$. Therefore $\{a \notin A$ and $a \in A\}$, which is impossible (i.e., which also is a contradiction). Final conclusion: $\#X \neq \#\wp(X)$. □

Let A be a subset of a set X. The *characteristic function* of the set A is the map $\chi_A\colon X \to \{0, 1\}$ such that

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in X\backslash A. \end{cases}$$

It is clear that the correspondence between the subsets of X and their characteristic functions is one to one. Hence $\#\wp(X)$ coincides with the cardinality of the collection of all characteristic functions of subsets of X. More generally, let 2^X denote the collection of all maps of a set X into the set $\{0,1\}$ (i.e., set $2^X = \{0,1\}^X$).

Theorem 1.5. *$\#\wp(X) = \#2^X$ for every set X.*

Proof. Let F be a function from the collection 2^X to the power set $\wp(X)$ that assigns to each map $\varphi: X \to \{0,1\}$ the inverse image of the singleton $\{1\}$ under φ. That is, consider the function $F: 2^X \to \wp(X)$ defined by $F(\varphi) = \varphi^{-1}(\{1\})$ for every φ in 2^X.

Claim 1. F is surjective.

Proof. If A is a subset of X, then the characteristic function of A, χ_A in 2^X, is such that $\chi_A^{-1}(\{1\}) = A$. Thus $F(\chi_A) = A$ for every $A \in \wp(X)$. Therefore, $\wp(X) = \bigcup_{A\in\wp(X)} F(\chi_A) \subseteq \bigcup_{\varphi\in 2^X} F(\varphi) = \mathcal{R}(F)$. □

Claim 2. F is injective.

Proof. Take $\varphi, \psi \in 2^X$. If $\varphi \neq \psi$, then $\varphi(x) \neq \psi(x)$ for some $x \in X$. Thus $\varphi(x) = 0$ and $\psi(x) = 1$ (or vice versa), so that $x \notin \varphi^{-1}(\{1\})$ and $x \in \psi^{-1}(\{1\})$. Hence $\varphi^{-1}(\{1\}) \neq \psi^{-1}(\{1\})$ so that $F(\varphi) \neq F(\psi)$. □

Conclusion: F is a one-to-one correspondence between 2^X and $\wp(X)$. □

Although the next theorem may come as no surprise, it is all but trivial. This actually is a rewriting, in terms of the concept of cardinality, of the rather important Theorem 1.2.

Theorem 1.6. (Cantor–Bernstein). *Let X and Y be any sets. If $\#X \leq \#Y$ and $\#Y \leq \#X$, then $\#X = \#Y$.*

The Cantor–Bernstein Theorem exhibits an antisymmetry property. Note that reflexivity and transitivity are readily verified (see Problem 1.22) which, together with Theorem 1.6, lead to a partial ordering property. Behind the antisymmetry property of Theorem 1.6 there is in fact a simple ordering property. To establish it (Theorem 1.7 below) we shall rely on the following axiom.

Zorn's Lemma. *Let X be a partially ordered set. If every simply ordered subset of X* (i.e., *if each chain in X*) *has an upper bound in X, then X has a maximal element.*

The label "Zorn's Lemma" is inappropriate but has already been consecrated in the literature. It should read "Zorn's Axiom" instead, for it really is an axiom equivalent to the Axiom of Choice.

Theorem 1.7. *For any two sets X and Y, either $\#X \leq \#Y$ or $\#Y \leq \#X$.*

Proof. Consider two sets X and Y. Let $\mathcal{I}$ be the collection of all injective functions from subsets of X to Y. That is,

$$\mathcal{I} = \big\{F \in Y^A\colon\ A \subseteq X \text{ and } F \text{ is injective}\big\}.$$

Recall that (Problem 1.17), as a subset of $\mathcal{F} = \bigcup_{A\in\wp(X)} Y^A$, $\mathcal{I}$ is a partially ordered set in the extension ordering, and every simply ordered subset of it has a supremum in $\mathcal{I}$. Thus, by Zorn's Lemma, $\mathcal{I}$ contains a maximal function. Let $F_0\colon A_0 \to Y$ be a maximal function of $\mathcal{I}$, where $A_0 \subseteq X$ and $\#F_0(A_0) = \#A_0$ (since F_0 is injective). Suppose $A_0 \neq X$ and $F_0(A_0) \neq Y$. Take $x_0 \in X\backslash A_0$ and $y_0 \in Y\backslash F_0(A_0)$, and consider the function $F_1\colon A_0 \cup \{x_0\} \to Y$ defined by

$$F_1(x) = \begin{cases} F_0(x) \in F_0(A_0), & x \in A_0, \\ y_0 \in Y\backslash F_0(A_0), & x = x_0 \in X\backslash A_0, \end{cases}$$

which is injective (because F_0 is injective and $y_0 \notin F_0(A_0)$). Since $F_1 \in \mathcal{I}$ and $F_0 = F_1|_{A_0}$, it follows that $F_0 \leq F_1$, which contradicts the fact that F_0 is a maximal function of $\mathcal{I}$ (for $F_0 \neq F_1$). Hence, either $A_0 = X$ or $F_0(A_0) = Y$. If $A_0 = X$, then $F_0\colon X \to Y$ is injective and so $\#X \leq \#Y$. If $F_0(A_0) = Y$, then $\#Y = \#F_0(A_0) = \#A_0 \leq \#X$ (for $A_0 \subseteq X$ — see Problem 1.21(a)). □

We have already seen that $\mathbb{N} \leftrightarrow \mathbb{N}_0$. Thus $\mathbb{N}$ and $\mathbb{N}_0$ have the same cardinality. It is usual to assign a special symbol (*aleph-naught*) to such a cardinal number: $\#\mathbb{N} = \#\mathbb{N}_0 = \aleph_0$. We have also seen (cf. proof of Theorem 1.3) that, if X is an infinite set, then there exists a subset of it, say A, which is equivalent to $\mathbb{N}$. Thus $\#\mathbb{N} = \#A \leq \#X$, and hence $\aleph_0$ *is the smallest infinite cardinal number* in the sense that $\aleph_0 \leq \#X$ for every infinite set X (see Problems 1.21(a) and 1.22). A set X such that $\#X = \aleph_0$ is said to be *countably infinite* (or *denumerable*). Therefore, *every infinite set has a countably infinite subset*. A set that is either finite or countably infinite (i.e., a set X such that $\#X \leq \aleph_0$) is said to *countable*; otherwise, it is said to be *uncountable* (or *uncountably infinite*, or *nondenumerable*).

Proposition 1.8. $\#(X\times X) = \#X$ *for every countably infinite set* X.

Proof. Suppose $\#X = \#\mathbb{N}$. According to Problems 1.26, 1.23(b), and 1.25(a), we get $\#X \leq \#(X\times X) \leq \#(\mathbb{N}\times\mathbb{N}) = \#\mathbb{N} = \#X$. Hence the identity $\#X = \#(X\times X)$ follows by the Cantor–Bernstein Theorem. □

Note that $\#X \leq \#(X\times X)$ for any set X (see Problem 1.26). Moreover, it is easy to show that $\#X < \#(X\times X)$ whenever X is a finite set with more than one element. Thus, if a set with more than one element X is such that $\#X = \#(X\times X)$, then it is an infinite set. The previous proposition ensured the converse for countably infinite sets. The next theorem (which is another application of Zorn's Lemma) ensures the converse for every infinite set. Therefore, the identity $\#X = \#(X\times X)$ actually characterizes the infinite sets (of any cardinality) among the sets with more than one element.

Theorem 1.9. *If X is an infinite set, then $\#X = \#(X\times X)$.*

Proof. First we verify the following auxiliary result.

Claim 0. Let C, D, and E be nonempty sets. If $\#(E\times E) = \#E$, then $\#(C\cup D) \le \#E$ whenever $\#C \le \#E$ and $\#D \le \#E$.

Proof. The claimed result is a straightforward application of Problems 1.26, 1.23(b), and 1.22: $\#(C\cup D) \le \#(C\times D) \le \#(E\times E) = \#E$. □

Now, back to the theorem statement. Let X be a set and let $\mathcal{J}$ be the collection of all injective functions from subsets of X to $X\times X$ such that the range of each function in $\mathcal{J}$ coincides with the Cartesian product of its domain with itself. That is,

$$\mathcal{J} \;=\; \big\{F \in (X\times X)^A\colon\ A\subseteq X,\ F \text{ is injective and } F(A) = A\times A\big\}.$$

Note that $\mathcal{J}$ is nonempty (at least the empty function is there). From now on suppose X is infinite. Thus X has a countably infinite subset, and so Proposition 1.8 ensures the existence of a function in $\mathcal{J}$ with infinite (at least countably infinite) domain. Recall that $\mathcal{J}$ is a partially ordered set in the extension ordering, and that every chain in $\mathcal{J}$ (i.e., every simply ordered subset of $\mathcal{J}$) has an injective supremum (see Problem 1.17).

Claim 1. Such a supremum in fact lies in $\mathcal{J}$.

Proof. Let $\{F_\gamma\}$ be an arbitrary chain in $\mathcal{J}$, and let $\mathcal{D}(F_\gamma)$ and $\mathcal{R}(F_\gamma)$ denote the domain and range of F_γ, respectively. Thus each F_γ is an injective function from A_γ to $X\times X$, with $A_\gamma = \mathcal{D}(F_\gamma) \subseteq X$ and $F_\gamma(A_\gamma) = A_\gamma\times A_\gamma$. Now let $\bigvee_\gamma F_\gamma\colon \bigcup_\gamma A_\gamma \to X\times X$ be the supremum of $\{F_\gamma\}$, and note that (see Problem 1.17) $\big(\bigvee_\gamma F_\gamma\big)\big(\bigcup_\gamma A_\gamma\big) = \mathcal{R}\big(\bigvee_\gamma F_\gamma\big) = \bigcup_\gamma \mathcal{R}(F\gamma) = \bigcup_\gamma (A_\gamma\times A_\gamma)$. Clearly, $\bigcup_\gamma (A_\gamma\times A_\gamma) \subseteq \big(\bigcup_\gamma A_\gamma\big)\times\big(\bigcup_\gamma A_\gamma\big)$. On the other hand, if $\{F_\lambda, F_\mu\}$ is an arbitrary pair from $\{F_\gamma\}$, then $F_\lambda \le F_\mu$ (or vice versa), so that $A_\lambda\times A_\mu \subseteq A_\mu\times A_\mu$ (because $A_\lambda \subseteq A_\mu$), and therefore $\big(\bigcup_\gamma A_\gamma\big)\times\big(\bigcup_\gamma A_\gamma\big) \subseteq \bigcup_\gamma (A_\gamma\times A_\gamma)$. Hence $\big(\bigvee_\gamma F_\gamma\big)\big(\bigcup_\gamma A_\gamma\big) = \big(\bigcup_\gamma A_\gamma\big)\times\big(\bigcup_\gamma A_\gamma\big)$, and so $\bigvee_\gamma F_\gamma \in \mathcal{J}$. □

Conclusion: Every chain in $\mathcal{J}$ has an upper bound (a supremum, actually) in $\mathcal{J}$. Thus, according to Zorn's Lemma, $\mathcal{J}$ contains a maximal element. Let $F_0\colon A_0 \to X\times X$ be a maximal function of $\mathcal{J}$ so that $A_0 \subseteq X$ and $F_0(A_0) = A_0\times A_0$. Since F_0 is injective,

$$\#(A_0\times A_0) \;=\; \#A_0,$$

where A_0 is an infinite (at least countably infinite) set.

Claim 2. If $\#A_0 < \#X$, then $\#A_0 < \#(X\backslash A_0)$.

Proof. If $\#(X\backslash A_0) \le \#A_0$, then (cf. Problems 1.26 and 1.23(b)) $\#X = \#(A_0\cup X\backslash A_0) \le \#(A_0\times(X\backslash A_0)) \le \#(A_0\times A_0) = \#A_0$. Thus $\#(X\backslash A_0) \le$

$\#A_0$ implies $\#X \le \#A_0$ (Problem 1.22). Equivalently, $\#A_0 < \#X$ implies $\#A_0 < \#(X\backslash A_0)$ by Theorem 1.7. □

Note that $\#A_0 \le \#X$ (for $A_0 \subseteq X$). Suppose $\#A_0 < \#X$. In this case $\#A_0 < \#(X\backslash A_0)$ by Claim 2. Thus there exists a proper subset of $X\backslash A_0$, say A_1, such that $\#A_0 = \#A_1$. Hence (cf. Problem 1.23(b)) $\#(A_i \times A_j) \le \#(A_0 \times A_0) = \#A_0$ for all possible combinations of i, j in $\{0, 1\}$, and therefore $\#[(A_0 \times A_1) \cup (A_1 \times A_0) \cup (A_1 \times A_1)] \le \#A_0$ according to Claim 0. Since the reverse inequality is trivially verified, it follows by Theorem 1.6 that $\#[(A_0 \times A_1) \cup (A_1 \times A_0) \cup (A_1 \times A_1)] = \#A_0$. Set

$$A \;=\; A_0 \cup A_1$$

and observe that $(A \times A)\backslash(A_0 \times A_0) = (A_0 \times A_1) \cup (A_1 \times A_0) \cup (A_1 \times A_1)$ because A_0 and A_1 are disjoint. Thus

$$\#[(A \times A)\backslash(A_0 \times A_0)] \;=\; \#A_0 \;=\; \#A_1,$$

which ensures the existence of an injective function $F_1 \colon A_1 \to X \times X$ such that $F_1(A_1) = (A \times A)\backslash(A_0 \times A_0)$. Now consider a function F from A to $X \times X$ defined as follows.

$$F(x) \;=\; \begin{cases} F_0(x) \in A_0 \times A_0, & x \in A_0, \\ F_1(x) \in (A \times A)\backslash(A_0 \times A_0), & x \in A_1 = A\backslash A_0. \end{cases}$$

$F \colon A \to X \times X$ is injective (because F_0 and F_1 are injective functions with disjoint ranges) and $F(A) = A \times A$ (for $F_0(A_0) \cup F_1(A_1) = A \times A$). Since $F \in \mathcal{J}$ and $F_0 = F|_{A_0}$, it follows that $F_0 \le F$, which contradicts the fact that F_0 is a maximal function of $\mathcal{J}$ (for $F_0 \ne F$). Therefore

$$\#A_0 \;=\; \#X,$$

and hence (cf. Problem 1.23(b) $\#(A_0 \times A_0) = \#(X \times X)$. Conclusion: $\#X = \#A_0 = \#(A_0 \times A_0) = \#(X \times X)$. □

Theorem 1.9 is a natural extension of Proposition 1.8 which in turn generalizes Problem 1.25(a). Another important and particularly useful result along this line is given by the next theorem and its corollary.

Theorem 1.10. *Let X be a set and consider an indexed family of sets $\{X_\gamma\}_{\gamma \in \Gamma}$. If $\#X_\gamma \le \#X$ for all $\gamma \in \Gamma$, then*

$$\#\Big(\bigcup_{\gamma \in \Gamma} X_\gamma\Big) \;\le\; \#\big(\Gamma \times X\big).$$

Proof. Take an indexed family of sets $\{X_\gamma\}_{\gamma \in \Gamma}$ and suppose there is a set X such that $\#X_\gamma \le \#X$ for all $\gamma \in \Gamma$. Thus, for each $\gamma \in \Gamma$, there is a surjective

function $F_\gamma\colon X \to X_\gamma$ of X onto X_γ. Let $G\colon \Gamma\times X \to \bigcup_{\gamma\in\Gamma} X_\gamma$ be the function defined by $G(\gamma, x) = F_\gamma(x)$ for every $\gamma \in \Gamma$ and every $x \in X$. Take any y in $\bigcup_{\gamma\in\Gamma} X_\gamma$ so that $y \in X_\gamma$ for some $\gamma \in \Gamma$. Since $F_\gamma\colon X \to X_\gamma$ is surjective, there exists $x \in X$ such that $y = F_\gamma(x)$. Thus $y = G(\gamma, x)$; that is, $y \in \mathcal{R}(G)$. Hence $G\colon \Gamma\times X \to \bigcup_{\gamma\in\Gamma} X_\gamma$ is surjective, and so $\#\big(\bigcup_{\gamma\in\Gamma} X_\gamma\big) \le \#(\Gamma\times X)$. □

Corollary 1.11. *A countable union of countable sets is countable.*

Proof. Consider any countable family of countable sets, say $\{X_n\}_{n\in N}$, so that $\#N \le \#\mathbb{N}$ and $\#X_n \le \#\mathbb{N}$ for all $n \in N$. According to Theorem 1.10 we have $\#\big(\bigcup_{n\in N} X_n\big) \le \#(N\times\mathbb{N})$. However, $\#(N\times\mathbb{N}) \le \#(\mathbb{N}\times\mathbb{N}) = \#\mathbb{N}$. Thus

$$\#\Big(\bigcup_{n\in N} X_n\Big) \le \#\mathbb{N}.$$

(See Problems 1.23(b), 1.25(a), and 1.22.) □

1.9 Remarks

We assume the reader is familiar with the definition of an *interval* of the real line $\mathbb{R}$. An interval of $\mathbb{R}$ is *nondegenerate* if it does not collapse to a singleton. It is easy to show that the cardinality of the real line $\mathbb{R}$ is the same as the cardinality of any nondegenerate interval of $\mathbb{R}$. A typical example: The function $F\colon [0,1] \to [-1,1]$ given by $F(x) = 2x - 1$ for every $x \in [0,1]$ is injective and surjective, and so is the function $G\colon (-1,1) \to \mathbb{R}$ defined by $G(x) = (1 - |x|)^{-1}x$ for every $x \in (-1,1)$. Thus $\#[0,1] = \#[-1,1]$ and $\#(-1,1) = \#\mathbb{R}$. Since $(-1,1) \subset [-1,1] \subset \mathbb{R}$, it follows that $\#\mathbb{R} = \#(-1,1) \le \#[-1,1] \le \#\mathbb{R}$, and hence (Cantor–Bernstein Theorem) $\#[-1,1] = \#\mathbb{R}$. Thus,

$$\#[0,1] = \#(-1,1) = \#[-1,1] = \#\mathbb{R}.$$

We can also prove that $\#\mathbb{R} = \#2^{\mathbb{N}}$. Indeed, take the function $F\colon 2^{\mathbb{N}} \to [0,1]$ that assigns to each sequence $\{\alpha_n\}$ in $2^{\mathbb{N}}$ (i.e., to each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ with values either 0 or 1) a real number in $[0,1]$, in ternary expansion, as follows.

$$F(\{\alpha_n\}) = 0.(2\alpha_1)(2\alpha_2)\ldots$$

for every $\{\alpha_n\} \in 2^{\mathbb{N}}$. It can be shown that F is injective.

Reason: Every real number $x \in [0,1]$ can be written as $\sum_{n=1}^{\infty} \beta_n p^{-n}$ for a given positive integer p greater than 1 (i.e., for a given *base* p). In this case $0.\beta_1\beta_2\ldots$ is a *p-ary* expansion of x, where $\{\beta_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative integers ranging from 0 to $p-1$. That is, $\{\beta_n\}_{n\in\mathbb{N}}$ is a sequence of *digits* with respect to the base p — e.g., if $p = 2, 3$, or 10, then $0.\beta_1\beta_2\ldots$ is a *binary*, *ternary*, or *denary* (i.e., *decimal*) expansion of x, respectively. A p-ary expansion (with respect to a base p) is not unique — e.g., $0.499\ldots$ and $0.500\ldots$ are decimal

expansions for $x = \frac{1}{2}$. However, if two p-ary expansions of x differ, then the absolute difference between the first digits in which they differ is equal to 1. Hence, if we take a p-ary expansion whose digits are either 0 or 2 (as we did), then it is *unique*. (This can only be done for a p-ary expansion with respect to a base $p \geq 3$, so that a ternary expansion is enough.) Thus F is injective.

Therefore, $\#2^{\mathbb{N}} \leq \#[0,1]$. On the other hand, let $G\colon 2^{\mathbb{N}} \to [0,1]$ be the function that assigns to each sequence $\{\alpha_n\}$ in $2^{\mathbb{N}}$ a real number in $[0,1]$, in binary expansion, as follows.

$$G(\{\alpha_n\}) \;=\; 0.\alpha_1\alpha_2\ldots$$

for every $\{\alpha_n\} \in 2^{\mathbb{N}}$. It can also be shown that G is surjective.

Reason: Every $x \in [0,1]$ can be written as $\sum_{n=1}^{\infty}\alpha_n 2^{-n}$ so that $0.\alpha_1\alpha_2\ldots$ is a binary expansion of it for some sequence $\{\alpha_n\} \in 2^{\mathbb{N}}$. Thus G is surjective.

Therefore, $\#[0,1] \leq \#2^{\mathbb{N}}$. Hence $\#[0,1] = \#2^{\mathbb{N}}$ by the Cantor–Bernstein Theorem, and so

$$\#\mathbb{R} \;=\; \#2^{\mathbb{N}}$$

(since $\#[0,1] = \#\mathbb{R}$). Using Theorems 1.4 and 1.5 we may conclude that

$$\#\mathbb{N} \;<\; \#\mathbb{R}.$$

Such a fundamental result can also be derived by the celebrated Cantor's *diagonal procedure* as follows. Clearly, $\#\mathbb{N} \leq \#\mathbb{R}$ (because $\mathbb{N} \subset \mathbb{R}$). Suppose $\#\mathbb{N} = \#\mathbb{R}$. This implies that $\#\mathbb{N} = \#[0,1]$ (recall: $\#[0,1] = \#\mathbb{R}$). Thus the interval $[0,1]$ can be indexed by $\mathbb{N}$ so that $[0,1] = \{x_n\}_{n\in\mathbb{N}}$. Write each x_n in decimal expansion:

$$x_n \;=\; 0.\alpha_{n_1}\alpha_{n_2}\ldots$$

where each α_{n_k} ($k \in \mathbb{N}$) is a nonnegative integer between 0 and 9. Now consider the point $x \in [0,1]$ with the following decimal expansion.

$$x \;=\; 0.\alpha_1\alpha_2\ldots$$

where, again, each α_n ($n \in \mathbb{N}$) is a nonnegative integer between 0 and 9 but $\alpha_1 \neq \alpha_{1_1}$, $\alpha_2 \neq \alpha_{2_2}$, and so on. That is, $\alpha_n \neq \alpha_{n_n}$ for each $n \in \mathbb{N}$ (e.g., take α_n diametrically opposite to α_{n_n} so that, for each $n \in \mathbb{N}$, $\alpha_n = \alpha_{n_n} + 5$ if $0 \leq \alpha_{n_n} \leq 4$ or $\alpha_n = \alpha_{n_n} - 5$ if $5 \leq \alpha_{n_n} \leq 9$). Thus $x \neq x_n$ for every $n \in \mathbb{N}$. Hence $x \notin [0,1]$, which is a contradiction. Therefore $\#\mathbb{N} \neq \#\mathbb{R}$. Equivalently (since $\#\mathbb{N} \leq \#\mathbb{R}$), $\#\mathbb{N} < \#\mathbb{R}$.

We have denoted $\#\mathbb{N}$ by $\aleph_0$. Let us now denote $\#2^{\mathbb{N}}$ by $2^{\aleph_0}$ so that

$$\#\mathbb{N} \;=\; \aleph_0 \;<\; 2^{\aleph_0} \;=\; \#\mathbb{R}.$$

Cantor conjectured in 1878 that there is no cardinal number between $\aleph_0$ and $2^{\aleph_0}$. Such a conjecture is called the Continuum Hypothesis.

Continuum Hypothesis. *There is no set whose cardinality is greater than* $\#\mathbb{N}$ *and smaller than* $\#\mathbb{R}$.

The Generalized Continuum Hypothesis is the conjecture that naturally generalizes the Continuum Hypothesis.

Generalized Continuum Hypothesis. *For any infinite set* X, *there is no cardinal number between* $\#X$ *and* $\#2^X$.

There are several different axiomatic set theories, each based on a somewhat different axiom system. The most popular is probably the axiom system *ZFC*. It comprises the axiom system *ZF* ("Z" for Zermelo and "F" for Fraenkel) plus the Axiom of Choice. The axioms of *ZF* are listed below.

Axiom of Empty Set: There exists the empty set $\varnothing$.

Axiom of Extension: $\{x \in A$ if and only if $x \in B\}$ defines $A = B$.

Axiom of Specification: There exists the set $\{x \in A\colon\ P(x)\}$.

Axiom of Pairing: There exists the set $\{a, b\} = \{x\colon\ x = a$ or $x = b\}$.

Axiom of Union: There exists the set $\bigcup_{A\in S} A = \{x\colon\ x \in A$ for some $A \in S\}$.

Axiom of Power Set: There exists the set $\wp(A) = \{B\colon\ x \in B$ implies $x \in A\}$.

Axiom of Replacement: There exists the set $F_\gamma = \{x\colon\ P_\gamma(x)\}$ for each $\gamma \in \Gamma$.

Axiom of Infinity: There exists an infinite set.

Axiom of Foundation: There exists $A \in S \neq \varnothing$ such that $A \cap S = \varnothing$.

The Axiom of Empty Set states that there is a unique set $\varnothing$ such that, for every x, $x \notin \varnothing$. After defining inclusion, the Axiom of Extension ensures that $A = B$ if and only if $\{A \subseteq B$ and $B \subseteq A\}$. In the Axiom of Specification (also called *Axiom of Separation*), $P(\ )$ is a meaningful predicate for elements of a set A. The Axiom of Pairing simply states the existence of a pair (of elements, which may be sets themselves), and the Axiom of Union states the existence of unions for every collection of sets. After defining inclusion, the Axiom of Power Set states that $\{B\colon\ B \subseteq A\}$ is a set: the collection of all subsets of a set is itself a set. The Axiom of Replacement (also called *Axiom of Substitution*) says that, if $P_\gamma(\ )$ is a meaningful predicate for each γ in a set Γ such that each $\{x\colon\ P_\gamma(x)\}$ is a set, then there exists a function F whose domain is Γ that assigns the set $F_\gamma = \{x\colon\ P_\gamma(x)\}$ to each $\gamma \in \Gamma$. The Axiom of Infinity is equivalent to saying that $\mathbb{N}$ is a set. The Axiom of Foundation (also called *Axiom of Regularity* or *Axiom of Restriction*) says that if $S \neq \varnothing$, then there exists $A \in S$ such that $B \notin A$ for every $B \in S$.

The axiom system *ZF* contains some redundancies (i.e., *ZF* is not an independent system of axioms). For instance. the Axiom of Empty Set may be viewed as a consequence of the Axiom of Specification. Indeed, if there exists a set A, then the set $\{x \in A\colon\ x \neq x\}$ is empty.

Observe that if $\{A\colon A \notin A\}$ is a set, say S, then $S \in S$ implies $S \notin S$; a contradiction (and $S \notin S$ implies $S \in S$; another contradiction). Therefore, $\{A\colon A \notin A\}$ is not a set — such a "thing" is too large to be a set: *there is no set of all sets that are not members of themselves*. Applying the Axiom of Specification we may consider the set $R = \{A \in S\colon A \notin A\}$ for any set S. If $R \in S$ and $R \in R$, then $R \notin R$; a contradiction. If $R \in S$ and $R \notin R$, then $R \in R$; another contradiction. Thus $R \notin S$. Since S is an arbitrary set, we conclude that there exists something (viz., R) that does not belong to anything (i.e., that does not belong to any S). This is sometimes stated by saying that *there is no universe*. The above arguments were known (in the pre-axiomatic era) as the *Russell paradox*.

It is worth noticing that the arguments in the preceding paragraph have nothing to do with the existence (or nonexistence) of a set A such that $A \in A$. However, it is natural to ask at this point whether there is any set A such that $A \in A$. The Axiom of Foundation is very little used, but it provides a way (set $S = \{A\}$) to ensure that *no set is a member of itself*. That is, $A \notin A$ always.

The Axiom of Choice actually is a genuine axiom to be added to *ZF*. Indeed, Gödel proved in 1939 that the Axiom of Choice is consistent with *ZF*, and Cohen proved in 1963 that the Axiom of Choice is independent of *ZF*. The situation of the Continuum Hypothesis with respect to *ZFC* is somewhat similar to that of the Axiom of Choice with respect to *ZF*, although the Continuum Hypothesis itself is not as primitive as the Axiom of Choice (even if the Axiom of Choice might be regarded as not primitive enough). Gödel proved in 1939 that the Generalized Continuum Hypothesis is consistent with *ZFC*, and Cohen proved in 1963 that the denial of the Continuum Hypothesis also is consistent with *ZFC*. Thus the Continuum Hypothesis and the Generalized Continuum Hypothesis are consistent with *ZFC* and also independent of *ZFC*: neither of them can be proved or disproved on the basis of *ZFC* alone (they are *undecidable* statements in *ZFC*). The Generalized Continuum Hypothesis in fact is stronger than the Axiom of Choice: Sierpinski showed in 1947 that the Generalized Continuum Hypothesis implies the Axiom of Choice.

We have already observed that the Axiom of Choice and Zorn's Lemma are equivalent. There is a myriad of axioms equivalent to the Axiom of Choice. Let us mention just two of them.

Hausdorff Maximal Principle. *Every partially ordered set contains a maximal chain* (i.e., *a maximal simply ordered subset*).

Zermelo Well-Ordering Principle. *Every set may be well ordered.*

In particular, the set $\mathbb{R}$ of all reals may be well ordered. This is a pure existence result, not exhibiting (either constructing or even defining) a well-ordering of $\mathbb{R}$. Indeed, Feferman showed in 1965 that no *defined* partial ordering can be proved in *ZFC* to well-order the set $\mathbb{R}$.

If X and Y are any sets, properly well ordered, and if there exists a one-to-one order-preserving correspondence between them (i.e., an injective and surjective mapping $\Phi\colon X \to Y$ such that $x_1 < x_2$ in $(X, \leq)$ if and only if $\Phi(x_1) < \Phi(x_2)$ in $(Y, \leq)$), then X and Y are said to have the same *ordinal number*. If two well-ordered sets have the same ordinal number, then they have the same cardinal number. However, unlike the notion of cardinal number, the notion of ordinal number depends on the well-orderings that well-order the sets.

Proposition 1.12. *There is an uncountable set X, well-ordered by a relation $\leq$ on it, with the following properties. X has a greatest element Ω and the set $\{x \in X\colon x < z\}$ is countable for every z in $X\backslash\{\Omega\}$.*

Proof. Let Y be an uncountable set. By the Well-Ordering Principle, there exists a well-ordering of Y. Take ζ not in Y, set $Z = Y \cup \{\zeta\}$, and extend the well-ordering of Y to Z by setting $y < \zeta$ for every $y \in Y$. Consider the set $A = \big\{\alpha \in Z\colon\ \{z \in Z\colon z < \alpha\}$ is an uncountable set$\big\}$. A is nonempty ($\zeta \in A$ because $Y = \{z \in Z\colon z < \zeta\}$ is uncountable), and hence it has a minimum (since $\varnothing \neq A \subseteq Z$ and Z is well-ordered). Set $\Omega = \min A$ so that $X = \{z \in Z\colon z \leq \Omega\}$ is the required set. □

Moreover, it can be shown that such a well-ordered set X is unique in the sense that, if Y is any well-ordered set with the same properties of X, then there exists a one-to-one order-preserving correspondence between X and Y (i.e., then X and Y have the same ordinal number). The greatest (or the last) element Ω in X is called the *least* or *first uncountable ordinal*, and the elements x of X such that $x < \Omega$ are called *countable ordinals*. The greatest elements of the finite subsets of X are called *finite ordinals*. If ω is the *first infinite ordinal* (i.e., the least nonfinite ordinal), then the set $\{x \in X\colon x < \omega\}$ of all finite ordinals and the set of all natural numbers $\mathbb{N}$ (equipped with its natural ordering) have the same ordinal number. It is usual to assign the symbol ω as the ordinal number of any well-ordered set that is in a one-to-one order-preserving correspondence with $\mathbb{N}$.

Suggested Reading

Binmore [1]
Brown and Pearcy [2]
Cohen [2]
Crossley et al. [1]
Dugundji [1]
Fraenkel, Bar-Hillel, and Levy [1]
Halmos [3]
Kelley [1]
Kolmogorov and Fomin [1]
Moore [1]
Royden [1]
Simmons [1]
Suppes [1]
Vaught [1]
Wilder [1]
Yandell [1]

Problems

Problem 1.1. Let A, B, and C be arbitrary sets. Prove the assertions below.

(a) $(A\backslash B)\cup(B\backslash A) = (A\cup B)\backslash(A\cap B)$ and $A\cap B = (A\cup B)\backslash(A\triangledown B)$.

(b) $(A\triangledown B)\cup(B\triangledown C) = (A\cup B\cup C)\backslash(A\cap B\cap C)$.

(c) De Morgan laws: If A and B are subsets of X, then
$X\backslash(A\cup B) = (X\backslash A)\cap(X\backslash B)$ and $X\backslash(A\cap B) = (X\backslash A)\cup(X\backslash B)$.

Problem 1.2. Consider a function $F\colon X\to Y$ from a set X to a set Y. Let A, A_1, and A_2 be arbitrary subsets of X, and let B, B_1, and B_2 be arbitrary subsets of Y. Verify the following propositions.

(a) $F(X)\backslash F(A) \subseteq F(X\backslash A)$.

(b) $F^{-1}(Y\backslash B) = X\backslash F^{-1}(B)$.

(c) $A_1\subseteq A_2 \implies F(A_1)\subseteq F(A_2)$.

(d) $B_1\subseteq B_2 \implies F^{-1}(B_1)\subseteq F^{-1}(B_2)$.

(e) $F(A_1\cup A_2) = F(A_1)\cup F(A_2)$.

(f) $F(A_1\cap A_2) \subseteq F(A_1)\cap F(A_2)$.

(g) $F^{-1}(B_1\cup B_2) = F^{-1}(B_1)\cup F^{-1}(B_2)$.

(h) $F^{-1}(B_1\cap B_2) = F^{-1}(B_1)\cap F^{-1}(B_2)$.

(i) $A\subseteq F^{-1}(F(A))$.

(j) $F(F^{-1}(B))\subseteq B$.

Problem 1.3. Consider the setup of Problem 1.2. Show that

(a) F is injective if and only if the inverse image under F of each singleton in $\mathcal{R}(F)$ is a singleton in X;

(b) F is injective if and only if $F(A_1\cap A_2) = F(A_1)\cap F(A_2)$ for every $A_1, A_2\subseteq X$;

(c) F is injective if and only if the images of disjoint sets in X are disjoint sets in Y;

(d) F is injective if and only if $A = F^{-1}(F(A))$ for every $A\subseteq X$;

(e) F is surjective if and only if the inverse image under F of each nonempty subset of Y is a nonempty subset of X;

(f) F is surjective if and only if $F(F^{-1}(B)) = B$ for every $B\subseteq Y$.

Problem 1.4. Verify that a function $F\colon X\to X$ is idempotent if and only if the range of F coincides with the set of all fixed points of F.

Problem 1.5. A function $L: Y \to X$ is said to be a *left inverse* of a function $F: X \to Y$ if $LF = I_X$, the identity on X. F is injective if and only if it has a left inverse. The restriction of a left inverse of F to the range of F is unique and coincides with the inverse of F on $\mathcal{R}(F)$ (recall: an injective function has an inverse on its range). Prove.

Problem 1.6. A function $R: Y \to X$ is said to be a *right inverse* of a function $F: X \to Y$ if $FR = I_Y$, the identity on Y. Show that F is surjective if and only if it has a right inverse. Note that any right inverse of F is injective (for it has a left inverse). Similarly, any left inverse of F is surjective (for it has a right inverse). Conclusion: There exists an injective mapping of X into Y if and only if there exists a surjective mapping of Y onto X.

Problem 1.7. A function $F: X \to Y$ is injective and surjective if and only if there is a function $G: Y \to X$ such that $GF = I_X$ (the identity on X) and $FG = I_Y$ (the identity on Y). Prove the above proposition and show that, in this case, this function G is unique and coincides with the inverse of F.

Problem 1.8. If $F: X \to Y$ is an invertible function, then so is its inverse $F^{-1}: Y \to X$ and $(F^{-1})^{-1} = F$. Moreover, if $A \subseteq X$, then $F|_A: A \to F(A)$ is invertible and $(F|_A)^{-1} = F^{-1}|_{F(A)}$. Prove.

Problem 1.9. Verify the following propositions.

(a) The composition of two injective functions is an injective function.

(b) The composition of two surjective functions is a surjective function.

(c) The composition of two invertible functions is an invertible function.

Note: When we speak of the composition $G \circ F$ of two functions F and G, it is assumed that the domain of G includes the range of F.

Problem 1.10. Let $F: X \to Y$ and $G: Y \to Z$ be invertible mappings.

(a) Show that $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$. That is, using the simplified notation $GF = G \circ F$, this means that if F and G are invertible, then $GF: X \to Z$ is invertible and the inverse $(GF)^{-1}: Z \to X$ is given by $(GF)^{-1} = F^{-1}G^{-1}$.

Consider the nth power $F^n: X \to X$ of $F: X \to X$ (i.e., the composition of F with itself n times) for any integer $n \geq 0$ (with $F^0 = I$; the identity on X).

(b) Show by induction that if F is invertible, then so is F^n for every $n \geq 0$, and $(F^n)^{-1} = (F^{-1})^n$ for each $n \geq 0$. In this case we write $F^{-n} = (F^{-1})^n$.

Problem 1.11. A function F of a set X into itself is an *involution* if $F^2 = I$.

(a) Verify that an involution is precisely an invertible function on X that coincides with its inverse: $F = F^{-1}$. Show that the composition of two involutions is again an involution if and only if they commute.

A sequence $\{F_n\}_{n=0}^{\infty}$ of functions of a set X into itself has the *semigroup property* if $F_0 = I$ and $F_{m+n} = F_m \circ F_n$ (i.e., $F_{m+n} = F_m F_n$) for all $m, n \geq 0$.

(b) Show that the only sequences with the semigroup property are the *power sequences* $\{F^n\}_{n=0}^{\infty}$ for some $F: X \to X$ (where $F^n = F \cdots F$, n times).

Hint: Set $F = F_1$. If $F_{m+n} = F_m \circ F_n$, then show by induction that $F_{n\,m} = (F_m)^n = (F_n)^m$ for every $m \geq 0$ and $n \geq 0$. Thus $F_n = F^n$.

Problem 1.12. Let $F: X \to Y$ be a one-to-one mapping of a set X onto a set Y. Let $G: X \to A$ be a one-to-one mapping of X onto a subset A of X. Prove: If A is a proper subset of X, then $F(A)$ is a proper subset of Y. Apply this to show that a finite set has no proper equivalent subset.

Hint: Consider the commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{F^{-1}} & Y \\ {\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle H} \\ A & \xrightarrow{F|_A} & F(A). \end{array}$$

Problem 1.13. Let X be a set with more than one element, and consider the following relations R_1, R_2, R_3, R_4, and R_5 on the power set $\wp(X)$ of X. For every pair $\{A, B\}$ of subsets of X

$A\,R_1\,B$ if $A \triangledown B = \varnothing$ (i.e., if $A = B$),

$A\,R_2\,B$ if $A \triangledown B$ is finite,

$A\,R_3\,B$ if A and B are singletons,

$A\,R_4\,B$ if $A \subseteq B$ or $B \subseteq A$ (i.e., if $A \backslash B = \varnothing$ or $B \backslash A = \varnothing$),

$A\,R_5\,B$ if $A \subseteq B$ (i.e., if $A \backslash B = \varnothing$).

Show that the table below properly classifies these relations according to reflexivity, transitivity, symmetry, and antisymmetry.

	Reflexive	Transitive	Symmetric	Antisymmetric
R_1	√	√	√	√
R_2	√	√	√	
R_3		√	√	
R_4	√		√	
R_5	√	√		√

Hint: To verify that R_2 is transitive use Problem 1.1(b) and recall that the union of two sets is finite if and only if each of them is finite.

Problem 1.14. Consider the functions $\Phi\colon \wp(X) \to \wp(X)$ and $H\colon X \to Y$ as in the proof of Theorem 1.2. Show that Φ is an increasing function with respect to the inclusion ordering of $\wp(X)$, and that H is injective and surjective.

Problem 1.15. Let Y^X denote the collection of all functions from a set X to a set Y. Suppose Y is partially ordered and let $\le$ be a partial ordering of Y. Now consider a relation on Y^X, also denoted by $\le$ and defined as follows. For any pair $\{F, G\}$ of functions in Y^X,

$$F \le G \quad \text{if} \quad F(x) \le G(x) \ \text{ for every } \ x \in X.$$

(a) Show that $\le$ is a partial ordering of Y^X.

(b) Prove: If $(Y, \le)$ is a lattice, then $(Y^X, \le)$ is a lattice.

Hint: Suppose $(Y, \le)$ is a lattice. Take F and G from Y^X and let U and V be functions in Y^X defined as follows. $U(x) = F(x) \vee G(x)$ and $V(x) = F(x) \wedge G(x)$ for every $x \in X$. Show that $F \vee G = U$ and $F \wedge G = V$.

Problem 1.16. Set $Y = \{0, 1\}$ and let $\chi_A\colon X \to \{0, 1\}$ be the characteristic function of an arbitrary subset A of a set X. Thus, for every $A \in \wp(X)$, χ_A lies in $Y^X = 2^X$. Let A and B be subsets of X, and consider the partial ordering of 2^X introduced in Problem 1.15. Prove the following propositions.

(a) $\chi_A \le \chi_B \iff A \subseteq B$,

(b) $\chi_A \vee \chi_B = \chi_{A \cup B}$,

(c) $\chi_A \wedge \chi_B = \chi_{A \cap B} = \chi_A \chi_B$,

(d) $A \cap B = \varnothing \iff \chi_{A \cap B} = 0 \iff \chi_{A \cup B} = \chi_A + \chi_B$.

Problem 1.17. Let $\mathcal{F}$ be the collection of all functions from subsets of a set X to a set Y. That is,

$$\mathcal{F} = \big\{F \in Y^A\colon\ A \subseteq X\big\} = \bigcup_{A \in \wp(X)} Y^A.$$

The unique function whose domain is the empty set $\varnothing$ is called the *empty function* in $\mathcal{F}$. Consider the following relation $\le$ on $\mathcal{F}$. For any pair $\{F, G\}$ of functions in $\mathcal{F}$, $F \le G$ if F is a restriction of G (equivalently, if G is an extension of F). That is, $F \le G$ if and only if $F\colon A \to Y$, $G\colon B \to Y$, $A \subseteq B \subseteq X$, and $F = G|_A$ (i.e., $F(x) = G(x)$ for every $x \in A$).

(a) Show that the relation $\le$ on $\mathcal{F}$ is a partial ordering (called the *extension ordering*).

A function $V\colon C \to Y$ in $\mathcal{F}$ is a lower bound for a pair of functions $F\colon A \to Y$ and $G\colon B \to Y$ in $\mathcal{F}$ if $C \subseteq B$, $C \subseteq A$, and $V = F|_C = G|_C$. A function $U\colon D \to Y$ in $\mathcal{F}$ is an upper bound for the pair $\{F, G\}$ if $A \subseteq D$, $B \subseteq D$, $U|_A = F$, and $U|_B = G$.

(b) Show that every pair of functions F and G in $\mathcal{F}$ has an infimum $F \wedge G$ in $\mathcal{F}$. (In particular, if the domain A of F and the domain B of G are disjoint, then $F \wedge G$ is the empty function — which function is $F \wedge G$ if A and B are not disjoint but $F(A)$ and $G(B)$ are disjoint?)

Let $\{F_\gamma\}$ be an indexed family of functions in $\mathcal{F}$. For each F_γ let $\mathcal{D}(F_\gamma)$ and $\mathcal{R}(F_\gamma)$ denote the domain and range of F_γ, respectively. Prove the following propositions.

(c) If $\{F_\gamma\}$ is simply ordered (i.e., if $\{F_\gamma\}$ is a chain in $\mathcal{F}$), then it has a supremum $\bigvee_\gamma F_\gamma$ in $\mathcal{F}$, whose domain and range are $\mathcal{D}\big(\bigvee_\gamma F_\gamma\big) = \bigcup_\gamma \mathcal{D}(F_\gamma)$ and $\mathcal{R}\big(\bigvee_\gamma F_\gamma\big) = \bigcup_\gamma \mathcal{R}(F_\gamma)$. Moreover, if each F_γ is injective, then so is $\bigvee_\gamma F_\gamma$.

(d) If the domains $\{\mathcal{D}(F_\gamma)\}$ are pairwise disjoint, then $\{F_\gamma\}$ has a supremum $\bigvee_\gamma F_\gamma \colon \bigcup_\gamma \mathcal{D}(F_\gamma) \to \bigcup_\gamma \mathcal{R}(F_\gamma)$ in $\mathcal{F}$. Moreover, if the ranges $\{\mathcal{R}(F_\gamma)\}$ are also pairwise disjoint, and if each F_γ is injective, then so is $\bigvee_\gamma F_\gamma$.

Problem 1.18. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of sets. Set $Y_1 = X_1$ and

$$Y_{n+1} \;=\; \Big(\bigcup_{k=1}^{n+1} X_k\Big)\backslash\Big(\bigcup_{k=1}^{n} X_k\Big)$$

for each $n \in \mathbb{N}$. Show by induction that

$$\bigcup_{k=1}^{n} Y_k \;=\; \bigcup_{k=1}^{n} X_k$$

for every $n \in \mathbb{N}$. (*Hint*: $A \cup (B\backslash A) = A \cup B$.) Verify that $Y_n \subseteq X_n$ for each $n \in \mathbb{N}$, and $Y_m \cap Y_n = \varnothing$ for every pair of distinct natural numbers m and n. Moreover, show that

$$\bigcup_{n=1}^{\infty} Y_n \;=\; \bigcup_{n=1}^{\infty} X_n.$$

The sequence $\{Y_n\}_{n\in\mathbb{N}}$ is referred to as the *disjointification* of $\{X_n\}_{n\in\mathbb{N}}$.

Problem 1.19. Let $\wp(X)$ be the power set of a set X and consider the inclusion ordering of $\wp(X)$. Let $\{X_n\}_{n=1}^{\infty}$ be an arbitrary $\wp(X)$-valued sequence (i.e., a sequence of subsets of a given set X). Recall that, for each $n \in \mathbb{N}$, $\{X_k\}_{k=n}^{\infty}$ is an indexed family of subsets of X, and hence a subset of the complete lattice $\wp(X)$. Let $\inf_{n\le k} X_k$ and $\sup_{n\le k} X_k$ denote $\inf\{X_k\}_{k=n}^{\infty}$ and $\sup\{X_k\}_{k=n}^{\infty}$, respectively, and set

$$Y_n = \inf_{n\le k} X_k = \bigcap_{k=n}^{\infty} X_k \quad \text{and} \quad Z_n = \sup_{n\le k} X_k = \bigcup_{k=n}^{\infty} X_k$$

so that $\{Y_n\}_{n=1}^{\infty}$ and $\{Z_n\}_{n=1}^{\infty}$ are $\wp(X)$-valued sequences as well.

(a) Verify that $\{Y_n\}_{n=1}^{\infty}$ is an increasing sequence and $\{Z_n\}_{n=1}^{\infty}$ is decreasing. The union $\bigcup_{n=1}^{\infty} Y_n$ and the intersection $\bigcap_{n=1}^{\infty} Z_n$, which are elements of $\wp(X)$, are usually denoted by

$$\bigcup_{n=1}^{\infty} Y_n = \liminf_n X_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} Z_n = \limsup_n X_n,$$

called *limit inferior* and *limit superior* of $\{X_n\}_{n=1}^{\infty}$, respectively. Show that

(b) $$\liminf_n X_n \subseteq \limsup_n X_n.$$

If $\liminf_n X_n = \limsup_n X_n$, then the sequence $\{X_n\}_{n=1}^{\infty}$ is said to *converge* to the *limit*

$$\lim_n X_n = \liminf_n X_n = \limsup_n X_n.$$

Prove the following propositions.

(c) If $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence, then $Y_n = X_n$ for each $n \in \mathbb{N}$ and $Z_n = \bigcup_{m=1}^{\infty} X_m = \sup_m X_m$ for every $n \in \mathbb{N}$, so that

$$\liminf_n X_n = \limsup_n X_n = \sup_n X_n.$$

(d) If $\{X_n\}_{n=1}^{\infty}$ is itself decreasing, then $Y_n = \bigcap_{m=1}^{\infty} X_m = \inf_m X_m$ for every $n \in \mathbb{N}$ and $Z_n = X_n$ for each $n \in \mathbb{N}$, so that

$$\liminf_n X_n = \limsup_n X_n = \inf_n X_n.$$

Therefore, an increasing sequence of sets converges to its union ($\sup_n X_n = \bigcup_n X_n$) and, dually, a decreasing sequence of sets converges to its intersection ($\inf_n X_n = \bigcap_n X_n$), and so every monotone sequence of sets converges. Thus, since $\{Y_n\}_{n=1}^{\infty}$ and $\{Z_n\}_{n=1}^{\infty}$ are always increasing and decreasing, respectively, they do converge: $\{Y_n\}_{n=1}^{\infty}$ converges to its union $\bigcup_{n=1}^{\infty} Y_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} X_k$, and $\{Z_n\}_{n=1}^{\infty}$ converges to its intersection $\bigcap_{n=1}^{\infty} Z_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} X_k$.

(e) Verify the following identities.

$$\liminf_n X_n = \lim_n \Big(\inf_{n \le k} X_k \Big) \quad \text{and} \quad \limsup_n X_n = \lim_n \Big(\sup_{n \le k} X_k \Big).$$

(f) Now show that

$$X \setminus \limsup_n X_n = \liminf_n (X \setminus X_n) \ \text{ and } \ X \setminus \liminf_n X_n = \limsup_n (X \setminus X_n).$$

Thus a sequence $\{X_n\}_{n=1}^{\infty}$ converges if and only if the sequence of its complements $\{X \setminus X_n\}_{n=1}^{\infty}$ converges and, in this case,

$$\lim_n (X \setminus X_n) = X \setminus \lim_n X_n.$$

Problem 1.20. Let A and B be subsets of the sets X and Y. Show that

$$A \leftrightarrow B \ \text{ and } \ X \setminus A \leftrightarrow Y \setminus B \quad \text{imply} \quad X \leftrightarrow Y.$$

(*Warning*: $X \leftrightarrow Y$ and $A \leftrightarrow B$ do not imply $X \backslash A \leftrightarrow Y \backslash B$.)

Problem 1.21. Let A and B be any sets. Prove that

(a) $\quad A \subseteq B \quad$ implies $\quad \#A \leq \#B \quad$ (*hint*: inclusion map),

(b) $\quad A \subseteq B,\ B$ is finite, and $\#A = \#B \quad$ imply $\quad A = B$;

and show that assertion (b) does not hold if B is infinite.

Problem 1.22. For any sets A, B, and C verify that

$$\#A \leq \#B \leq \#C \quad \text{implies} \quad \#A \leq \#C.$$

Moreover, if $\#A < \#B \leq \#C$ or $\#A \leq \#B < \#C$, then $\#A < \#C$.

Hint: Cantor–Bernstein Theorem.

Problem 1.23. Suppose the sets A, B, C, and D are such that

$$\#A \leq \#C \quad \text{and} \quad \#B \leq \#D.$$

Prove the following propositions.

(a) $\#(A \cup B) \leq \#(C \cup D)$ whenever $C \cap D = \varnothing$,

(b) $\#(A \times B) \leq \#(C \times D)$.

Moreover, if $\#A = \#C$ and $\#B = \#D$, then the cardinalities in (b) coincide.

Problem 1.24. Let Y^X denote the collection of all mappings of a set X into a set Y. Show that, if Y has more than one element, then

$$\#\wp(X) \leq \#Y^X \text{ for every set } X.$$

(In fact, $\#\wp(X) = \#2^X = \#Y^X$ if X is infinite and $2 \leq \#Y \leq \#X$.)

Problem 1.25. Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ have their standard meanings and set $\aleph_0 = \#\mathbb{N}$ as usual. Verify the following identities.

(a) $\#(\mathbb{N} \times \mathbb{N}) = \aleph_0$.

Hint: The function $F: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined, for each pair $m, n \in \mathbb{N}$, by $F(m,n) = \frac{(m+n-1)(m+n-2)}{2} + m$ is injective and surjective. The array

$$\begin{array}{cccccccc} n \uparrow & & & & & & & \\ 5 & 11 & \cdot & \cdot & \cdot & \cdot & & \\ 4 & 7 & 12 & \cdot & \cdot & \cdot & & \\ 3 & 4 & 8 & 13 & \cdot & \cdot & & \\ 2 & 2 & 5 & 9 & 14 & \cdot & & \\ 1 & 1 & 3 & 6 & 10 & 15 & & \\ & & & & & & & m \\ & 1 & 2 & 3 & 4 & 5 & & \rightarrow \end{array}$$

may be suggestive.

(b) $\#\mathbb{Z} = \aleph_0$.

Hint: Let $\mathbb{N}_e$ denote the set of all nonnegative even integers (including zero), and let $\mathbb{N}_o$ denote the set of all positive odd integers. Recall that $\#\mathbb{N}_0 = \#\mathbb{N}$, and note that $\#\mathbb{N}_0 = \#\mathbb{N}_e$ and $\#\mathbb{N} = \#\mathbb{N}_o$ (reason: the functions $F\colon \mathbb{N}_0 \to \mathbb{N}_e$ and $G\colon \mathbb{N} \to \mathbb{N}_o$, defined by $F(n) = 2n$ for every $n \in \mathbb{N}_0$ and $G(n) = 2n - 1$ for every $n \in \mathbb{N}$, are injective and surjective). Set $\mathbb{N}_- = \{k \in \mathbb{Z}\colon -k \in \mathbb{N}\}$ so that $\#\mathbb{N}_- = \#\mathbb{N}_o$. Use Problem 1.23(a) to show that $\#\mathbb{Z} \le \#\mathbb{N}$.

(c) $\#\mathbb{Q} = \aleph_0$.

Hint: The function $F\colon \mathbb{Z}\times\mathbb{N} \to \mathbb{Q}$ defined by $F(k, n) = \frac{k}{n}$ for every $k \in \mathbb{Z}$ and every $n \in \mathbb{N}$ is surjective. Use Problem 1.23(b).

Problem 1.26. The purpose of this problem is to prove that, if X and Y are sets with more than one element, then

$$\#(X \cup Y) \le \#(X\times Y).$$

(a) First verify that the above assertion holds whenever X and Y are both finite sets with more than one element.

Consider the relations $\sim_X$ and $\sim_Y$ on the Cartesian product $X\times Y$ defined as follows. For every pair of pairs (x_1, y_1) and (x_2, y_2) in $X\times Y$,

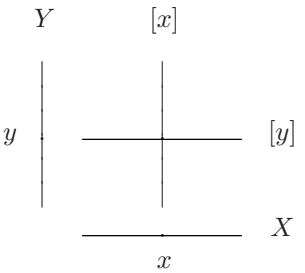
$$\begin{aligned} (x_1, y_1) \sim_X (x_2, y_2) &\iff x_1 = x_2 \\ (x_1, y_1) \sim_Y (x_2, y_2) &\iff y_1 = y_2. \end{aligned}$$

(b) Show that $\sim_X$ and $\sim_Y$ are both equivalence relations on $X\times Y$.

Now take $x \in X$ and $y \in Y$ arbitrary and consider the equivalence classes, $[x] \subseteq X\times Y$ and $[y] \subseteq X\times Y$, of the point $(x, y) \in X\times Y$ with respect to $\sim_X$ and $\sim_Y$, respectively.

(c) Show that $[x] \leftrightarrow Y$ and $[y] \leftrightarrow X$.

Hint:



Next suppose one of the sets X or Y, say X, is infinite and consider the singleton $\{(x, y)\}$ on $(x, y) \in X\times Y$.

(d) Verify that $\#X = \#([y]\backslash\{(x,y)\})$ and $\#Y = \#[x]$.

(e) Finally, apply the results of Problems 1.21(a), 1.22, and 1.23(a) to conclude that $\#(X \cup Y) \le \#(X \times Y)$.

Problem 1.27. Prove the following propositions.

(a) The union of two sets is countable if and only if each of them is countable.

Hint: Problems 1.22, 1.23, 1.25, and 1.26.

(b) Let X be an infinite set, and let A and B be arbitrary subsets of X. The relation $\sim$ on $\wp(X)$ defined by

$$A \sim B \quad \text{if} \quad A \triangledown B \text{ is countable}$$

is an equivalence relation — compare with Problem 1.13.

Problem 1.28. Let E be an infinite set. Suppose A and B are sets such that

$$\#A \le \#E \quad \text{and} \quad \#B \le \#E.$$

(a) The following propositions, which are naturally linked to Problems 1.23 and 1.26, are in fact corollaries of Theorem 1.9. Prove them.

$$\#(A \cup B) \le \#E \quad \text{and} \quad \#(A \times B) \le \#E.$$

(b) Now use Problem 1.23(b) and Theorem 1.9 to show by induction that, if X is an infinite set, then

$$\#X^n = \#X \text{ for every } n \in \mathbb{N}.$$

Problem 1.29. Let $\mathcal{I}$ be the collection of all injective functions from subsets of a set X to itself. That is, set

$$\mathcal{I} = \big\{F \in X^A\colon\ A \subseteq X \text{ and } F \text{ is injective}\big\}.$$

As a subset of $\bigcup_{A \in \wp(X)} X^A$, $\mathcal{I}$ is partially ordered in the extension ordering (cf. Problem 1.17). Let $\mathcal{J}$ be the collection of all those functions F in $\mathcal{I}$ for which the range $\mathcal{R}(F)$ is disjoint with the domain $\mathcal{D}(F)$:

$$\mathcal{J} = \big\{F \in \mathcal{I}\colon\ \mathcal{R}(F) \subseteq X\backslash\mathcal{D}(F)\big\}.$$

Problem 1.17 also tells us that every chain $\{F_\gamma\}$ in $\mathcal{J}$ has a supremum $\bigvee_\gamma F_\gamma$ in $\mathcal{I}$, and also that $\mathcal{D}\big(\bigvee_\gamma F_\gamma\big) = \bigcup_\gamma \mathcal{D}(F_\gamma)$ and $\mathcal{R}\big(\bigvee_\gamma F_\gamma\big) = \bigcup_\gamma \mathcal{R}(F_\gamma)$.

(a) Show that $\bigvee_\gamma F_\gamma$ in fact lies in $\mathcal{J}$.

Hint: Take F_λ and F_μ arbitrary from $\{F_\gamma\} \subseteq \mathcal{J}$ so that $F_\lambda \le F_\mu$ (or vice versa). Note that $\mathcal{R}(F_\lambda) \cap \mathcal{D}(F_\mu) \subseteq \mathcal{R}(F_\mu) \cap \mathcal{D}(F_\mu) = \varnothing$, and conclude that $\bigcup_\gamma \mathcal{R}(F_\gamma) \cap \bigcup_\gamma \mathcal{D}(F_\gamma)$ is empty too.

Thus every chain in $\mathcal{J}$ has an upper bound (a supremum, actually) in $\mathcal{J}$, and so $\mathcal{J}$ contains a maximal function by Zorn's Lemma. Let F_0 be a maximal function of $\mathcal{J}$ and let A_0 be the domain of F_0, so that $F_0(A_0) \subseteq X \backslash A_0$. Suppose

$$\#A_0 \;<\; \#(X \backslash A_0).$$

(b) Show that if X is an infinite set, then there exist two distinct points, say x_0 and x_1, in $(X \backslash A_0) \backslash F_0(A_0)$.

Hint: Under the above assumption $\#F_0(A_0) < \#(X \backslash A_0)$ (why?) and $X \backslash A_0$ is infinite — recall: the union of finite sets is finite.

Now set $A_1 = A_0 \cup \{x_0\}$ and consider the function $F_1 \colon A_1 \to X$ defined by

$$F_1(x) \;=\; \begin{cases} F_0(x) \in F_0(A_0), & x \in A_0, \\ x_1 \in X \backslash F_0(A_0), & x = x_0 \in X \backslash A_0. \end{cases}$$

(c) Show that $F_1 \in \mathcal{J}$.

Since $F_0 = F_1|_{A_0}$ it follows that $F_0 \leq F_1$, which contradicts the fact that F_0 is a maximal of $\mathcal{J}$ (for $F_0 \neq F_1$). Therefore (by Theorem 1.7),

$$\#(X \backslash A_0) \;\leq\; \#A_0.$$

Next verify that $\#A_0 = \#F_0(A_0) \leq \#(X \backslash A_0)$ and conclude:

$$\#A_0 \;=\; \#(X \backslash A_0)$$

(cf. Cantor–Bernstein Theorem). Finally, using Problem 1.28(a), show that

$$\#A_0 \;=\; \#X.$$

Outcome: *If X is an infinite set, then it has a subset, say A_0, such that*

$$\#A_0 \;=\; \#X \;=\; \#(X \backslash A_0).$$

Problem 1.30. Let A be an arbitrary set and write $A' = A \times \{A\}$.

(a) Show that $\#A' = \#A$.

Hint: The function that assigns to each a in A the ordered pair (a, A) in the Cartesian product A' is a one-to-one mapping of A onto A'.

It is plain that $A \cap A' = \varnothing$. Moreover, if B is any set such that $B \neq A$, then $A' \cap B' = \varnothing$ where $B' = B \times \{B\}$. Conclusion 1: If A and B are any sets, then there exist sets C and D such that

$$\#A = \#C, \quad \#B = \#D, \quad \text{and} \quad C \cap D = \varnothing.$$

Now suppose C_1, C_2, D_1 and D_2 are sets with the following properties. $C_1 \cap D_1 = \varnothing$, $C_2 \cap D_2 = \varnothing$, $\#C_1 = \#C_2$, and $\#D_1 = \#D_2$.

(b) Show that $\#(C_1 \cup D_1) = \#(C_2 \cup D_2)$.

Conclusion 2: $\#(C \cup D)$ is independent of the particular pair of sets $\{C, D\}$ employed in Conclusion 1. We are now prepared to define the sum of cardinal numbers. If A and B are sets, then

$$\#A + \#B \;=\; \#(C \cup D)$$

for any pair of sets $\{C, D\}$ such that

$$\#A = \#C, \quad \#B = \#D, \quad \text{and} \quad C \cap D = \varnothing.$$

In particular, if $A \cap B = \varnothing$, then $\#A + \#B = \#(A \cup B)$.

(c) Use Problem 1.29 to show that $\#X + \#X = \#X$ for every infinite set X.

The definition of product of cardinal numbers is much simpler:

$$\#A \cdot \#B \;=\; \#(A \times B)$$

for any sets A and B. According to Theorem 1.9, $\#X \cdot \#X = \#X$ for every infinite set X.

(d) Prove: If X and Y are two sets, at least one of which is infinite, then

$$\#X + \#Y \;=\; \#X \cdot \#Y \;=\; \max\{\#X, \#Y\}.$$

Hint: If $\#Y \leq \#X$, then verify that $\#X \leq \#X + \#Y \leq \#X + \#X$ and also that $\#X \leq \#X \cdot \#Y \leq \#X \cdot \#X$.

Problem 1.31. Let A be an arbitrary nonempty set. The Zermelo Well-Ordering Principle says that A can be well ordered so that there is a well-ordering, say $\leq$, of A. Let $P(\ )$ be a predicate that is meaningful for every $a \in A$ (i.e., let $P(a)$ be a proposition for each $a \in A$). Suppose that

(i) $P(a_0)$ holds true for the minimum element a_0 of A, and that

(ii) $P(a'')$ holds whenever $P(a')$ holds for every $a' < a''$.

Then show that $P(a)$ holds true for every $a \in A$. This is called the *Principle of Transfinite Induction.*

Hint: Prove by contradiction. (Compare with the Principle of Mathematical Induction of Section 1.1.)

2

Algebraic Structures

The main algebraic structure involved with the subject of this book is that of a "linear space" (or "vector space"). A linear space is a set endowed with an extra structure in addition to its set-theoretic structure (i.e., an extra structure that goes beyond the notions of inclusion, union, complement, function, and ordering, for instance). Roughly speaking, linear spaces are sets where two operations, called "addition" and "scalar multiplication", are properly defined so that we can refer to the "sum" of two points in a linear space, as well as to the "product" of a point in it by a "scalar". Although the reader is supposed to have already had a contact with linear algebra and, in particular, with "finite-dimensional vector spaces", we shall proceed from the very beginning. Our approach avoids the parochially "finite-dimensional" constructions (whenever this is possible), and focuses either on general results that do not depend on the "dimensionality" of the linear space, or on abstract "infinite-dimensional" linear spaces.

2.1 Linear Spaces

A *binary operation* on a set X is a mapping of $X \times X$ into X. If F is a function from $X \times X$ to X, then we generally write $z = F(x, y)$ to indicate that z in X is the value of F at the point (x, y) in $X \times X$. However, to emphasize the rule of the binary operation (the outcome of a binary operation on two points of X is again a point of X), it is convenient (and customary) to adopt a different notation. Moreover, in order to emphasize the abstract character of a binary operation, it is also common to use a noncommittal symbol to denote it. Thus, if $\star$ is a binary operation on X (so that $\star\colon X \times X \to X$), then we shall write $z = x \star y$ instead of $z = \star(x, y)$ to indicate that z in X is the value of $\star$ at the point (x, y) in $X \times X$. If a binary operation $\star$ on X has the property that

$$x \star (y \star z) \;=\; (x \star y) \star z$$

for every x, y, and z in X, then it is said to be *associative*. In this case we shall drop the parentheses and write $x \star y \star z$. If there exists an element e in X such that

C.S. Kubrusly, *The Elements of Operator Theory*, DOI 10.1007/978-0-8176-4998-2_2,

© Springer Science+Business Media, LLC 2011

$$x \star e = e \star x = x$$

for every $x \in X$, then e is said to be the *neutral element* (or the *identity element*) with respect to the binary operation $\star$ on X. It is easy to show that, if a binary operation $\star$ has a neutral element e, then e is *unique*. If an associative binary operation $\star$ on X has a neutral element e in X, and if for some $x \in X$ there exists $x^{-1} \in X$ such that

$$x \star x^{-1} = x^{-1} \star x = e,$$

then x^{-1} is called the *inverse* of x with respect to $\star$. It is also easy to show that, if the inverse of x exists with respect to an associative binary operation $\star$, then it is *unique*. A *group* is a nonempty set X on which is defined a binary operation $\star$ such that

(a) $\star$ is associative,

(b) $\star$ has a neutral element in X, and

(c) every x in X has an inverse in X with respect to $\star$.

If a binary operation $\star$ on X has the property that

$$x \star y = y \star x$$

for every x and y in X, then it is said to be *commutative*. If X is a group with respect to a binary operation $\star$, and if

(d) $\star$ is commutative,

then X is said to be an *Abelian* (or *commutative*) *group*.

Example 2.A. Let X be a set with more than three elements. The collection of all injective mappings of X onto itself (i.e., the collection of all invertible mappings on X) is a non-Abelian group with respect to the composition operation $\circ$. The neutral element (or the identity element) of such a group is, of course, the identity map on X.

An *additive Abelian group* is an Abelian group X for which the underlying binary operation is interpreted as an *addition* and denoted by $+$ (instead of $\star$). In this case the element $x + y$ (which lies in X for every x and y in X) is called the *sum* of x and y. The (unique) neutral element with respect to addition is denoted by 0 (instead of e) and called *zero*. The (unique) inverse of x under addition is denoted by $-x$ (instead of x^{-1}) and is called the *negative* of x. Thus $x + 0 = 0 + x = x$ and $x + (-x) = (-x) + x = 0$ for every $x \in X$. Moreover, the operation of *subtraction* is defined by $x - y = x + (-y)$, and $x - y$ is called the *difference* between x and y.

If $\diamond\colon X \times X \to X$ is another binary operation on X, and if

$$x \diamond (y \star z) = (x \diamond y) \star (x \diamond z) \quad \text{and} \quad (y \star z) \diamond x = (y \diamond x) \star (z \diamond x)$$

for every x, y, and z in X, then $\diamond$ is said to be *distributive* with respect to $\star$. The above properties are called the *distributive laws*. A *ring* is an additive Abelian group X with a second binary operation on X, called *multiplication* and denoted by $\cdot$, such that

(e) the multiplication operation is associative and

(f) distributive with respect to the addition operation.

In this case the element $x \cdot y$ (which lies in X for every x and y in X) is called the *product* of x and y (alternative notation: xy instead of $x \cdot y$). A *commutative ring* is a ring for which

(g) the multiplication operation is commutative.

A *ring with identity* is a ring X such that

(h) the multiplication operation has a neutral element in X.

In this case such a (unique) neutral element in X with respect to the multiplication operation is denoted by 1 (so that $x \cdot 1 = 1 \cdot x = x$ for every $x \in X$) and is called the *identity*.

Example 2.B. The power set $\wp(X)$ of a nonempty set X is a commutative ring with identity if addition is interpreted as symmetric difference (or Boolean sum) and multiplication as intersection (i.e., $A + B = A \triangledown B$ and $A \cdot B = A \cap B$ for all subsets A and B of X). Here the neutral element under addition (i.e., the zero) is the empty set $\varnothing$, and the neutral element under multiplication (i.e., the identity) is X itself.

A ring with identity is *nontrivial* if it has another element besides the identity. (The set $\{0\}$ with the operations $0 + 0 = 0 \cdot 0 = 0$ is the trivial ring whose only element is the identity.) If a ring with identity is nontrivial, then the neutral element under addition and the neutral element under multiplication never coincide (i.e., $0 \neq 1$). In fact, $x \cdot 0 = 0 \cdot x = 0$ for every x in X whenever X is a ring (with or without identity). Incidentally (or not) this also shows that, in a nontrivial ring with identity, zero has no inverse with respect to the multiplication operation (i.e., there is no x in X such that $0 \cdot x = x \cdot 0 = 1$). A ring X with identity is called a *division ring* if

(i) each nonzero x in X has an inverse in X with respect to the multiplication operation.

That is, if $x \neq 0$ in X, then there exists a (unique) $x^{-1} \in X$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$.

Example 2.C. Let the addition and multiplication operations have their ordinary ("numerical") meanings. The set of all natural numbers $\mathbb{N}$ is not a group under addition; neither is the set of all nonnegative integers $\mathbb{N}_0$. However, the set of all integers $\mathbb{Z}$ is a commutative ring with identity, but not a

division ring. The sets $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ (of all rational, real, and complex numbers, respectively), when equipped with their respective operations of addition and multiplication, are all commutative division rings. These are infinite commutative division rings, but there are finite commutative division rings (e.g., if we declare that $1+1=0+0=0$, $0+1=1+0=1$, $0\cdot 0=0\cdot 1=1\cdot 0=0$, and $1\cdot 1=1$, then $\{0,1\}$ is a commutative division ring).

Roughly speaking, commutative division rings are the *number systems* of mathematics, and so they deserve a name of their own. A *field* is a nontrivial commutative division ring. The elements of a field are usually called *scalars*. We shall be particularly concerned with the fields $\mathbb{R}$ and $\mathbb{C}$ (the *real field* and the *complex field*). An arbitrary field will be denoted by $\mathbb{F}$. Summing up: A field $\mathbb{F}$ is a set with more than one element (at least 0 and 1 are distinct elements of it) equipped with two binary operations, called addition and multiplication, that satisfy all the properties (a) through (i) — clearly, with $\star$ replaced by $+$ in properties (a) through (d).

Definition 2.1. A *linear space* (or *vector space*) over a field $\mathbb{F}$ is a nonempty set $\mathcal{X}$ (whose elements are called *vectors*) satisfying the following axioms.

Vector Addition. $\mathcal{X}$ is an additive Abelian group under a binary operation $\dot{+}$ called *vector addition*.

Scalar Multiplication. There is given a mapping of $\mathbb{F}\times\mathcal{X}$ into $\mathcal{X}$ that assigns to each scalar α in $\mathbb{F}$ and each vector x in $\mathcal{X}$ a vector αx in $\mathcal{X}$. Such a mapping defines an operation, called *scalar multiplication*, with the following properties. For all scalars α and β in $\mathbb{F}$, and all vectors x and y in $\mathcal{X}$,

$$\begin{aligned} 1x &= x, \\ \alpha(\beta x) &= (\alpha\cdot\beta)x, \\ \alpha(x\dot{+}y) &= \alpha x\dot{+}\alpha y, \\ (\alpha+\beta)x &= \alpha x\dot{+}\beta x. \end{aligned}$$

Some remarks on notation and terminology. The *underlying set* of a linear space is the nonempty set upon which the linear space is built. We shall use the same notation $\mathcal{X}$ for both the linear space and its underlying set, even though the underlying set alone has no algebraic structure of its own. A set $\mathcal{X}$ needs a binary operation on it, a field, and another operation involving such a field with $\mathcal{X}$ to acquire the necessary algebraic structure that will grant it the status of a linear space. The scalar 1 in the above definition stands, of course, for the identity in the field $\mathbb{F}$ with respect to the multiplication $\cdot$ in $\mathbb{F}$, and $+$ denotes the addition in $\mathbb{F}$. Observe that $+$ (addition in the field $\mathbb{F}$) and $\dot{+}$ (addition in the group $\mathcal{X}$) are different binary operations. However, once the difference has been pointed out, we shall use the same symbol $+$ to denote both addition in the field $\mathbb{F}$ and addition in the group $\mathcal{X}$. Moreover, we shall also drop the dot from the multiplication notation in $\mathbb{F}$, and write $\alpha\beta$ instead

of $\alpha \cdot \beta$. The neutral element under the vector addition in $\mathcal{X}$ (i.e., the vector zero) is referred to as the *origin* of the linear space $\mathcal{X}$. Again, we shall use one and the same symbol 0 to denote both the origin in $\mathcal{X}$ and the scalar zero in $\mathbb{F}$. A linear space over $\mathbb{R}$ is called a *real linear space*, and a linear space over $\mathbb{C}$ is called a *complex linear space*.

Example 2.D. $\mathbb{R}$ itself is a linear space over $\mathbb{R}$. That is, the plain set $\mathbb{R}$ when equipped with the ordinary binary operations of addition and multiplication becomes a field, also denoted by $\mathbb{R}$. If vector addition is identified with scalar addition, then it becomes a real linear space, denoted again by $\mathbb{R}$. More generally, for each $n \in \mathbb{N}$, let $\mathbb{F}^n$ denote the Cartesian product of n copies of a field $\mathbb{F}$ (i.e., the set of all ordered n-tuples of scalars in $\mathbb{F}$). Now define vector addition and scalar multiplication coordinatewise, as usual:

$$x + y = (\xi_1 + \upsilon_1, \ldots, \xi_n + \upsilon_n) \quad \text{and} \quad \alpha x = (\alpha\xi_1, \ldots, \alpha\xi_n)$$

for every $x = (\xi_1, \ldots, \xi_n)$ and $y = (\upsilon_1, \ldots, \upsilon_n)$ in $\mathbb{F}^n$ and every α in $\mathbb{F}$. This makes $\mathbb{F}^n$ into a linear space over $\mathbb{F}$. In particular, $\mathbb{R}^n$ (the Cartesian product of n copies of $\mathbb{R}$) and $\mathbb{C}^n$ (the Cartesian product of n copies of $\mathbb{C}$) become real and complex linear spaces, respectively, whenever vector addition and scalar multiplication are defined coordinatewise. However, if we restrict scalar multiplication to real multiplication only, then $\mathbb{C}^n$ can also be made into a real linear space.

Example 2.E. Let S be a nonempty set, and let $\mathbb{F}$ be an arbitrary field. Consider the set

$$\mathcal{X} = \mathbb{F}^S$$

of all functions from S to $\mathbb{F}$ (i.e., the set of all *scalar-valued functions* on S, where "scalar-valued" stands for "$\mathbb{F}$-valued"). Let vector addition and scalar multiplication be defined pointwise. That is, if x and y are functions in $\mathcal{X}$ and α is a scalar in $\mathbb{F}$, then $x + y$ and αx are functions in $\mathcal{X}$ defined by

$$(x + y)(s) = x(s) + y(s) \quad \text{and} \quad (\alpha x)(s) = \alpha(x(s))$$

for every $s \in S$. Now it is easy to show that $\mathcal{X}$, when equipped with these two operations, in fact is a linear space over $\mathbb{F}$. Particular cases: $\mathbb{F}^{\mathbb{N}}$ (the set of all *scalar-valued sequences*) and $\mathbb{F}^{[0,1]}$ (the set of all scalar-valued functions on the interval $[0, 1]$) are linear spaces over $\mathbb{F}$, whenever vector addition and scalar multiplication are defined pointwise. Note that the linear space $\mathbb{F}^n$ in the previous example also is a particular case of the present example, where the coordinatewise operations are identified with the pointwise operations (recall: $\mathbb{F}^n = \mathbb{F}^{\mathbb{I}_n}$ where $\mathbb{I}_n = \{i \in \mathbb{N}\colon\ i \leq n\}$).

Example 2.F. What was the role played by the field $\mathbb{F}$ in the previous example? Answer: Vector addition and scalar multiplication in $\mathbb{F}^S$ were defined

pointwise by using addition and multiplication in $\mathbb{F}$. This suggests the following generalization of Example 2.E. Let S be a nonempty set, and let $\mathcal{Y}$ be an arbitrary linear space (over a field $\mathbb{F}$). Consider the set

$$\mathcal{X} = \mathcal{Y}^S$$

of all functions from S to $\mathcal{Y}$ (i.e., the set of all $\mathcal{Y}$-valued functions on S). Let vector addition and scalar multiplication in $\mathcal{X}$ be defined pointwise by using vector addition and scalar multiplication in $\mathcal{Y}$. That is, if f and g are functions in $\mathcal{X}$ (so that $f(s)$ and $g(s)$ are elements of $\mathcal{Y}$ for each $s \in S$) and α is a scalar in $\mathbb{F}$, then $f + g$ and αf are functions in $\mathcal{X}$ defined by

$$(f + g)(s) = f(s) + g(s) \quad \text{and} \quad (\alpha f)(s) = \alpha(f(s))$$

for every $s \in S$. As before, it is easily verified that $\mathcal{X}$, when equipped with these operations, becomes a linear space over the same field $\mathbb{F}$. The origin of $\mathcal{X}$ is the *null function* $0\colon S \to \mathcal{Y}$ (which is defined by $0(s) = 0$ for all $s \in S$). Examples 2.D and 2.E can be thought of as particular cases of this one.

Example 2.G. Let $\mathcal{X}$ be a linear space over $\mathbb{F}$, and let x, x', y, and y' be arbitrary vectors in $\mathcal{X}$. An equivalence relation $\sim$ on $\mathcal{X}$ is *compatible with vector addition* if

$$x' \sim x \text{ and } y' \sim y \quad \text{imply} \quad x' + y' \sim x + y.$$

Similarly, it is said to be *compatible with scalar multiplication* if, for x and x' in $\mathcal{X}$ and α in $\mathbb{F}$,

$$x' \sim x \quad \text{implies} \quad \alpha x' \sim \alpha x.$$

If an equivalence relation $\sim$ on a linear space $\mathcal{X}$ is compatible with both vector addition and scalar multiplication, then we shall say that $\sim$ is a *linear equivalence relation*. Now consider $\mathcal{X}/\!\sim$, the quotient space of $\mathcal{X}$ modulo $\sim$ (i.e., the collection of all equivalence classes $[x]$ with respect to $\sim$), and suppose the equivalence relation $\sim$ on $\mathcal{X}$ is linear. In this case a binary operation $+$ on $\mathcal{X}/\!\sim$ can be defined by setting

$$[x] + [y] = [x + y]$$

for every $[x]$ and $[y]$ in $\mathcal{X}/\!\sim$. Indeed, since $\sim$ is compatible with vector addition, it follows that $[x + y]$ does not depend on which particular members x and y of the equivalence classes $[x]$ and $[y]$ were taken. Thus the operation $+$ actually is a function from $(\mathcal{X}/\!\sim \times \mathcal{X}/\!\sim)$ to $\mathcal{X}/\!\sim$. This defines vector addition in $\mathcal{X}/\!\sim$. Scalar multiplication in $\mathcal{X}/\!\sim$ can be similarly defined by setting

$$\alpha[x] = [\alpha x]$$

for every $[x]$ in $\mathcal{X}/\!\sim$ and α in $\mathbb{F}$. Therefore, if $\sim$ is a linear equivalence relation on a linear space $\mathcal{X}$ over a field $\mathbb{F}$, then $\mathcal{X}/\!\sim$ becomes a linear space over $\mathbb{F}$ when vector addition and scalar multiplication in $\mathcal{X}/\!\sim$ are defined this way.

It is clear by the definition of a linear space $\mathcal{X}$ that $x + y + z$ is a well-defined vector in $\mathcal{X}$ whenever x, y, and z are vectors in $\mathcal{X}$. Similarly, if $\{x_i\}_{i=1}^n$ is a *finite* set of vectors in $\mathcal{X}$, then the sum $x_1 + \cdots + x_n$, denoted by $\sum_{i=1}^n x_i$, is again a vector in $\mathcal{X}$. (The notion of *infinite sums* needs *topology* and we shall consider these topics in Chapters 4 and 5.)

2.2 Linear Manifolds

A *linear manifold* of a linear space $\mathcal{X}$ over $\mathbb{F}$ is a nonempty subset $\mathcal{M}$ of $\mathcal{X}$ with the following properties.

$$x + y \in \mathcal{M} \quad \text{and} \quad \alpha x \in \mathcal{M}$$

for every pair of vectors x, y in $\mathcal{M}$ and every scalar α in $\mathbb{F}$. It is readily verified that a linear manifold $\mathcal{M}$ of a linear space $\mathcal{X}$ over a field $\mathbb{F}$ is itself a linear space over the same field $\mathbb{F}$. The origin 0 of $\mathcal{X}$ is the origin of every linear manifold $\mathcal{M}$ of $\mathcal{X}$. The *zero linear manifold* is $\{0\}$, consisting of the single vector 0. If a linear manifold $\mathcal{M}$ is a proper subset of $\mathcal{X}$, then it is said to be a *proper linear manifold*. A *nontrivial linear manifold* $\mathcal{M}$ of a linear space $\mathcal{X}$ is a nonzero proper linear manifold of it ($\{0\} \neq \mathcal{M} \neq \mathcal{X}$).

Example 2.H. Let $\mathcal{M}$ be a linear manifold of a linear space $\mathcal{X}$ and consider a relation $\sim$ on $\mathcal{X}$ defined as follows. If x and x' are vectors in $\mathcal{X}$, then

$$x' \sim x \quad \text{if} \quad x' - x \in \mathcal{M}.$$

That is, $x' \sim x$ if x' is *congruent to x modulo $\mathcal{M}$* — notation: $x' \equiv x (\text{mod } \mathcal{M})$. Since $\mathcal{M}$ is a linear manifold of $\mathcal{X}$, the relation $\sim$ in fact is an equivalence relation on $\mathcal{X}$ (reason: $0 \in \mathcal{M}$ — reflexivity, $x' - x'' = (x' - x) + (x - x'') \in \mathcal{M}$ whenever $x' - x$ and $x - x''$ lie in $\mathcal{M}$ — transitivity, and $x - x' \in \mathcal{M}$ whenever $x' - x \in \mathcal{M}$ — symmetry). The equivalence class (with respect to $\sim$)

$$[x] = \{x' \in \mathcal{X}\colon\ x' \sim x\} = \{x' \in \mathcal{X}\colon\ x' = x + z \text{ for some } z \in \mathcal{M}\}$$

of a vector x in $\mathcal{X}$ is called the *coset of x modulo $\mathcal{M}$* — notation: $[x] = x + \mathcal{M}$. The set of all cosets $[x]$ modulo $\mathcal{M}$ for every $x \in \mathcal{X}$ (i.e., the collection of all equivalence classes $[x]$ with respect to the equivalence relation $\sim$ for every x in $\mathcal{X}$) is precisely the quotient space $\mathcal{X}/\sim$ of $\mathcal{X}$ modulo $\sim$. Following the terminology introduced in Example 2.G, $\sim$ is a *linear equivalence relation* on the linear space $\mathcal{X}$. Indeed, if $x' - x \in \mathcal{M}$ and $y' - y \in \mathcal{M}$, then $(x' + y') - (x + y) = (x' - x) + (y' - y) \in \mathcal{M}$ and $\alpha x' - \alpha x = \alpha(x' - x) \in \mathcal{M}$ for every scalar α, so that $x' \sim x$ and $y' \sim y$ imply $x' + y' \sim x + y$ and $\alpha x' \sim \alpha x$. Therefore, with vector addition and scalar multiplication defined by

$$[x] + [y] = [x + y] \quad \text{and} \quad \alpha[x] = [\alpha x],$$

$\mathcal{X}/\sim$ is made into a linear space over the same scalar field. This is usually denoted by $\mathcal{X}/\mathcal{M}$ (instead of $\mathcal{X}/\sim$), and called the *quotient space of* $\mathcal{X}$ *modulo* $\mathcal{M}$. The origin of $\mathcal{X}/\mathcal{M}$ is, of course, $[0] = \mathcal{M}$. Let $\pi\colon \mathcal{X} \to \mathcal{X}/\mathcal{M}$ be the natural mapping of $\mathcal{X}$ onto the quotient space $\mathcal{X}/\mathcal{M}$ as defined in Section 1.4:

$$\pi(x) \,=\, [x] \,=\, x + \mathcal{M} \quad \text{ for every } \quad x \in \mathcal{X}.$$

The concept of "linear transformation" will be defined in Section 2.5. It can be easily shown that π is a linear transformation between the linear spaces $\mathcal{X}$ and $\mathcal{X}/\mathcal{M}$. The *null space* of π, viz., $\mathcal{N}(\pi) = \{x \in \mathcal{X}\colon\ \pi(x) = [0]\}$, is given by

$$\mathcal{N}(\pi) \,=\, \mathcal{M}.$$

Indeed, if $\pi(x) = [0] = \mathcal{M}$, then $x + \mathcal{M} = [x] = [0] = \mathcal{M}$, and so $x \in \mathcal{M}$. On the other hand, if $u \in \mathcal{M}$, then $\pi(u) = u + \mathcal{M} = 0 + \mathcal{M} = \mathcal{M} = [0]$.

If $\mathcal{M}$ and $\mathcal{N}$ are linear manifolds of a linear space $\mathcal{X}$, then the *sum* of $\mathcal{M}$ and $\mathcal{N}$, denoted by $\mathcal{M} + \mathcal{N}$, is the subset of $\mathcal{X}$ made up of all sums $x + y$ where x is a vector in $\mathcal{M}$ and y is a vector in $\mathcal{N}$:

$$\mathcal{M} + \mathcal{N} \,=\, \big\{z \in \mathcal{X}\colon\ z = x + y,\ x \in \mathcal{M} \text{ and } y \in \mathcal{N}\big\}.$$

It is trivially verified that $\mathcal{M} + \mathcal{N}$ is a linear manifold of $\mathcal{X}$. If $\{\mathcal{M}_i\}_{i=1}^n$ is a finite family of linear manifolds of a linear space $\mathcal{X}$, then the sum $\sum_{i=1}^n \mathcal{M}_i$ is the linear manifold $\mathcal{M}_1 + \cdots + \mathcal{M}_n$ of $\mathcal{X}$ consisting of all sums $\sum_{i=1}^n x_i$ where each vector x_i lies in $\mathcal{M}_i$. More generally, if $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ is an arbitrary indexed family of linear manifolds of a linear space $\mathcal{X}$, then the *sum* $\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma$ is defined as the set of all sums $\sum_{\gamma\in\Gamma} x_\gamma$ with $x_\gamma \in \mathcal{M}_\gamma$ for each index γ and $x_\gamma = 0$ except for some finite set of indices (i.e., $\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma$ is the set made up of *all finite sums* with each summand being a vector in one of the linear manifolds $\mathcal{M}_\gamma$). Clearly, $\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma$ is itself a linear manifold of $\mathcal{X}$, and $\mathcal{M}_\alpha \subseteq \sum_{\gamma\in\Gamma}\mathcal{M}_\gamma$ for every $\mathcal{M}_\alpha \in \{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$.

A linear manifold of a linear space $\mathcal{X}$ is never empty: the origin of $\mathcal{X}$ is always there. Note that the intersection $\mathcal{M} \cap \mathcal{N}$ of two linear manifolds $\mathcal{M}$ and $\mathcal{N}$ of a linear space $\mathcal{X}$ is itself a linear manifold of $\mathcal{X}$. In fact, if $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ is an arbitrary collection of linear manifolds of a linear space $\mathcal{X}$, then the intersection $\bigcap_{\gamma\in\Gamma}\mathcal{M}_\gamma$ is again a linear manifold of $\mathcal{X}$. Moreover, $\bigcap_{\gamma\in\Gamma}\mathcal{M}_\gamma \subseteq \mathcal{M}_\alpha$ for every $\mathcal{M}_\alpha \in \{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$.

Consider the collection $\mathcal{L}at(\mathcal{X})$ of all linear manifolds of a linear space $\mathcal{X}$. Since $\mathcal{L}at(\mathcal{X})$ is a subcollection of the power set $\wp(\mathcal{X})$, it follows that $\mathcal{L}at(\mathcal{X})$ is partially ordered in the inclusion ordering. If $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ is a subcollection of $\mathcal{L}at(\mathcal{X})$, then $\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma$ in $\mathcal{L}at(\mathcal{X})$ is an upper bound for $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ and $\bigcap_{\gamma\in\Gamma}\mathcal{M}_\gamma$ in $\mathcal{L}at(\mathcal{X})$ is a lower bound for $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$. If $\mathcal{U} \in \mathcal{L}at(\mathcal{X})$ is an upper bound for $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ (i.e., $\mathcal{M}_\gamma \subseteq \mathcal{U}$ for all $\gamma \in \Gamma$), then $\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma \subseteq \mathcal{U}$. Thus

$$\sum_{\gamma\in\Gamma} \mathcal{M}_\gamma \,=\, \sup\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}.$$

Similarly, if $\mathcal{V} \in \mathcal{L}at(\mathcal{X})$ is a lower bound for $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ (i.e., if $\mathcal{V} \subseteq \mathcal{M}_\gamma$ for all $\gamma \in \Gamma$), then $\mathcal{V} \subseteq \bigcap_{\gamma\in\Gamma}\mathcal{M}_\gamma$. Thus

$$\bigcap_{\gamma\in\Gamma} \mathcal{M}_\gamma \;=\; \inf\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}.$$

Conclusion: $\mathcal{L}at(\mathcal{X})$ is a complete lattice. *The collection of all linear manifolds of a linear space is a complete lattice in the inclusion ordering*. If $\{\mathcal{M},\mathcal{N}\}$ is a pair of elements of $\mathcal{L}at(\mathcal{X})$, then $\mathcal{M} \vee \mathcal{N} = \mathcal{M} + \mathcal{N}$ and $\mathcal{M} \wedge \mathcal{N} = \mathcal{M} \cap \mathcal{N}$.

Let A be an arbitrary subset of a linear space $\mathcal{X}$, and consider the subcollection (a sublattice, actually) $\mathcal{L}_A$ of the complete lattice $\mathcal{L}at(\mathcal{X})$,

$$\mathcal{L}_A \;=\; \big\{\mathcal{M} \in \mathcal{L}at(\mathcal{X})\colon\; A \subseteq \mathcal{M}\big\},$$

consisting of all linear manifolds of $\mathcal{X}$ that include A. Set

$$\operatorname{span} A \;=\; \inf \mathcal{L}_A \;=\; \bigcap \mathcal{L}_A,$$

which is called the (linear) *span* of A. Since $A \subseteq \bigcap \mathcal{L}_A$ (for $A \subseteq \mathcal{M}$ for every $\mathcal{M} \in \mathcal{L}_A$), it follows that $\inf \mathcal{L}_A = \min \mathcal{L}_A$ so that $\operatorname{span} A \in \mathcal{L}_A$. Thus $\operatorname{span} A$ *is the smallest linear manifold of* $\mathcal{X}$ *that includes* A, which coincides with *the intersection of all linear manifolds of* $\mathcal{X}$ *that include* A. It is readily verified that $\operatorname{span} \varnothing = \{0\}$, $\operatorname{span} \mathcal{M} = \mathcal{M}$ for every $\mathcal{M} \in \mathcal{L}at(\mathcal{X})$, and $A \subseteq \operatorname{span} A = \operatorname{span}(\operatorname{span} A)$ for every $A \in \wp(\mathcal{X})$. Moreover, if A and B are subsets of $\mathcal{X}$, then

$$A \subseteq B \quad \text{ implies } \quad \operatorname{span} A \subseteq \operatorname{span} B.$$

If $\mathcal{M}$ and $\mathcal{N}$ are linear manifolds of a linear space $\mathcal{X}$, then $\mathcal{M} \cup \mathcal{N} \subseteq \mathcal{M} + \mathcal{N}$. Moreover, if $\mathcal{K}$ is a linear manifold of $\mathcal{X}$ such that $\mathcal{M} \cup \mathcal{N} \subseteq \mathcal{K}$, then $x + y \in \mathcal{K}$ for every $x \in \mathcal{M}$ and every $y \in \mathcal{N}$, and hence $\mathcal{M} + \mathcal{N} \subseteq \mathcal{K}$. Thus $\mathcal{M} + \mathcal{N}$ is the smallest linear manifold of $\mathcal{X}$ that includes $\mathcal{M} \cup \mathcal{N}$, which means that

$$\mathcal{M} + \mathcal{N} \;=\; \operatorname{span}(\mathcal{M} \cup \mathcal{N}).$$

More generally, let $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary subcollection of $\mathcal{L}at(\mathcal{X})$, and suppose $\mathcal{K} \in \mathcal{L}at(\mathcal{X})$ is such that $\bigcup_{\gamma\in\Gamma}\mathcal{M}_\gamma \subseteq \mathcal{K}$. Then every (finite) sum $\sum_{\gamma\in\Gamma} x_\gamma$ with each x_γ in $\mathcal{M}_\gamma$ is a vector in $\mathcal{K}$. Thus $\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma \subseteq \mathcal{K}$. Since $\bigcup_{\gamma\in\Gamma}\mathcal{M}_\gamma \subseteq \sum_{\gamma\in\Gamma}\mathcal{M}_\gamma$, it follows that $\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma$ is the smallest element of $\mathcal{L}at(\mathcal{X})$ that includes $\bigcup_{\gamma\in\Gamma}\mathcal{M}_\gamma$. Equivalently,

$$\sum_{\gamma\in\Gamma} \mathcal{M}_\gamma \;=\; \operatorname{span}\Big(\bigcup_{\gamma\in\Gamma} \mathcal{M}_\gamma\Big).$$

2.3 Linear Independence

Let A be a nonempty subset of a linear space $\mathcal{X}$. A vector $x \in \mathcal{X}$ is a *linear combination* of vectors in A if there exist a *finite* set $\{x_i\}_{i=1}^n$ of vectors in A and a *finite* family of scalars $\{\alpha_i\}_{i=1}^n$ such that

$$x = \sum_{i=1}^{n} \alpha_i x_i.$$

Warning: A linear combination is, by definition, *finite*. That is, a linear combination of vectors in a set A is a weighted sum of a finite subset of vectors in A, weighted by a finite family of scalars, no matter whether A is a finite or an infinite set. Since $\mathcal{X}$ is a linear space, any linear combination of vectors in A is a vector in $\mathcal{X}$.

Proposition 2.2. *The set of all linear combinations of vectors in a nonempty subset A of a linear space $\mathcal{X}$ coincides with the linear manifold* $\operatorname{span} A$.

Proof. Let A be an arbitrary subset of a linear space $\mathcal{X}$, consider the collection $\mathcal{L}_A$ of all linear manifolds of $\mathcal{X}$ that include A, and recall that

$$\operatorname{span} A = \min \mathcal{L}_A.$$

Suppose A is nonempty and let $\langle A \rangle$ denote the set of all linear combinations of vectors in A. It is plain that $A \subseteq \langle A \rangle$ (every vector in A is a trivial linear combination of vectors in A), and that $\langle A \rangle$ is a linear manifold of $\mathcal{X}$ (if $x, y \in \langle A \rangle$, then $x + y$ and αx lie in $\langle A \rangle$). Therefore,

$$\langle A \rangle \in \mathcal{L}_A.$$

Moreover, if $\mathcal{M}$ is an arbitrary linear manifold of $\mathcal{X}$, and if $x \in \mathcal{X}$ is a linear combination of vectors in $\mathcal{M}$, then $x \in \mathcal{M}$ (because $\mathcal{M}$ is itself a linear space). Thus $\langle \mathcal{M} \rangle \subseteq \mathcal{M}$. Since $\mathcal{M} \subseteq \langle \mathcal{M} \rangle$, it follows that $\langle \mathcal{M} \rangle = \mathcal{M}$ for every linear manifold $\mathcal{M}$ of $\mathcal{X}$. Furthermore, if $\mathcal{M} \in \mathcal{L}_A$, then $A \subseteq \mathcal{M}$ and so $\langle A \rangle \subseteq \langle \mathcal{M} \rangle$ (reason: $\langle A \rangle \subseteq \langle B \rangle$ whenever A and B are nonempty subsets of $\mathcal{X}$ such that $A \subseteq B$). Thus

$$\mathcal{M} \in \mathcal{L}_A \quad \text{implies} \quad \langle A \rangle \subseteq \mathcal{M}.$$

Conclusion: $\langle A \rangle$ is the smallest element of $\mathcal{L}_A$. That is,

$$\langle A \rangle = \operatorname{span} A. \qquad \square$$

Following the notation introduced in the proof of Proposition 2.2, $\langle A \rangle = \operatorname{span} A$ whenever $A \neq \varnothing$. Set $\langle \varnothing \rangle = \operatorname{span} \varnothing$ so that $\langle \varnothing \rangle = \{0\}$, and hence $\langle A \rangle$ is well defined for every subset A of $\mathcal{X}$. We shall use one and the same notation, viz., $\operatorname{span} A$, for both of them: the set of all linear combinations of vectors in A and the (linear) span of A. For this reason $\operatorname{span} A$ is also referred to as the *linear manifold generated* (or *spanned*) by A. If a linear manifold $\mathcal{M}$ of $\mathcal{X}$ (which may be $\mathcal{X}$ itself) is such that $\operatorname{span} A = \mathcal{M}$ for some subset A of $\mathcal{X}$, then we say that A *spans* $\mathcal{M}$.

A subset A of a linear space $\mathcal{X}$ is said to be *linearly independent* if each vector x in A is not a linear combination of vectors in $A \backslash \{x\}$. Equivalently, A

is linearly independent if $x \notin \operatorname{span}(A \backslash \{x\})$ for every $x \in A$. If a set A is not linearly independent, then it is said to be *linearly dependent*. Note that the empty set $\varnothing$ of a linear space $\mathcal{X}$ is linearly independent (there is no vector in $\varnothing$ that is a linear combination of vectors in $\varnothing$). Any singleton $\{x\}$ of $\mathcal{X}$ such that $x \neq 0$ is linearly independent. Indeed, $\operatorname{span}(\{x\} \backslash \{x\}) = \operatorname{span} \varnothing = \{0\}$ so that $x \notin \operatorname{span}(\{x\} \backslash \{x\})$ if $x \neq 0$. However, $0 \in \operatorname{span}(\{0\} \backslash \{0\}) = \{0\}$, and hence the singleton $\{0\}$ is not linearly independent. In fact, every subset of $\mathcal{X}$ that contains the origin of $\mathcal{X}$ is not linearly independent (indeed, if $0 \in A \subseteq \mathcal{X}$ and A has another vector, say $x \neq 0$, then $0 = 0x$). Thus, if a vector x is an element of a linearly independent subset of a linear space $\mathcal{X}$, then $x \neq 0$.

Proposition 2.3. *Let A be a nonempty subset of a linear space $\mathcal{X}$. The following assertions are pairwise equivalent.*

(a) *A is linearly independent.*

(b) *Each nonzero vector in* $\operatorname{span} A$ *has a unique representation as a linear combination of vectors in A.*

(c) *Every finite subset of A is linearly independent.*

(d) *There is no proper subset of A whose span coincides with* $\operatorname{span} A$.

Proof. The statement (b) can be rewritten as follows.

(b′) For every nonzero $x \in \operatorname{span} A$ there exist a unique finite family of scalars $\{\alpha_i\}_{i=1}^n$ and a unique finite subset $\{a_i\}_{i=1}^n$ of A such that $x = \sum_{i=1}^n \alpha_i a_i$.

Proof of (a)⇒(b). Suppose $A \neq \varnothing$ is linearly independent. Take an arbitrary nonzero $x \in \operatorname{span} A$, and consider two representations of it as a linear combination of vectors in A:

$$x = \sum_{i=1}^{n} \beta_i b_i = \sum_{i=1}^{m} \gamma_i c_i,$$

where each b_i and each c_i are vectors in A (and hence nonzero because A is linearly independent). Since $x \neq 0$ we may assume that the scalars β_i and γ_i are all nonzero. Set $B = \{b_i\}_{i=1}^n$ and $C = \{c_i\}_{i=1}^m$, both finite nonempty subsets of A. Take an arbitrary $b \in B$ and note that b is a linear combination of vectors in $(B \backslash \{b\}) \cup C$. However, since $b \in A$ and A is linearly independent, it follows that b is not a linear combination of any subset of $A \backslash \{b\}$. Thus $b \in C$. Similarly, take an arbitrary $c \in C$ and conclude that $c \in B$ by using the same argument. Hence $B \subseteq C \subseteq B$. That is, $B = C$. Therefore $x = \sum_{i=1}^n \beta_i b_i = \sum_{i=1}^n \gamma_i b_i$, which implies that $\sum_{i=1}^n (\beta_i - \gamma_i) b_i = 0$. Since each b_i is not a linear combination of vectors in $B \backslash \{b_i\}$, it follows that $\beta_i = \gamma_i$ for every i. Summing up: The two representations of x coincide.

Proof of (b)⇒(a). If A is nonempty and every nonzero vector x in $\operatorname{span} A$ has a unique representation as a linear combination of vectors in A, then the unique representation of an arbitrary a in A as a linear combination of vectors

in A is a itself (recall: $A \subseteq \operatorname{span} A$). Therefore, every $a \in A$ is not a linear combination of vectors in $A\backslash\{a\}$, which means that A is linearly independent.

Proof of (a)⇔(c). If A is linearly independent, then every subset of it clearly is linearly independent. If A is not linearly independent, then either $A = \{0\}$ or there exists $x \in A$ which is a linear combination of vectors, say $\{x_i\}_{i=1}^n$ for some $n \in \mathbb{N}$, in $A\backslash\{x\} \neq \varnothing$. In the former case A is itself a finite subset of A which is not linearly independent. In the latter case $\{x_i\}_{i=1}^n \cup \{x\}$ is a finite subset of A that is not linearly independent. Conclusion: If every finite subset of A is linearly independent, then A is itself linearly independent.

Proof of (a)⇒(d). Recalling that $B \subseteq A$ implies $\operatorname{span} B \subseteq \operatorname{span} A$, the statement (d) can be rewritten as follows.

(d′) $B \subset A$ implies $\operatorname{span} B \subset \operatorname{span} A$.

Suppose A is nonempty and linearly independent. Let B be an arbitrary proper subset of A. If $B = \varnothing$, then (d′) holds trivially ($\varnothing \neq A \neq \{0\}$ so that $\operatorname{span} \varnothing \subset \operatorname{span} A$). Thus suppose $B \neq \varnothing$ and take any $x \in A\backslash B$. If $x \in \operatorname{span} B$, then x is a linear combination of vectors in B. This implies that $B \cup \{x\}$ is a subset of A that is not linearly independent, and so A itself is not linearly independent, which is a contradiction. Therefore, $x \notin \operatorname{span} B$ for every $x \in A\backslash B$ whenever $\varnothing \neq B \subset A$. Since $x \in \operatorname{span} A$ (for $x \in A$) and $\operatorname{span} B \subseteq \operatorname{span} A$ (for $B \subset A$), it follows that $\operatorname{span} B \subset \operatorname{span} A$ so that (d′) holds true.

Proof of (d)⇒(a). If A is not linearly independent, then either $A = \{0\}$ or there is an $x \in A$ which is a linear combination of vectors in $A\backslash\{x\}$. In the former case $B = \varnothing$ is the unique proper subset of A and $\operatorname{span} B = \{0\} = \operatorname{span} A$. In the latter case $B = A\backslash\{x\}$ is a proper subset of A such that $\operatorname{span} B = \operatorname{span} A$. (Indeed, $\operatorname{span} B \subseteq \operatorname{span} A$ as $B \subseteq A$, and $\operatorname{span} A \subseteq \operatorname{span} B$ as vectors in A are linear combinations of vectors in $A\backslash\{x\}$.) Thus (d′) implies (a). □

2.4 Hamel Basis

A linearly independent subset of a linear space $\mathcal{X}$ that spans $\mathcal{X}$ is called a *Hamel basis* (or a *linear basis*) for $\mathcal{X}$. In other words, a subset B of a linear space $\mathcal{X}$ is a Hamel basis for $\mathcal{X}$ if

(i) B is linearly independent, and
(ii) $\operatorname{span} B = \mathcal{X}$.

Let $B = \{x_\gamma\}_{\gamma\in\Gamma}$ be an indexed Hamel basis for a linear space $\mathcal{X}$. If x is a nonzero vector in $\mathcal{X}$, then Proposition 2.3 ensures the existence of a unique (similarly indexed) family of scalars $\{\alpha_\gamma\}_{\gamma\in\Gamma}$ (which may depend on x) such that $\alpha_\gamma = 0$ for all but a finite set of indices γ and $x = \sum_{\gamma\in\Gamma} \alpha_\gamma x_\gamma$. The weighted sum $\sum_{\gamma\in\Gamma} \alpha_\gamma x_\gamma$ (i.e., the unique representation of x as a linear combination of vectors in B, or the unique (linear) representation of x in

terms of B) is called the *expansion* of x on B, and the coefficients of it (i.e., the unique indexed family of scalars $\{\alpha_\gamma\}_{\gamma\in\Gamma}$) are called the *coordinates* of x with respect to the indexed basis B. If $x = 0$, then its unique expansion on B is the trivial one whose coefficients are all null.

Since $\varnothing$ is linearly independent, and since $\operatorname{span}\varnothing = \{0\}$, it follows that the empty set $\varnothing$ is a Hamel basis for the zero linear space $\{0\}$. Now suppose $\mathcal{X}$ is a nonzero linear space. Every singleton $\{x\}$ in $\mathcal{X}$ such that $x \neq 0$ is linearly independent. Thus every nonzero linear space has many linearly independent subsets. If a linearly independent subset A of $\mathcal{X}$ is not already a Hamel basis for $\mathcal{X}$, then we can construct a larger linearly independent subset of $\mathcal{X}$.

Proposition 2.4. *If A is a linearly independent subset of a linear space $\mathcal{X}$, and if there exists a vector x in $\mathcal{X}\backslash\operatorname{span} A$, then $A\cup\{x\}$ is another linearly independent subset of $\mathcal{X}$.*

Proof. Suppose there exists $x \in \mathcal{X}\backslash\operatorname{span} A$. Note that $x \neq 0$, and so $\mathcal{X} \neq \{0\}$. If $A = \varnothing$, then the result is trivially verified ($\{x\} = \varnothing \cup \{x\}$ is linearly independent). Thus suppose A is nonempty and set $C = A\cup\{x\} \subset \mathcal{X}$. Since $x \notin \operatorname{span} A$, it follows that $x \notin \operatorname{span}(C\backslash\{x\})$. Take an arbitrary $a \in A$. Suppose it is a linear combination of vectors in $C\backslash\{a\}$. Since $a \neq \alpha x$ for every scalar α (for $x \notin \operatorname{span} A$ and $a \neq 0$ because A is linearly independent), we get

$$a \;=\; \alpha_0 x + \sum_{i=1}^{n} \alpha_i a_i,$$

where each a_i is a vector in $A\backslash\{a\}$ and each α_i is a nonzero scalar (recall: $0 \neq a \neq \sum_{i=1}^{n}\alpha_i a_i$ because A is linearly independent). Thus x is a linear combination of vectors in A, which contradicts the assumption that $x \notin \operatorname{span} A$. Therefore, every $a \in A$ is not a linear combination of vectors in $C\backslash\{a\}$. Conclusion: Every $c \in C$ is not a linear combination of vectors in $C\backslash\{c\}$, which means that C is linearly independent. □

Can we proceed this way, enlarging linearly independent subsets of $\mathcal{X}$ in order to form a chain of linearly independent subsets, so that an "ultimate" linearly independent subset becomes a Hamel basis for $\mathcal{X}$? Yes, we can; and it seems reasonable that the Axiom of Choice (or any statement equivalent to it as, for instance, Zorn's Lemma) might be called into play. In fact, every linearly independent subset of any linear space $\mathcal{X}$ is included in some Hamel basis for $\mathcal{X}$, so that every linear space has a large supply of Hamel bases.

Theorem 2.5. *If A is a linearly independent subset of a linear space $\mathcal{X}$, then there exists a Hamel basis B for $\mathcal{X}$ such that $A \subseteq B$.*

Proof. Suppose A is a linearly independent subset of a linear space $\mathcal{X}$. Set

$$\mathcal{I}_A \;=\; \big\{B \in \wp(\mathcal{X}):\ B \text{ is linearly independent and } A \subseteq B\big\},$$

the collection of all linearly independent subsets of $\mathcal{X}$ that include A. Recall that, as a nonempty subcollection (since $A \in \mathcal{I}_A$) of the power set $\wp(\mathcal{X})$, $\mathcal{I}_A$ is partially ordered in the inclusion ordering.

Claim 1. $\mathcal{I}_A$ has a maximal element.

Proof. If $\mathcal{X} = \{0\}$, then $A = \varnothing$ and $\mathcal{I}_A = \{A\} = \{\varnothing\} \neq \varnothing$, so that the claimed result is trivially verified. Thus suppose $\mathcal{X} \neq \{0\}$. In this case, the nonempty collection $\mathcal{I}_A$ contains a nonempty set (e.g., if $A = \varnothing$, then every nonzero singleton in $\mathcal{X}$ belongs to $\mathcal{I}_A$; if $A \neq \varnothing$, then $A \in \mathcal{I}_A$). Now consider an arbitrary chain $\mathcal{C}$ in $\mathcal{I}_A$ containing a nonempty set. Recall that $\bigcup\mathcal{C}$ denotes the union of all sets in $\mathcal{C}$. Take an arbitrary finite nonempty subset of $\bigcup\mathcal{C}$, say, a set $D \subseteq \bigcup\mathcal{C}$ such that $\#D = n$ for some $n \in \mathbb{N}$. Each element of D belongs to a set in $\mathcal{C}$ (for $D \subseteq \bigcup\mathcal{C}$). Since $\mathcal{C}$ is a chain, we can arrange the elements of D as follows. $D = \{x_i\}_{i=1}^n$ such that $x_i \in C_i \in \mathcal{C}$ for each index i, where $C_1 \subseteq \cdots \subseteq C_n$. Thus $D \subseteq C_n$. Since C_n is linearly independent (because $C_n \in \mathcal{C} \subseteq \mathcal{I}_A$), it follows that D is linearly independent. Conclusion: Every finite subset of $\bigcup\mathcal{C}$ is linearly independent. Therefore $\bigcup\mathcal{C}$ is linearly independent by Proposition 2.3. Moreover, since $A \subseteq C$ for all $C \in \mathcal{C}$ (for $\mathcal{C} \subseteq \mathcal{I}_A$), it also follows that $A \subseteq \bigcup\mathcal{C}$. Hence $\bigcup\mathcal{C} \in \mathcal{I}_A$. Since $\bigcup\mathcal{C}$ clearly is an upper bound for $\mathcal{C}$, we may conclude: Every chain in $\mathcal{I}_A$ has an upper bound in $\mathcal{I}_A$. Thus $\mathcal{I}_A$ has a maximal element by Zorn's Lemma. □

Claim 2. $B \in \mathcal{I}_A$ is maximal in $\mathcal{I}_A$ if and only if B is a Hamel basis for X.

Proof. Again, if $\mathcal{X} = \{0\}$, then $B = A = \varnothing$ is the only (and so a maximal) element in $\mathcal{I}_A$ and $\operatorname{span} B = \mathcal{X}$, so that the claimed result holds trivially. Thus suppose $\mathcal{X} \neq \{0\}$, which implies that $\mathcal{I}_A$ contains nonempty sets, and take an arbitrary B in $\mathcal{I}_A$. If $\operatorname{span} B \neq \mathcal{X}$ (i.e., if $\operatorname{span} B \subset \mathcal{X}$), then take $x \in \mathcal{X}\backslash\operatorname{span} B$ so that $B \cup \{x\} \in \mathcal{I}_A$ (i.e., $B \cup \{x\}$ is linearly independent by Proposition 2.4, and $A \subset B \cup \{x\}$ because $A \subseteq B$). Hence B is not maximal in $\mathcal{I}_A$. Therefore, if B is maximal in $\mathcal{I}_A$, then $\operatorname{span} B = \mathcal{X}$. On the other hand, if $\operatorname{span} B = \mathcal{X}$, then $B \neq \varnothing$ (for $\mathcal{X} \neq \{0\}$) and every vector in $\mathcal{X}$ is a linear combination of vectors in B. Thus $B \cup \{x\}$ is not linearly independent for every $x \in \mathcal{X}\backslash B$. This implies that there is no $B' \in \mathcal{I}_A$ such that $B \subset B'$, which means that B is maximal in $\mathcal{I}_A$. Conclusion: If $B \in \mathcal{I}_A$, then B is maximal in $\mathcal{I}_A$ if and only if $\operatorname{span} B = \mathcal{X}$. According to the definition of Hamel basis, B in $\mathcal{I}_A$ is such that $\operatorname{span} B = \mathcal{X}$ if and only if B is a Hamel basis for $\mathcal{X}$. □

Claims 1 and 2 ensure that, for each linearly independent subset A of $\mathcal{X}$, there exists a Hamel basis B for $\mathcal{X}$ such that $A \subseteq B$. □

Since the empty set is a linearly independent subset of any $\mathcal{X}$, Theorem 2.5 ensures (by setting $A = \varnothing$) that *every linear space has a Hamel basis*. In this case $\mathcal{I}_\varnothing$ is the collection of all linearly independent subsets of $\mathcal{X}$, and so Claim 2 says that *a Hamel basis for a linear space is precisely a maximal linearly independent subset of it* (i.e., a Hamel basis is a maximal element of $\mathcal{I}_\varnothing$).

The idea behind the previous theorem was that of enlarging a linearly independent subset of $\mathcal{X}$ to get a Hamel basis for $\mathcal{X}$. Another way of facing the same problem (i.e., another way to obtain a Hamel basis for linear space $\mathcal{X}$) is to begin with a set that spans $\mathcal{X}$ and then to weed out from it a linearly independent subset that also spans $\mathcal{X}$.

Theorem 2.6. *If a subset A of a linear space $\mathcal{X}$ spans $\mathcal{X}$, then there exists a Hamel basis B for $\mathcal{X}$ such that $B \subseteq A$.*

Proof. Let A be a subset of a linear space $\mathcal{X}$ such that $\operatorname{span} A = \mathcal{X}$, and consider the collection $\mathcal{I}'_A$ of all linearly independent subsets of A:

$$\mathcal{I}'_A = \big\{B \in \wp(\mathcal{X})\colon\ B \text{ is linearly independent and } B \subseteq A\big\}.$$

If $\mathcal{X} = \{0\}$, then either $A = \varnothing$ or $A = \{0\}$. In any case $\mathcal{I}'_A = \{\varnothing\}$ trivially has a maximal element. If $\mathcal{X} \neq \{0\}$, then A has a nonzero vector (for $\operatorname{span} A = \mathcal{X}$) and every nonzero singleton $\{x\} \subseteq A$ is a element of $\mathcal{I}'_A$. Thus, proceeding exactly as in the proof of Theorem 2.5 (Claim 1), we can show that $\mathcal{I}'_A$ has a maximal element. Let A_0 be a maximal element of $\mathcal{I}'_A$. If A is linearly independent, then we are done (i.e., A is itself a Hamel basis for $\mathcal{X}$ since $\operatorname{span} A = \mathcal{X}$). Thus suppose A is not linearly independent so that A_0 is a proper subset of A. Take an arbitrary $a \in A\backslash A_0$ and consider the set $A_0 \cup \{a\} \subseteq A$, which is not linearly independent because A_0 is maximal in $\mathcal{I}'_A$. Since A_0 is linearly independent, it follows that a is a linear combination of vectors in A_0. Thus $A\backslash A_0 \subseteq \operatorname{span} A_0$, and hence $A = A_0 \cup (A\backslash A_0) \subseteq \operatorname{span} A_0$. Therefore $\operatorname{span} A \subseteq \operatorname{span}(\operatorname{span} A_0) = \operatorname{span} A_0 \subseteq \operatorname{span} A$, which implies that $\operatorname{span} A_0 = \operatorname{span} A = \mathcal{X}$. Conclusion: A_0 is a Hamel basis for $\mathcal{X}$. □

Since $\mathcal{X}$ trivially spans $\mathcal{X}$, Theorem 2.6 holds for $A = \mathcal{X}$. In this case $\mathcal{I}'_{\mathcal{X}}$ is the collection of all linearly independent subsets of $\mathcal{X}$ (i.e., $\mathcal{I}'_{\mathcal{X}} = \mathcal{I}_{\varnothing}$), and the theorem statement again says that every linear space has a Hamel basis.

An ever-present purpose in mathematics is a quest for hidden invariants. The concept of Hamel basis supplies a fundamental invariant for a given linear space $\mathcal{X}$, namely, the cardinality of all Hamel bases for $\mathcal{X}$.

Theorem 2.7. *Every Hamel basis for a linear space has the same cardinality.*

Proof. If $\mathcal{X} = \{0\}$, then the result holds trivially. Suppose $\mathcal{X} \neq \{0\}$ and let B and C be arbitrary Hamel bases for $\mathcal{X}$ (so that they are nonempty and do not contain the origin). Proposition 2.3 ensures that for every nonzero vector x in $\mathcal{X}$ there is a unique finite subset of the Hamel basis C, say C_x, such that x is a linear combination of all vectors in $C_x \subseteq C$. Now take an arbitrary $c \in C$ and consider the unique representation of it as a linear combination of vectors in the Hamel basis B. Thus c is a linear combination of all vectors in $\{b\} \cup B'$ for some (nonzero) $b \in B$ and some finite subset B' of B. Hence $c = \beta b + d$, where β is a nonzero scalar and d is a vector in $\mathcal{X}$ different from c (for $\beta b \neq 0$). If $d = 0$, then $c = \beta b$ so that $C_b = \{c\}$, and hence $c \in C_b$ trivially. Suppose

$d \neq 0$. Recalling again that C also is a Hamel basis for $\mathcal{X}$, consider the unique representation of the nonzero vector d as a linear combination of vectors in C so that $\beta b = c - d \neq 0$ is a linear combination of vectors in C. Thus b is itself a linear combination of all vectors in $\{c\} \cup C'$ for some subset C' of C. Since such a representation is unique, $\{c\} \cup C' = C_b$. Therefore $c \in C_b$. Summing up: For every $c \in C$ there exists $b \in B$ such that $c \in C_b$. Hence

$$C \subseteq \bigcup_{b \in B} C_b.$$

Now we shall split the proof into two parts, one dealing with the case of finite Hamel bases, and the other with infinite Hamel bases.

Claim 0. If a subset E of a linear space $\mathcal{X}$ has exactly n elements and spans $\mathcal{X}$, then every subset of $\mathcal{X}$ with more than n elements is not linearly independent.

Proof. Assume the linear space $\mathcal{X}$ is nonzero (i.e., $\mathcal{X} \neq \{0\}$) to avoid trivialities. Take an integer $n \in \mathbb{N}$ and let $E = \{e_i\}_{i=1}^n$ be a subset of $\mathcal{X}$ with n elements such that $\operatorname{span} E = \mathcal{X}$. Take any subset of $\mathcal{X}$ with $n+1$ elements, say $D = \{d_i\}_{i=1}^{n+1}$. Suppose D is linearly independent. Now consider the set

$$S_1 = \{d_1\} \cup E$$

which clearly spans $\mathcal{X}$ (because E already does). Since $\operatorname{span} E = \mathcal{X}$, it follows that d_1 is a linear combination of vectors in E. Moreover, $d_1 \neq 0$ because D is linearly independent. Thus $d_1 = \sum_{i=1}^n \alpha_i e_i$ where at least one, say α_k, of the scalars $\{\alpha_i\}_{i=1}^n$ is nonzero. Therefore, if we delete e_k from S_1, then the set

$$S_1' = S_1 \backslash \{e_k\} = \{d_1\} \cup E \backslash \{e_k\}$$

still spans $\mathcal{X}$. That is, in forming this new set S_1' that spans $\mathcal{X}$ we have traded off one vector in D for one vector in E. Next rename the elements of S_1' by setting $s_i = e_i$ for each $i \neq k$ and $s_k = d_1$, so that $S_1' = \{s_i\}_{i=1}^n$. Since D has at least two elements, set

$$S_2 = \{d_2\} \cup S_1' = \{d_1, d_2\} \cup E \backslash \{e_k\}$$

which again spans $\mathcal{X}$ (for S_1' spans $\mathcal{X}$). Since $\operatorname{span} S_1' = \mathcal{X}$, it follows that d_2 is a linear combination of vectors in S_1', say $d_2 = \sum_{i=1}^n \beta_i s_i$ for some family of scalars $\{\beta_i\}_{i=1}^n$. Moreover, $0 \neq d_2 \neq \beta_k s_k = \beta_k d_1$ because D is linearly independent. Thus there is at least one nonzero scalar in $\{\beta_i\}_{i=1}^n$ different from β_k, say β_j. Hence, if we delete s_j from S_2 (recall: $s_j = e_j \neq e_k$), then the set

$$S_2' = S_2 \backslash \{e_j\} = \{d_1, d_2\} \cup E \backslash \{e_k, e_j\}$$

still spans $\mathcal{X}$. Continuing this way, we eventually get down to the set

$$S_n' = \{d_i\}_{i=1}^n \cup E \backslash \{e_i\}_{i=1}^n = D \backslash \{d_{n+1}\}$$

which once again spans $\mathcal{X}$. Thus d_{n+1} is a linear combination of vectors in $D\backslash\{d_{n+1}\}$, which contradicts the assumption that D is linearly independent. Conclusion: Every subset of $\mathcal{X}$ with $n+1$ elements is not linearly independent. Recalling that every subset of a linearly independent set is again linearly independent, it follows that every subset of $\mathcal{X}$ with more than n elements is not linearly independent. □

Claim 1. If B is finite, then $\#C = \#B$.

Proof. Recall that C_b is finite for every b in B. If B is finite, then $\bigcup_{b\in B} C_b$ is a finite union of finite sets. Hence any subset of it is finite. In particular, C is finite. Since C is linearly independent, it follows by Claim 0 that $\#C \le \#B$. Dually (swap the Hamel bases B and C), $\#B \le \#C$. Hence $\#C = \#B$. □

Claim 2. If B is infinite, then $\#C = \#B$.

Proof. If B is infinite, and since C_b is finite for every b in B, it follows that $\#C_b \le \#B$ for all b in B. Thus, according to Theorems 1.9 and 1.10,

$$\#\Big(\bigcup_{b\in B} C_b\Big) \;\le\; \#\big(B\times B\big) \;=\; \#B$$

because B is infinite. Hence $\#C \le \#B$ (recall: $C \subseteq \bigcup_{b\in B} C_b$ and use Problems 1.21(a) and 1.22). Moreover, Claim 1 ensures that B and C are finite together. Thus C must be infinite as B is infinite. Since C is infinite we may reverse the argument (swapping again the Hamel bases B and C) and get $\#B \le \#C$. Hence $\#C = \#B$ by the Cantor–Bernstein Theorem (Theorem 1.6). □

Claims 1 and 2 ensure that, if B and C are Hamel bases for a linear space $\mathcal{X}$, then B and C have the same cardinal number. □

Such an invariant (i.e., the cardinality of any Hamel basis) is called the *dimension* (or the *linear dimension*) of the linear space $\mathcal{X}$, denoted by $\dim \mathcal{X}$. Thus $\dim \mathcal{X} = \#B$ for any Hamel basis B for $\mathcal{X}$. If the dimension of $\mathcal{X}$ is finite (equivalently, if any Hamel basis for $\mathcal{X}$ is a finite set), then we say that $\mathcal{X}$ is a *finite-dimensional linear space*. Otherwise (i.e., if any Hamel basis for $\mathcal{X}$ is an infinite set) we say that $\mathcal{X}$ is an *infinite-dimensional linear space*.

Example 2.I. The *Kronecker delta* (or *Kronecker function*) is the mapping in $2^{\mathbb{Z}\times\mathbb{Z}}$ (i.e., the function from $\mathbb{Z}\times\mathbb{Z}$ to $\{0,1\}$) defined by

$$\delta_{ij} \;=\; \begin{cases} 1, & i = j, \\ 0, & i \ne j, \end{cases}$$

for all integers i, j. Now consider the linear space $\mathbb{F}^n$ (for an arbitrary positive integer n, over an arbitrary field $\mathbb{F}$ — see Example 2.D). The subset $B = \{e_i\}_{i=1}^n$ of $\mathbb{F}^n$ consisting of the n-tuples $e_i = (\delta_{i1}, \ldots, \delta_{in})$, with 1 at the ith position and zeros elsewhere, constitute a Hamel basis for $\mathbb{F}^n$. This is called

the *canonical basis* (or the *natural basis*) for $\mathbb{F}^n$. Thus $\dim \mathbb{F}^n = n$. As we shall see later, $\mathbb{F}^n$ in fact is a prototype for every finite-dimensional linear space (of dimension n) over a field $\mathbb{F}$.

Example 2.J. Let $\mathbb{F}^{\mathbb{N}}$ be the linear space (over a field $\mathbb{F}$) of all scalar-valued sequences (see Example 2.E), and let $\mathcal{X}$ be the subset of $\mathbb{F}^{\mathbb{N}}$ defined as follows. $x = \{\xi_k\}_{k\in\mathbb{N}}$ belongs to $\mathcal{X}$ if and only if $\xi_k = 0$ except for some finite set of indices k in $\mathbb{N}$. That is, $\mathcal{X}$ is the set consisting of *all $\mathbb{F}$-valued sequences with a finite number of nonzero entries*, which clearly is a linear manifold of $\mathbb{F}^{\mathbb{N}}$, and hence a linear space itself over $\mathbb{F}$. For each integer $j \in \mathbb{N}$ let e_j be an $\mathbb{F}$-valued sequence with just one nonzero entry (equal to 1) at the jth position; that is, $e_j = \{\delta_{jk}\}_{k\in\mathbb{N}} \in \mathcal{X}$ for every $j \in \mathbb{N}$. Now set $B = \{e_j\}_{j\in\mathbb{N}} \subset \mathcal{X}$. It is readily verified that B is linearly independent and that $\operatorname{span} B = \mathcal{X}$ (every vector in $\mathcal{X}$ is a linear combination of vectors in B). Thus B is a Hamel basis for $\mathcal{X}$. Since B is countably infinite, $\mathcal{X}$ is an infinite-dimensional linear space with $\dim \mathcal{X} = \aleph_0$. Therefore (see Problem 2.6(b)), $\mathbb{F}^{\mathbb{N}}$ is an infinite-dimensional linear space. Note that B is not a Hamel basis for $\mathbb{F}^{\mathbb{N}}$ (reason: $\operatorname{span} B = \mathcal{X}$ and $\mathcal{X}$ is properly included in $\mathbb{F}^{\mathbb{N}}$). The next example shows that $\aleph_0 < \dim \mathbb{F}^{\mathbb{N}}$ whenever $\mathbb{F} = \mathbb{Q}$, $\mathbb{F} = \mathbb{R}$, or $\mathbb{F} = \mathbb{C}$.

Example 2.K. Let $\mathbb{C}^{\mathbb{N}}$ be the complex linear space of all complex-valued sequences. For each real number $t \in (0,1)$ consider the real sequence $x_t = \{t^{k-1}\}_{k\in\mathbb{N}} = \{t^k\}_{k\in\mathbb{N}_0} = (1, t, t^2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ whose entries are the nonnegative powers of t. Set $A = \{x_t\}_{t\in(0,1)} \subseteq \mathbb{C}^{\mathbb{N}}$. We claim that A is linearly independent. A bit of elementary real analysis (rather than pure algebra) supplies a very simple proof as follows. Suppose A is not linearly independent. Then there exists $s \in (0,1)$ such that x_s is a linear combination of vectors in $A\backslash\{x_s\}$. That is, $x_s = \sum_{i=1}^n \alpha_i x_{t_i}$ for some $n \in \mathbb{N}$, where $\{\alpha_i\}_{i=1}^n$ is a family of nonzero complex numbers and $\{x_{t_i}\}_{i=1}^n$ is a (finite) subset of A such that $x_{t_i} \neq x_s$ for every $i = 1, \ldots, n$. Hence $n > 1$ (reason: if $n = 1$, then $x_s = \alpha_1 x_{t_1}$ so that $s^k = \alpha_1 t_1^k$ for every $k \in \mathbb{N}_0$, which implies that $x_s = x_{t_1}$). As the set $\{t_i\}_{i=1}^n$ consists of distinct points from $(0,1)$, suppose it is decreasingly ordered (reorder it if necessary) so that $t_i < t_1$ for each $i = 2, \ldots, n$. Since $s^k = \sum_{i=1}^n \alpha_i t_i^k$, we get $(s/t_1)^k = \alpha_1 + \sum_{i=2}^n \alpha_i (t_i/t_1)^k$ for every $k \in \mathbb{N}_0$. But $\lim_k \sum_{i=2}^n \alpha_i (t_i/t_1)^k = 0$, because each t_i/t_1 lies in $(0,1)$, and hence $\lim_k (s/t_1)^k = \alpha_1$. Thus $\alpha_1 = 0$ (recall: $x_s \neq x_{t_1}$ so that $s \neq t_1$) which is a contradiction. Conclusion: A is linearly independent. Therefore, by Theorem 2.5 there is a Hamel basis B for $\mathbb{C}^{\mathbb{N}}$ including A. Since $A \subseteq B$ and $\#A = \#(0,1) = 2^{\aleph_0}$, it follows that $2^{\aleph_0} \leq \#B$. However, $\#\mathbb{C} = \#\mathbb{R} = 2^{\aleph_0} \leq \#B = \dim \mathbb{C}^{\mathbb{N}}$, and so $\#\mathbb{C}^{\mathbb{N}} = \dim \mathbb{C}^{\mathbb{N}}$ (Problem 2.8). Conclusion: $\mathbb{C}^{\mathbb{N}}$ is an infinite-dimensional linear space such that

$$2^{\aleph_0} \leq \dim \mathbb{C}^{\mathbb{N}} = \#\mathbb{C}^{\mathbb{N}}.$$

Note that the whole argument does apply for $\mathbb{C}$ replaced by $\mathbb{R}$, so that

$$2^{\aleph_0} \leq \dim \mathbb{R}^{\mathbb{N}} = \#\mathbb{R}^{\mathbb{N}};$$

but it does not apply to the rational field $\mathbb{Q}$ (the interval $(0,1)$ is not a subset of $\mathbb{Q}$, and hence the set A is not included in $\mathbb{Q}^{\mathbb{N}}$). However, the final conclusion does hold for the linear space $\mathbb{Q}^{\mathbb{N}}$. Indeed, if $\mathbb{F}$ is an arbitrary infinite field, then $2^{\aleph_0} = \#2^{\mathbb{N}} \leq \#\mathbb{F}^{\mathbb{N}} = \max\{\#\mathbb{F}, \dim \mathbb{F}^{\mathbb{N}}\}$ according to Problems 1.24 and 2.8. Therefore, since $\#\mathbb{Q} = \aleph_0 < 2^{\aleph_0}$ (Problem 1.25(c)), it follows that

$$2^{\aleph_0} \leq \dim \mathbb{Q}^{\mathbb{N}} = \#\mathbb{Q}^{\mathbb{N}}.$$

2.5 Linear Transformations

A mapping $L\colon \mathcal{X} \to \mathcal{Y}$ of a linear space $\mathcal{X}$ over a field $\mathbb{F}$ into a linear space $\mathcal{Y}$ over the same field $\mathbb{F}$ is *homogeneous* if

$$L(\alpha x) = \alpha L x$$

for every vector $x \in \mathcal{X}$ and every scalar $\alpha \in \mathbb{F}$. The scalar multiplication on the left-hand side is an operation on $\mathcal{X}$ and that on the right-hand side is an operation on $\mathcal{Y}$ (so that the linear spaces $\mathcal{X}$ and $\mathcal{Y}$ must indeed be defined over the same field $\mathbb{F}$). L is *additive* if

$$L(x_1 + x_2) = L(x_1) + L(x_2)$$

for all vectors x_1, x_2 in $\mathcal{X}$. Again, the vector addition on the left-hand side is an operation on $\mathcal{X}$ while the one on the right-hand side is an operation on $\mathcal{Y}$. If $\mathcal{X}$ and $\mathcal{Y}$ are linear spaces over the same scalar field, and if L is a homogeneous and additive mapping of $\mathcal{X}$ into $\mathcal{Y}$, then L is a *linear transformation*: a linear transformation is a homogeneous and additive mapping between linear spaces over the same scalar field. When we say that $L\colon \mathcal{X} \to \mathcal{Y}$ is a linear transformation, it is implicitly assumed that $\mathcal{X}$ and $\mathcal{Y}$ are linear spaces over the same field $\mathbb{F}$. If $\mathcal{X} = \mathcal{Y}$ and $L\colon \mathcal{X} \to \mathcal{X}$ is a linear transformation, then we refer to L as a *linear transformation on* $\mathcal{X}$. Trivial example: The identity $I\colon \mathcal{X} \to \mathcal{X}$ (such that $I(x) = x$ for every $x \in \mathcal{X}$) is a linear transformation on $\mathcal{X}$. Recall that a field $\mathbb{F}$ can be made into a linear space over $\mathbb{F}$ itself (see Example 2.D). If $\mathcal{X}$ is a linear space over $\mathbb{F}$, then a linear transformation $f\colon \mathcal{X} \to \mathbb{F}$ is called a *linear functional*: a linear functional is a scalar-valued linear transformation (i.e., a linear transformation of a linear space $\mathcal{X}$ into its scalar field).

If $y \in \mathcal{Y}$ is the value of a linear transformation $L\colon \mathcal{X} \to \mathcal{Y}$ at $x \in \mathcal{X}$, then we shall often write $y = Lx$ (instead of $y = L(x)$). Since $\mathcal{Y}$ is a linear space, it has an origin. The *null space* (or *kernel*) of a linear transformation $L\colon \mathcal{X} \to \mathcal{Y}$ is the subset

$$\mathcal{N}(L) = \big\{x \in \mathcal{X}\colon\ Lx = 0\big\} = L^{-1}(\{0\})$$

of $\mathcal{X}$ consisting of all vectors in $\mathcal{X}$ mapped into the origin of $\mathcal{Y}$ by L. Since $\mathcal{X}$ also is a linear space, it has an origin too. The origin of $\mathcal{X}$ is always in $\mathcal{N}(L)$

(i.e., $L0 = 0$ for every linear transformation L). The *null transformation* (denoted by O) is the mapping $O\colon \mathcal{X} \to \mathcal{Y}$ such that $Ox = 0$ for every $x \in \mathcal{X}$, which certainly is a linear transformation. In fact, if $L\colon \mathcal{X} \to \mathcal{Y}$ is a linear transformation, then $L = O$ if and only if $\mathcal{N}(L) = \mathcal{X}$. Equivalently, $L = O$ if and only if $\mathcal{R}(L) = \{0\}$. The null space $\mathcal{N}(L) = L^{-1}(\{0\})$ of any linear transformation $L\colon \mathcal{X} \to \mathcal{Y}$ is a linear manifold of $\mathcal{X}$, and the range $\mathcal{R}(L) = L(\mathcal{X})$ of L is a linear manifold of $\mathcal{Y}$ (Problem 2.10(a)). These are indeed particular cases of Problem 2.11: *The linear image of a linear manifold is a linear manifold*, and *the inverse image of a linear manifold under a linear transformation is again a linear manifold*.

The theorem below supplies an elegant and useful, although very simple, necessary and sufficient condition that a linear transformation be injective.

Theorem 2.8. *A linear transformation L is injective if and only if $\mathcal{N}(L) = \{0\}$.*

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces over the same scalar field and consider a linear transformation $L\colon \mathcal{X} \to \mathcal{Y}$. If L is injective, then $L^{-1}(L(\{0\})) = \{0\}$ (see Problem 1.3(d)). But $L(\{0\}) = \{0\}$ (since $L0 = 0$) so that $L^{-1}(\{0\}) = \{0\}$, which means $\mathcal{N}(L) = \{0\}$. On the other hand, suppose $\mathcal{N}(L) = \{0\}$. Take x_1 and x_2 arbitrary in $\mathcal{X}$, and note that $Lx_1 - Lx_2 = L(x_1 - x_2)$ since L is linear. Thus, if $Lx_1 = Lx_2$, then $L(x_1 - x_2) = 0$ and hence $x_1 = x_2$ (i.e., $x_1 - x_2 = 0$ because $\mathcal{N}(L) = \{0\}$). Therefore L is injective. □

The collection $\mathcal{Y}^S$ of all mappings of a set S into a linear space $\mathcal{Y}$ over a field $\mathbb{F}$ is itself a linear space over $\mathbb{F}$ (see Example 2.F). Now suppose $\mathcal{X}$ is a linear space (over the same field $\mathbb{F}$) and let $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$ denote the collection of all linear transformations of $\mathcal{X}$ into $\mathcal{Y}$. Since $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is a linear manifold of $\mathcal{Y}^{\mathcal{X}}$, it follows that $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is a linear space over the same field $\mathbb{F}$ (see Problem 2.13). Set $\mathcal{L}[\mathcal{X}] = \mathcal{L}[\mathcal{X}, \mathcal{X}]$ for short so that $\mathcal{L}[\mathcal{X}] \subset \mathcal{X}^{\mathcal{X}}$ is the linear space of all linear transformations on $\mathcal{X}$. The linear space $\mathcal{L}[\mathcal{X}, \mathbb{F}]$ of all linear functionals defined on a linear space $\mathcal{X}$, which is a linear manifold of the linear space $\mathbb{F}^{\mathcal{X}}$ (see Example 2.E), is called the *algebraic dual* (or *algebraic conjugate*) of $\mathcal{X}$ and is denoted by $\mathcal{X}'$. (Dual spaces will be considered in Chapter 4.)

Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces over the same scalar field. Let $L|_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{Y}$ be the restriction of a linear transformation $L\colon \mathcal{X} \to \mathcal{Y}$ to a linear manifold $\mathcal{M}$ of $\mathcal{X}$. Since $\mathcal{M}$ is a linear space, it is easy to show that $L|_{\mathcal{M}}$ is a linear transformation: *The restriction of a linear transformation to a linear manifold is again a linear transformation* (Problem 2.14). The next result ensures the converse: If $\mathcal{M}$ is a linear manifold of $\mathcal{X}$ and $L \in \mathcal{L}[\mathcal{M}, \mathcal{Y}]$, then there exists $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ such that $L = T|_{\mathcal{M}}$. T is called a *linear extension* of L over $\mathcal{X}$.

Theorem 2.9. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces over the same field $\mathbb{F}$, and let $\mathcal{M}$ be a linear manifold of $\mathcal{X}$. If $L\colon \mathcal{M} \to \mathcal{Y}$ is a linear transformation, then there exists a linear extension $T\colon \mathcal{X} \to \mathcal{Y}$ of L defined on the whole space $\mathcal{X}$.*

Proof. Set

$$\mathcal{K} \;=\; \big\{K \in \mathcal{L}[\mathcal{N},\mathcal{Y}]\colon\ \mathcal{N} \in \mathcal{L}at(\mathcal{X}),\ \mathcal{M} \subseteq \mathcal{N} \text{ and } L = K|_{\mathcal{M}}\big\},$$

the collection of all linear transformations from linear manifolds of $\mathcal{X}$ to $\mathcal{Y}$ which are extensions of L. Note that $\mathcal{K}$ is nonempty (at least L is there). Moreover, as a subcollection of $\mathcal{F} = \bigcup_{A\in\wp(\mathcal{X})}\mathcal{Y}^A$, $\mathcal{K}$ is partially ordered in the extension ordering (see Problem 1.17). Problem 1.17 also tells us that every chain $\{K_\gamma\}$ in $\mathcal{K}$ has a supremum $\bigvee_\gamma K_\gamma$ in $\mathcal{F}$ with domain $\mathcal{D}\big(\bigvee_\gamma K_\gamma\big) = \bigcup_\gamma \mathcal{D}(K_\gamma)$ and range $\mathcal{R}\big(\bigvee_\gamma K_\gamma\big) = \bigcup_\gamma \mathcal{R}(K_\gamma)$. Since $\mathcal{D}(K_\gamma) \in \mathcal{L}at(\mathcal{X})$ (each K_γ is a linear transformation defined on a linear manifold of $\mathcal{X}$), and since $\mathcal{L}at(\mathcal{X})$ is a complete lattice, it follows that $\mathcal{D}\big(\bigvee_\gamma K_\gamma\big)$ is a linear manifold of $\mathcal{X}$ (i.e., $\bigcup_\gamma \mathcal{D}(K_\gamma) \in \mathcal{L}at(\mathcal{X})$). Similarly, $\mathcal{R}\big(\bigvee_\gamma K_\gamma\big)$ is a linear manifold of $\mathcal{Y}$.

Claim. The supremum $\bigvee_\gamma K_\gamma$ lies in $\mathcal{K}$.

Proof. Take u and v arbitrary in $\mathcal{D}\big(\bigvee_\gamma K_\gamma\big)$, so that $u \in \mathcal{D}(K_\lambda)$ for some $K_\lambda \in \{K_\gamma\}$ and $v \in \mathcal{D}(K_\mu)$ for some $K_\mu \in \{K_\gamma\}$. Since $\{K_\gamma\}$ is a chain, it follows that $K_\lambda \le K_\mu$ (or vice versa), so that $\mathcal{D}(K_\lambda) \subseteq \mathcal{D}(K_\mu)$. Thus $\alpha u + \beta v \in D(K_\mu)$ and hence $K_\mu(\alpha u + \beta v) = \alpha K_\mu u + \beta K_\mu v$ for every $\alpha, \beta \in \mathbb{F}$ (recall: each K_γ is linear). However $\big(\bigvee_\gamma K_\gamma\big)|_{\mathcal{D}(K_\mu)} = K_\mu$, which implies that $\big(\bigvee_\gamma K_\gamma\big)(\alpha u + \beta v) = \alpha\big(\bigvee_\gamma K_\gamma\big)u + \beta\big(\bigvee_\gamma K_\gamma\big)v$. That is, $\bigvee_\gamma K_\gamma\colon \mathcal{D}\big(\bigvee_\gamma K_\gamma\big) \to \mathcal{Y}$ is linear. Moreover, since each K_γ is such that $K_\gamma|_{\mathcal{M}} = L$, and since $\{K_\gamma\}$ is a chain, it follows that $\big(\bigvee_\gamma K_\gamma\big)|_{\mathcal{M}} = L$. Conclusion: $\bigvee_\gamma K_\gamma \in \mathcal{K}$. □

Therefore, every chain in $\mathcal{K}$ has a supremum (and so an upper bound) in $\mathcal{K}$. Thus, according to Zorn's Lemma, $\mathcal{K}$ contains a maximal element, say $K_0\colon \mathcal{N}_0 \to \mathcal{Y}$. We shall show that $\mathcal{N}_0 = \mathcal{X}$, and hence K_0 is a linear extension of L over $\mathcal{X}$. The proof goes by contradiction. Suppose $\mathcal{N}_0 \ne \mathcal{X}$. Take $x_1 \in \mathcal{X}\backslash\mathcal{N}_0$ (so that $x_1 \ne 0$ because $\mathcal{N}_0$ is a linear manifold of $\mathcal{X}$) and consider the sum of $\mathcal{N}_0$ and the one-dimensional linear manifold of $\mathcal{X}$ spanned by $\{x_1\}$,

$$\mathcal{N}_1 \;=\; \mathcal{N}_0 + \operatorname{span}\{x_1\},$$

which is a linear manifold of $\mathcal{X}$ properly including $\mathcal{M}$ (because $\mathcal{M} \subseteq \mathcal{N}_0 \subset \mathcal{N}_1$). Since $\mathcal{N}_0 \cap \operatorname{span}\{x_1\} = \{0\}$, it follows that every x in $\mathcal{N}_1$ has a unique representation as a sum of a vector in $\mathcal{N}_0$ and a vector in $\operatorname{span}\{x_1\}$. That is, for each $x \in \mathcal{N}_1$ there is a unique pair (x_0, α) in $\mathcal{N}_0 \times \mathbb{F}$ such that $x = x_0 + \alpha x_1$. (Indeed, if $x = x_0 + \alpha x_1 = x_0' + \alpha' x_1$, then $x_0 - x_0' = (\alpha' - \alpha)x_1$ lies in $\mathcal{N}_0 \cap \operatorname{span}\{x_1\} = \{0\}$ so that $x_0' = x_0$ and $\alpha' = \alpha$ — recall: $x_1 \ne 0$.) Take any y in $\mathcal{Y}$ (e.g., $y = 0$) and consider the mapping $K_1\colon \mathcal{N}_1 \to \mathcal{Y}$ defined by

$$K_1 x \;=\; K_0 x_0 + \alpha y$$

for every $x \in \mathcal{N}_1$. Observe that K_1 is linear (it inherits the linearity of K_0) and $K_0 = K_1|_{\mathcal{N}_0}$ (so that $K_0 \le K_1$). Since $\mathcal{M} \subseteq \mathcal{N}_0 \subset \mathcal{N}_1$, it follows that

$L = K_0|_{\mathcal{M}} = K_1|_{\mathcal{M}}$. Thus $K_1 \in \mathcal{K}$, which contradicts the fact that K_0 is maximal in $\mathcal{K}$ (for $K_0 \neq K_1$). Therefore, $\mathcal{N}_0 = \mathcal{X}$. □

Let $\mathcal{X}$ and $\mathcal{Y}$ be nonzero linear spaces over the same field. Take $x \neq 0$ in $\mathcal{X}$ and $y \neq 0$ in $\mathcal{Y}$, set $\mathcal{M} = \text{span}\{x\}$ in $\mathcal{L}at(\mathcal{X})$, and let $L\colon \mathcal{M} \to \mathcal{Y}$ be defined by $Lu = \alpha y$ for every $u = \alpha x \in \mathcal{M}$. Clearly, L is linear and $L \neq O$. Thus Theorem 2.9 ensures that, *if $\mathcal{X}$ and $\mathcal{Y}$ are nonzero linear spaces over the same field, then there exist many $T \neq O$ in $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$* (at least as many as one-dimensional linear manifolds in $\mathcal{L}at(\mathcal{X})$).

2.6 Isomorphisms

Two exemplars of a mathematical structure are indistinguishable, in the context of the theory in which that structure is embedded, if there exists a one-to-one correspondence between them that preserves such a structure. This is a central concept in mathematics. From the point of view of the linear space theory, two linear spaces are essentially the same if there exists a one-to-one correspondence between them that preserves all the linear relations — they may differ in the set-theoretic nature of their elements but, as far as the linear space (algebraic) structure is concerned, they are indistinguishable. In other words, two linear spaces $\mathcal{X}$ and $\mathcal{Y}$ over the same scalar field are regarded as essentially the same linear space if there exists a one-to-one correspondence between them that preserves vector addition and scalar multiplication; that is, if there exists at least one invertible linear transformation from $\mathcal{X}$ to $\mathcal{Y}$ whose inverse from $\mathcal{Y}$ to $\mathcal{X}$ also is linear. The theorem below shows that the inverse of an invertible linear transformation is always linear.

Theorem 2.10. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces over the same field $\mathbb{F}$. If $L\colon \mathcal{X} \to \mathcal{Y}$ is an invertible linear transformation, then its inverse $L^{-1}\colon \mathcal{Y} \to \mathcal{X}$ is a linear transformation.*

Proof. Suppose $L\colon \mathcal{X} \to \mathcal{Y}$ is an invertible linear transformation. Recall that a function is invertible if it is injective and surjective. Take y_1 and y_2 arbitrary in $\mathcal{Y}$ so that there exist x_1 and x_2 in $\mathcal{X}$ such that $y_1 = Lx_1$ and $y_2 = Lx_2$ (because $\mathcal{Y} = \mathcal{R}(L)$ — i.e., L is surjective). Since L is injective (i.e., $L^{-1}L$ is the identity on $\mathcal{X}$ — see Problems 1.5 and 1.7) and additive, it follows that

$$\begin{aligned} L^{-1}(y_1 + y_2) &= L^{-1}(Lx_1 + Lx_2) = L^{-1}L(x_1 + x_2) = x_1 + x_2 \\ &= L^{-1}Lx_1 + L^{-1}Lx_2 = L^{-1}y_1 + L^{-1}y_2, \end{aligned}$$

and hence L^{-1} is additive. Similarly, since L is injective and homogeneous,

$$L^{-1}(\alpha y) = L^{-1}(\alpha Lx) = L^{-1}L(\alpha x) = \alpha x = \alpha L^{-1}Lx = \alpha L^{-1}y$$

for every $y \in \mathcal{Y} = \mathcal{R}(L)$ and every $\alpha \in \mathbb{F}$, which implies that L^{-1} is homogeneous. Thus L^{-1} is a linear transformation. □

An *isomorphism* between linear spaces (over the same scalar field) is an injective and surjective linear transformation. Equivalently, an *isomorphism* between linear spaces is an invertible linear transformation. Two linear spaces $\mathcal{X}$ and $\mathcal{Y}$ over the same field $\mathbb{F}$ are *isomorphic* if there exists an isomorphism (i.e., a linear one-to-one correspondence) of $\mathcal{X}$ onto $\mathcal{Y}$. Thus, according to Theorem 2.8, a linear transformation $L: \mathcal{X} \to \mathcal{Y}$ of a linear space $\mathcal{X}$ into a linear space $\mathcal{Y}$ is an isomorphism if and only if $\mathcal{N}(L) = \{0\}$ and $\mathcal{R}(L) = \mathcal{Y}$. In particular, if $\mathcal{N}(L) = \{0\}$, then $\mathcal{X}$ and the range of L ($\mathcal{R}(L) = L(\mathcal{X})$) are isomorphic linear spaces.

We noticed in Example 2.I that $\mathbb{F}^n$ is a "prototype" for every n-dimensional linear space over $\mathbb{F}$. What this really means is that every n-dimensional linear space over a field $\mathbb{F}$ is *isomorphic* to $\mathbb{F}^n$, and hence two n-dimensional linear spaces over the same scalar field are isomorphic. In fact, such an isomorphism between linear spaces with the same dimension holds in general, either for finite-dimensional or for infinite-dimensional linear spaces. We shall prove this below (Theorem 2.12), but first we need the following auxiliary result.

Proposition 2.11. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces over the same field, and let B be a Hamel basis for $\mathcal{X}$. For each mapping $F: B \to \mathcal{Y}$ there exists a unique linear transformation $T: \mathcal{X} \to \mathcal{Y}$ such that $T|_B = F$.*

Proof. If $B = \{x_\gamma\}_{\gamma \in \Gamma}$ is a Hamel basis for $\mathcal{X}$, indexed by an index set Γ (recall: any set can be thought of as an indexed set), then every vector x in $\mathcal{X}$ has a unique expansion on B, viz.,

$$x = \sum_{\gamma \in \Gamma} \alpha_\gamma x_\gamma,$$

where $\{\alpha_\gamma\}_{\gamma \in \Gamma}$ is a similarly indexed family of scalars with $\alpha_\gamma = 0$ for all but a finite set of indices γ (the coordinates of x with respect to the basis B). Set

$$Tx = \sum_{\gamma \in \Gamma} \alpha_\gamma F(x_\gamma)$$

for every $x \in \mathcal{X}$. This defines a mapping $T: \mathcal{X} \to \mathcal{Y}$ of $\mathcal{X}$ into $\mathcal{Y}$ which is homogeneous, additive, and equals F when restricted to B. That is, T is a linear transformation such that $T|_B = F$. Moreover, if $L: \mathcal{X} \to \mathcal{Y}$ is a linear transformation of $\mathcal{X}$ into $\mathcal{Y}$ such that $L|_B = F$, then $L = T$. Indeed, for every $x \in \mathcal{X}$,

$$Lx = L\Big(\sum_{\gamma \in \Gamma} \alpha_\gamma x_\gamma\Big) = \sum_{\gamma \in \Gamma} \alpha_\gamma F(x_\gamma) = T\Big(\sum_{\gamma \in \Gamma} \alpha_\gamma x_\gamma\Big) = Tx. \qquad \square$$

Theorem 2.12. *Two linear spaces $\mathcal{X}$ and $\mathcal{Y}$ over the same scalar field are isomorphic if and only if* $\dim \mathcal{X} = \dim \mathcal{Y}$.

Proof. (a) Let $L: \mathcal{X} \to \mathcal{Y}$ be an isomorphism of $\mathcal{X}$ onto $\mathcal{Y}$, and let B_X be a Hamel basis for $\mathcal{X}$. Set $B_Y = L(B_X)$, a subset of $\mathcal{Y}$.

Claim 1. B_Y is linearly independent.

Proof. Recall that L is an injective and surjective linear transformation. If B_Y is not linearly independent, then there exists $y \in B_Y$, which is a linear combination of vectors in $B_Y \backslash \{y\}$, say $y = \sum_{i=1}^{n} \alpha_i y_i$ where each y_i is a vector in $B_Y \backslash \{y\}$. Thus $x = L^{-1}y$ in $B_X = L^{-1}(B_Y)$ is a linear combination of vectors in $B_X \backslash \{x\}$. (Indeed, $x = \sum_{i=1}^{n} \alpha_i x_i$ where each $x_i = L^{-1}y_i$ is a vector in $B_X = L^{-1}(B_Y)$ different from $x = L^{-1}y$ — recall: each y_i is a vector in B_Y different from y, and L is injective.) But this contradicts the fact that B_X is linearly independent. Conclusion: B_Y is linearly independent. □

Claim 2. B_Y spans $\mathcal{Y}$.

Proof. Take $y \in \mathcal{Y}$ arbitrary so that $y = Lx$ for some $x \in \mathcal{X}$ (because L is surjective). Since $\operatorname{span} B_X = \mathcal{X}$, it follows that x is a linear combination of vectors in B_X. Hence $y = Lx$ is a linear combination of vectors in $B_Y = L(B_X)$ (since L is linear) so that $\operatorname{span} B_Y = \mathcal{Y}$. □

Therefore, B_Y is a Hamel basis for $\mathcal{Y}$. Moreover, $\# B_Y = \# B_X$ because L sets a one-to-one correspondence between B_X and B_Y. (In fact, the restriction $L|_{B_X}: B_X \to B_Y$ is injective and surjective, since L is injective and $B_Y = L(B_X)$ by definition.) Thus $\dim \mathcal{X} = \dim \mathcal{Y}$.

(b) Let B_X and B_Y be Hamel bases for $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $\dim \mathcal{X} = \dim \mathcal{Y}$, then $\# B_Y = \# B_X$, which means that there exists a one-to-one mapping $F: B_X \to B_Y$ of B_X onto B_Y. Let $T: \mathcal{X} \to \mathcal{Y}$ be the unique linear transformation such that $T|_{B_X} = F$ (see Proposition 2.11), and hence $T(B_X) = F(B_X) = B_Y$.

Claim 3. T is injective.

Proof. If $\mathcal{X} = \{0\}$, then the result holds trivially. Thus suppose $\mathcal{X} \neq \{0\}$. Take any nonzero vector x in $\mathcal{X}$ and consider its (unique) representation as a linear combination of vectors in B_X. Therefore, Tx has a representation as a linear combination of vectors in $B_Y = T(B_X)$ because T is linear. Since B_Y is linearly independent, it follows that $Tx \neq 0$. That is, $\mathcal{N}(T) = \{0\}$ which means, by Theorem 2.8, that T is injective. □

Claim 4. T is surjective.

Proof. Take an arbitrary vector $y \in \mathcal{Y}$ and consider its expansion on B_Y, say $y = \sum_{i=1}^{n} \alpha_i y_i$ with each y_i in B_Y. Thus $y = \sum_{i=1}^{n} \alpha_i T(x_i)$ with each x_i in B_X because $B_Y = T(B_X)$. But T is linear so that $y = T\big(\sum_{i=1}^{n} \alpha_i x_i\big)$, where $\sum_{i=1}^{n} \alpha_i x_i$ is a vector in $\mathcal{X}$ (since $\mathcal{X}$ is a linear space). Hence $y \in \mathcal{R}(T)$. □

Therefore, $T: \mathcal{X} \to \mathcal{Y}$ is an isomorphism of $\mathcal{X}$ onto $\mathcal{Y}$. □

Example 2.L. Let $\mathcal{X}$ and $\mathcal{Y}$ be *finite-dimensional* linear spaces over the same field $\mathbb{F}$, with $\dim \mathcal{X} = n$ and $\dim \mathcal{Y} = m$. Let $B_X = \{x_j\}_{j=1}^{n}$ and $B_Y = \{y_i\}_{i=1}^{m}$ be Hamel bases for $\mathcal{X}$ and $\mathcal{Y}$, respectively. Take an arbitrary vector x in $\mathcal{X}$ and consider its unique expansion on B_X,

$$x = \sum_{j=1}^{n} \xi_j x_j,$$

where the family of scalars $\{\xi_j\}_{j=1}^{n}$ consists of the coordinates of x with respect to B_X. Now let $A\colon \mathcal{X} \to \mathcal{Y}$ be any linear transformation so that

$$Ax = \sum_{j=1}^{n} \xi_j Ax_j,$$

where Ax_j is a vector in $\mathcal{Y}$ for each j. Consider its unique expansion on B_Y,

$$Ax_j = \sum_{i=1}^{m} \alpha_{ij} y_i,$$

where $\{\alpha_{ij}\}_{i=1}^{m}$ is a family of scalars (the coordinates of each Ax_j with respect to B_Y). Set $y = Ax$ in $\mathcal{Y}$ and consider the unique expansion of y on B_Y,

$$y = \sum_{i=1}^{m} \upsilon_i y_i.$$

Again, $\{\upsilon_i\}_{i=1}^{m}$ is a family of scalars consisting of the coordinates of y with respect to B_Y. Thus the identity $y = Ax$ can be written as

$$\sum_{i=1}^{m} \upsilon_i y_i = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \xi_j \alpha_{ij} \Big) y_i.$$

Since the expansion of y on B_Y is unique, it follows that

$$\upsilon_i = \sum_{j=1}^{n} \alpha_{ij} \xi_j$$

for every $i = 1, \dots, m$. This gives an expression for each coordinate of Ax as a function of the coordinates of x. In terms of standard matrix notation, and according to the ordinary matrix operations, the matrix equation

$$\begin{pmatrix} \upsilon_1 \\ \vdots \\ \upsilon_m \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

represents the identity $y = Ax$ (the vector y is the value of the linear transformation A at the point x), and the $m \times n$ array of scalars

$$[A] = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix}$$

is the *matrix* that represents the linear transformation $A\colon \mathcal{X} \to \mathcal{Y}$ with respect to the bases B_X and B_Y. The matrix $[A]$ of a linear transformation A depends on the bases B_X and B_Y. If the bases are changed, then the matrix that represents the linear transformation may change as well. Different matrices representing the same linear transformation are simply different representations of it with respect to different bases. However, for fixed bases B_X and B_Y the representation $[A]$ of A is unique. Uniqueness is not all. It is easy to show that

(a) the set $\mathbb{F}_{m\times n}$ of all $m\times n$ matrices with entries in $\mathbb{F}$ is a linear space over $\mathbb{F}$ when equipped with the ordinary (entrywise) operations of matrix addition and scalar multiplication.

Moreover, for fixed bases B_X and B_Y,

(b) $\mathbb{F}_{m\times n}$ is isomorphic to $\mathcal{L}[\mathcal{X},\mathcal{Y}]$.

Indeed, if we fix the bases B_X and B_Y, then the relation between $\mathcal{L}[\mathcal{X},\mathcal{Y}]$ and $\mathbb{F}_{m\times n}$ defined by "$[A]$ represents A with respect to B_X and B_Y" in fact is a function from $\mathcal{L}[\mathcal{X},\mathcal{Y}]$ to $\mathbb{F}_{m\times n}$. It is readily verified that such a function, say $\Phi\colon \mathcal{L}[\mathcal{X},\mathcal{Y}] \to \mathbb{F}_{m\times n}$, is homogeneous, additive, injective, and surjective. In other words, Φ is an isomorphism. For this reason we may and shall identify a linear transformation $A \in \mathcal{L}[\mathbb{F}^n,\mathbb{F}^m]$ with its matrix $[A] \in \mathbb{F}_{m\times n}$ relative to the *canonical* bases for $\mathbb{F}^n$ and $\mathbb{F}^m$ (which were introduced in Example 2.I).

Example 2.M. Let $\mathbb{F}$ denote either the real field or the complex field. For every nonnegative integer n let $\mathcal{P}_n[0,1]$ be the collection of all polynomials in the variable $t \in [0,1]$ with coefficients in $\mathbb{F}$ of degree not greater than n:

$$\mathcal{P}_n[0,1] \;=\; \big\{p \in \mathbb{F}^{[0,1]}\colon\, p(t) = \textstyle\sum_{i=0}^{n}\alpha_i t^i,\ t \in [0,1], \text{ with each } \alpha_i \text{ in } \mathbb{F}\big\}.$$

Recall that the degree of a nonzero polynomial p is m if $p(t) = \sum_{i=0}^{m}\alpha_i t^i$ with $\alpha_m \neq 0$ (e.g., the degree of a constant polynomial is zero), and the degree of the zero polynomial is undefined (thus not greater than any $n \in \mathbb{N}_0$). It is readily verified that $\mathcal{P}_n[0,1]$ is a linear manifold of the linear space $\mathbb{F}^{[0,1]}$ (see Example 2.E), and hence a linear space over $\mathbb{F}$. Now consider the mapping $L\colon \mathbb{F}^{n+1} \to \mathcal{P}_n[0,1]$ defined as follows. For each $x = (\xi_0,\dots,\xi_n) \in \mathbb{F}^{n+1}$ let $p = Lx$ in $\mathcal{P}_n[0,1]$ be given by

$$p(t) \;=\; \sum_{i=0}^{n}\xi_i t^i$$

for every $t \in [0,1]$. It is easy to show that L is a linear transformation. Moreover, $\mathcal{N}(L) = \{0\}$ (i.e., if $p(t) = \sum_{i=0}^{n}\xi_i t^i = 0$ for every $t \in [0,1]$, then $x = (\xi_0,\dots,\xi_n) = 0$ — a nonzero polynomial has only a finite number of zeros) so that L is injective (see Theorem 2.8). Furthermore, every polynomial p in $\mathcal{P}_n[0,1]$ is of the form $p(t) = \sum_{i=0}^{n}\xi_i t^i$ for some $x = (\xi_0,\dots,\xi_n)$ in $\mathbb{F}^{n+1}$, which means that $\mathcal{P}_n[0,1] \subseteq \mathcal{R}(L)$. Hence $\mathcal{P}_n[0,1] = \mathcal{R}(L)$; that is, L is also surjective. Therefore, the linear transformation L is an isomorphism between the

linear spaces $\mathbb{F}^{n+1}$ and $\mathcal{P}_n[0,1]$. Thus, since $\dim \mathbb{F}^{n+1} = n+1$ (see Example 2.I), it follows by Theorem 2.12 that

$$\dim \mathcal{P}_n[0,1] \;=\; n+1.$$

Next consider the collection $\mathcal{P}[0,1]$ of all polynomials in the variable $t \in [0,1]$ with coefficients in $\mathbb{F}$ of any degree:

$$\mathcal{P}[0,1] \;=\; \bigcup_{n\in\mathbb{N}_0} \mathcal{P}_n[0,1].$$

Note that $\mathcal{P}[0,1]$ contains the zero polynomial together with every polynomial of finite degree. It is again readily verified that, as a linear manifold of $\mathbb{F}^{[0,1]}$, $\mathcal{P}[0,1]$ is itself a linear space over $\mathbb{F}$. The functions $p_j\colon [0,1] \to \mathbb{F}$, defined by $p_j(t) = t^j$ for every $t \in [0,1]$, clearly belong to $\mathcal{P}[0,1]$ for each $j \in \mathbb{N}_0$. Consider the set $B = \{p_j\}_{j\in\mathbb{N}_0} \subset \mathcal{P}[0,1]$. Since any polynomial in $\mathcal{P}[0,1]$ is, by definition, a (finite) linear combination of vectors in B, we get $\mathcal{P}[0,1] \subseteq \operatorname{span} B$. Hence B spans $\mathcal{P}[0,1]$ (i.e., $\operatorname{span} B = \mathcal{P}[0,1]$). We claim that B is also linearly independent. Indeed, suppose B is not linearly independent. Then there exists in B a linear combination p_k of vectors in $B\backslash\{p_k\}$. That is, $p_k = \sum_{i=1}^m \alpha_i p_{j_i}$ for some $m \in \mathbb{N}$, where $\{\alpha_i\}_{i=1}^m$ is a family of nonzero scalars and $\{p_{j_i}\}_{i=1}^m$ is a finite subset of B such that $p_{j_i} \neq p_k$ (i.e., $j_i \neq k$) for every $i = 1, \ldots, m$. Thus $p = p_k - \sum_{i=1}^m \alpha_i p_{j_i}$ is the origin of $\mathcal{P}[0,1]$, which means that

$$p(t) \;=\; t^k - \sum_{i=1}^m \alpha_i t^{j_i} \;=\; 0$$

for all $t \in [0,1]$. But this is a contradiction because p is a polynomial of degree equal to $\max\big\{\{k\} \cup \{j_i\}_{i=1}^m\big\} \geq 1$. Conclusion: B is linearly independent. Therefore the set $B = \{p_j\}_{j\in\mathbb{N}_0}$ is a Hamel basis for $\mathcal{P}[0,1]$, and hence

$$\dim \mathcal{P}[0,1] \;=\; \aleph_0$$

(since $\#B = \#\mathbb{N}_0 = \aleph_0$). Thus $\mathcal{P}[0,1]$ is isomorphic to the linear space $\mathcal{X}$ of all $\mathbb{F}$-valued sequences with a finite number of nonzero entries (which was introduced in Example 2.J).

2.7 Isomorphic Equivalence

Two linear spaces over the same scalar field are regarded as essentially the same linear space if they are isomorphic. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be linear spaces over the same field $\mathbb{F}$. It is clear that $\mathcal{X}$ is isomorphic to itself (reflexivity), and $\mathcal{Y}$ is isomorphic to $\mathcal{X}$ whenever $\mathcal{X}$ is isomorphic to $\mathcal{Y}$ (symmetry). Moreover, since the composition of two isomorphisms is again an isomorphism (see Problems 1.9(c) and 2.15), it follows that, if $\mathcal{X}$ is isomorphic to $\mathcal{Y}$ and $\mathcal{Y}$ is isomorphic to

$\mathcal{Z}$, then $\mathcal{X}$ is isomorphic to $\mathcal{Z}$ (transitivity). Thus, if the notion of isomorphic linear spaces is restricted to a given set (for instance, to the collection of all linear manifolds $\mathcal{L}at(\mathcal{X})$ of a linear space $\mathcal{X}$), then it is an equivalence relation on that set. We shall now define an equivalence between linear transformations. As usual, let $GF\colon \mathcal{X} \to \mathcal{Z}$ denote the composition $G \circ F$ of a mapping $G\colon \mathcal{Y} \to \mathcal{Z}$ and a mapping $F\colon \mathcal{X} \to \mathcal{Y}$.

Definition 2.13. Let $\mathcal{X}$, $\widetilde{\mathcal{X}}$, $\mathcal{Y}$, and $\widetilde{\mathcal{Y}}$ be linear spaces over the same scalar field, where $\mathcal{X}$ is isomorphic to $\widetilde{\mathcal{X}}$ and $\mathcal{Y}$ is isomorphic to $\widetilde{\mathcal{Y}}$. Two linear transformations $T\colon \mathcal{X} \to \mathcal{Y}$ and $L\colon \widetilde{\mathcal{X}} \to \widetilde{\mathcal{Y}}$ are *isomorphically equivalent* if there exist isomorphisms $X\colon \mathcal{X} \to \widetilde{\mathcal{X}}$ and $Y\colon \mathcal{Y} \to \widetilde{\mathcal{Y}}$ such that

$$YT = LX.$$

That is, L and T are isomorphically equivalent if there are isomorphisms X and Y such that $T = Y^{-1}LX$ (or $L = YTX^{-1}$), which means that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ T\ } & \mathcal{Y} \\ {\scriptstyle X}\big\downarrow & & \big\uparrow{\scriptstyle Y^{-1}} \\ \widetilde{\mathcal{X}} & \xrightarrow{\ L\ } & \widetilde{\mathcal{Y}} \end{array}$$

commutes. *Warning*: If $\mathcal{X}$ is isomorphic to $\widetilde{\mathcal{X}}$ and $\mathcal{Y}$ is isomorphic to $\widetilde{\mathcal{Y}}$, then there exists an uncountable supply of isomorphisms between $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ and between $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$. If we take arbitrary linear transformations $T\colon \mathcal{X} \to \mathcal{Y}$ and $L\colon \widetilde{\mathcal{X}} \to \widetilde{\mathcal{Y}}$, it may happen that the above diagram does not commute (i.e., it may happen that $YT \neq LX$) for all isomorphisms of $\mathcal{X}$ onto $\widetilde{\mathcal{X}}$ and all isomorphisms of $\mathcal{Y}$ onto $\widetilde{\mathcal{Y}}$. In this case T and L are not isomorphically equivalent. However, if there exists at least one pair of isomorphisms X and Y for which $YT = LX$, then T and L are isomorphically equivalent.

Isomorphic equivalence deserves its name. In fact, every T in $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is isomorphically equivalent to itself (reflexivity), and L in $L[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}]$ is isomorphically equivalent to T in $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$ whenever T is isomorphically equivalent to L (symmetry). Moreover, if T in $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is isomorphically equivalent to L in $\mathcal{L}[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}]$ and L is isomorphically equivalent to K in $\mathcal{L}[\widehat{\mathcal{X}}, \widehat{\mathcal{Y}}]$ (so that $\mathcal{X}$, $\widetilde{\mathcal{X}}$, and $\widehat{\mathcal{X}}$ are isomorphic linear spaces, as well as $\mathcal{Y}$, $\widetilde{\mathcal{Y}}$, and $\widehat{\mathcal{Y}}$), then it is easy to show that T is isomorphically equivalent to K (transitivity). Indeed, if $\mathcal{X} = \widetilde{\mathcal{X}}$ and $\mathcal{Y} = \widetilde{\mathcal{Y}}$, and if we restrict the concept of isomorphic equivalence to the set $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$ of all linear transformations of $\mathcal{X}$ into $\mathcal{Y}$, then isomorphic equivalence actually is an equivalence relation on $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$.

An important particular case is obtained when $\mathcal{X} = \mathcal{Y}$ and $\widetilde{\mathcal{X}} = \widetilde{\mathcal{Y}}$ so that T lies in $\mathcal{L}[\mathcal{X}]$ and L lies in $\mathcal{L}[\widetilde{\mathcal{X}}]$. Let $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ be isomorphic linear spaces. Two linear transformations $T\colon \mathcal{X} \to \mathcal{X}$ and $L\colon \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$ are *similar* if there exists an isomorphism $W\colon \mathcal{X} \to \widetilde{\mathcal{X}}$ such that

$$WT = LW.$$

To put it another way, if there is an isomorphism W such that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{T} & \mathcal{X} \\ {\scriptstyle W}\downarrow & & \downarrow{\scriptstyle W} \\ \widetilde{\mathcal{X}} & \xrightarrow{L} & \widetilde{\mathcal{X}} \end{array}$$

commutes. It should be noticed now that the concept of similarity will be redefined later in Chapter 4 where the linear spaces are endowed with an additional (topological) structure. Such a redefinition will assume that all linear transformations involved in the definition of similarity are "continuous" (including the inverse of W).

Example 2.N. Consider the setup of Example 2.L, where $\mathcal{X}$ and $\mathcal{Y}$ are *finite-dimensional* linear spaces over the same field $\mathbb{F}$. Let $X\colon \mathcal{X} \to \mathbb{F}^n$ and $Y\colon \mathcal{Y} \to \mathbb{F}^m$ be two mappings defined by

$$Xx = (\xi_1, \ldots, \xi_n) \quad \text{and} \quad Yy = (\upsilon_1, \ldots, \upsilon_m)$$

for every $x \in \mathcal{X}$ and every $y \in \mathcal{Y}$, where $\{\xi_j\}_{j=1}^n$ and $\{\upsilon_i\}_{i=1}^m$ consist of the coordinates of x and y with respect to the bases B_X and B_Y, respectively. It is readily verified that X and Y are both isomorphisms (for fixed bases B_X and B_Y). Let $\mathbb{F}_{n\times 1}$ denote the linear space (over the field $\mathbb{F}$) of all $n\times 1$ matrices (or, if you like, the linear space of all "column n-vectors" with entries in $\mathbb{F}$ — Example 2.L). Now consider the map $W_n\colon \mathbb{F}^n \to \mathbb{F}_{n\times 1}$ that assigns to each n-tuple $(\xi_1, \ldots, \xi_n)$ in $\mathbb{F}^n$ the $n\times 1$ matrix

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = W_n(\xi_1, \ldots, \xi_n)$$

in $\mathbb{F}_{n\times 1}$ whose entries are the (similarly ordered) coordinates of the ordered n-tuple with respect to the *canonical* basis for $\mathbb{F}^n$. It is easy to show that W_n is an isomorphism between $\mathbb{F}^n$ and $\mathbb{F}_{n\times 1}$. This is called the *natural isomorphism* of $\mathbb{F}^n$ onto $\mathbb{F}_{n\times 1}$. Note that any $m\times n$ matrix (with entries in $\mathbb{F}$) can be viewed as a linear transformation from $\mathbb{F}_{n\times 1}$ to $\mathbb{F}_{m\times 1}$: the action of an $m\times n$ matrix $[\alpha_{ij}] \in \mathbb{F}_{m\times n}$ on an $n\times 1$ matrix $[\xi_j] \in \mathbb{F}_{n\times 1}$ is simply the matrix product

$$[\alpha_{ij}][\xi_j] = \begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \ldots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix},$$

which is an $m\times 1$ matrix in $\mathbb{F}_{m\times 1}$. According to Example 2.L let $[A] \in \mathbb{F}_{m\times n}$ be the unique matrix representing the linear transformation $A \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ with

respect to the bases B_X and B_Y. Now, if this matrix is viewed as a linear transformation of $\mathbb{F}_{n\times 1}$ into $\mathbb{F}_{m\times 1}$, then the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ A\ } & \mathcal{Y} \\ {\scriptstyle X}\downarrow & & \uparrow{\scriptstyle Y^{-1}} \\ \mathbb{F}^n & & \mathbb{F}^m \\ {\scriptstyle W_n}\downarrow & & \uparrow{\scriptstyle W_m^{-1}} \\ \mathbb{F}_{n\times 1} & \xrightarrow{\ [A]\ } & \mathbb{F}_{m\times 1} \end{array}$$

commutes. This shows that the linear transformation $A\colon \mathcal{X} \to \mathcal{Y}$ is isomorphically equivalent to its matrix $[A]$ with respect to the bases B_X and B_Y when this matrix is viewed as a linear transformation $[A]\colon \mathbb{F}_{n\times 1} \to \mathbb{F}_{m\times 1}$. That is,

$$(W_m Y)A \,=\, [A](W_n X).$$

2.8 Direct Sum

Let $\{\mathcal{X}_i\}_{i=1}^n$ be a finite family of linear spaces over the same field $\mathbb{F}$ (not necessarily linear manifolds of the same linear space). The *direct sum* of $\{\mathcal{X}_i\}_{i=1}^n$, denoted by $\bigoplus_{i=1}^n \mathcal{X}_i$, is the set of all ordered n-tuples $(x_1, \ldots, x_n)$, with each x_i in $\mathcal{X}_i$, where vector addition and scalar multiplication are defined as follows.

$$(x_1, \ldots, x_n) \oplus (y_1, \ldots, y_n) \,=\, (x_1 + y_1, \ldots, x_n + y_n),$$
$$\alpha(x_1, \ldots, x_n) \,=\, (\alpha x_1, \ldots, \alpha x_n)$$

for every $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in $\bigoplus_{i=1}^n \mathcal{X}_i$ and every α in $\mathbb{F}$. It is easy to verify that the direct sum $\bigoplus_{i=1}^n \mathcal{X}_i$ of the linear spaces $\{\mathcal{X}_i\}_{i=1}^n$ is a linear space over $\mathbb{F}$ when vector addition (denoted by $\oplus$) and scalar multiplication are defined as above. The underlying set of the linear space $\bigoplus_{i=1}^n \mathcal{X}_i$ is the Cartesian product $\prod_{i=1}^n \mathcal{X}_i$ of the underlying sets of each linear space $\mathcal{X}_i$. The origin of $\bigoplus_{i=1}^n \mathcal{X}_i$ is the ordered n-tuple $(0_1, \ldots, 0_n)$ of the origins of each $\mathcal{X}_i$.

If $\mathcal{M}$ and $\mathcal{N}$ are linear manifolds of a linear space $\mathcal{X}$, then we may consider both their ordinary sum $\mathcal{M} + \mathcal{N}$ (defined as in Section 2.2) and their direct sum $\mathcal{M} \oplus \mathcal{N}$. These are different linear spaces over the same field. There is however a *natural mapping* $\Phi\colon \mathcal{M} \oplus \mathcal{N} \to \mathcal{M} + \mathcal{N}$, defined by

$$\Phi((x_1, x_2)) \,=\, x_1 + x_2,$$

which assigns to each pair (x_1, x_2) in $\mathcal{M} \oplus \mathcal{N}$ their sum in $\mathcal{M} + \mathcal{N} \subseteq \mathcal{X}$. It is readily verified that Φ is a surjective linear transformation of the linear space $\mathcal{M} \oplus \mathcal{N}$ onto the linear space $\mathcal{M} + \mathcal{N}$, but Φ is not always injective. We shall

establish below a necessary and sufficient condition that Φ be injective, viz., $\mathcal{M} \cap \mathcal{N} = \{0\}$. In such a case the mapping Φ is an isomorphism (called the *natural isomorphism*) of $\mathcal{M} \oplus \mathcal{N}$ onto $\mathcal{M} + \mathcal{N}$, so that the direct sum $\mathcal{M} \oplus \mathcal{N}$ and the ordinary sum $\mathcal{M} + \mathcal{N}$ become isomorphic linear spaces.

Theorem 2.14. *Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of a linear space $\mathcal{X}$. The following assertions are pairwise equivalent.*

(a) $\mathcal{M} \cap \mathcal{N} = \{0\}$.

(b) *For each x in $\mathcal{M} + \mathcal{N}$ there exists a unique u in $\mathcal{M}$ and a unique v in $\mathcal{N}$ such that $x = u + v$.*

(c) *The natural mapping $\Phi: \mathcal{M} \oplus \mathcal{N} \to \mathcal{M} + \mathcal{N}$ is an isomorphism.*

Proof. Take an arbitrary x in $\mathcal{M} + \mathcal{N}$. If $x = u_1 + v_1 = u_2 + v_2$, with u_1, u_2 in $\mathcal{M}$ and v_1, v_2 in $\mathcal{N}$, then $u_1 - u_2 = v_1 - v_2$ is in $\mathcal{M} \cap \mathcal{N}$ (for $u_1 - u_2 \in \mathcal{M}$ and $v_1 - v_2 \in \mathcal{N}$). Thus $\mathcal{M} \cap \mathcal{N} = \{0\}$ implies that $u_1 = u_2$ and $v_1 = v_2$, and hence (a)⇒(b). On the other hand, if $\mathcal{M} \cap \mathcal{N} \neq \{0\}$, then there exists a nonzero vector w in $\mathcal{M} \cap \mathcal{N}$. Take any nonzero vector x in $\mathcal{M} + \mathcal{N}$ so that $x = u + v$ with u in $\mathcal{M}$ and v in $\mathcal{N}$. Thus $x = (u + w) + (v - w)$, where $u + w$ is in $\mathcal{M}$ and $v - w$ is in $\mathcal{N}$. Since $w \neq 0$, it follows that $u + w \neq u$, and hence the representation of x as a sum $u + v$ with u in $\mathcal{M}$ and v in $\mathcal{N}$ is not unique. Thus, if (a) does not hold, then (b) does not hold. Equivalently, (b)⇒(a). Finally, recall that the natural mapping Φ is linear and surjective. Since Φ is injective if and only if (b) holds (by definition), it follows that (b)⇔(c). □

Two linear manifolds $\mathcal{M}$ and $\mathcal{N}$ of a linear space $\mathcal{X}$ are said to be *disjoint* (or *algebraically disjoint*) if $\mathcal{M} \cap \mathcal{N} = \{0\}$. (Note that, as linear manifolds of a linear space $\mathcal{X}$, $\mathcal{M}$ and $\mathcal{N}$ can never be "disjoint" in the set-theoretical sense — the origin of $\mathcal{X}$ always belongs to both of them.) Therefore, if $\mathcal{M}$ and $\mathcal{N}$ are disjoint linear manifolds of a linear space $\mathcal{X}$, then we may and shall identify their ordinary sum $\mathcal{M} + \mathcal{N}$ with their direct sum $\mathcal{M} \oplus \mathcal{N}$. Such an identification is carried out by the natural isomorphism $\Phi: \mathcal{M} \oplus \mathcal{N} \to \mathcal{M} + \mathcal{N}$ (Theorem 2.14). When we identify $\mathcal{M} \oplus \mathcal{N}$ with $\mathcal{M} + \mathcal{N}$, which is a linear manifold of $\mathcal{X}$, we are automatically identifying the pairs $(u, 0)$ and $(0, v)$ in $\mathcal{M} \oplus \mathcal{N}$ with u in $\mathcal{M}$ and with v in $\mathcal{N}$, respectively. More generally, we shall be identifying the direct sums $\mathcal{M} \oplus \{0\}$ and $\{0\} \oplus \mathcal{N}$ with $\mathcal{M}$ and $\mathcal{N}$, respectively. For instance, if $x \in \mathcal{M} \oplus \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$, then Theorem 2.14 ensures that there exists a unique u in $\mathcal{M}$ and a unique v in $\mathcal{N}$ such that $x = (u, v)$. Hence $x = (u, 0) \oplus (0, v)$ where $(u, 0) \in \mathcal{M} \oplus \{0\}$ and $(0, v) \in \{0\} \oplus \mathcal{N}$ (recall: $\mathcal{M} \oplus \{0\}$ and $\{0\} \oplus \mathcal{N}$ are both linear manifolds of $\mathcal{M} \oplus \mathcal{N}$). Now identify $(u, 0)$ with $\Phi((u, 0)) = u$ and $(0, v)$ with $\Phi((0, v)) = v$, and write $x = u \oplus v$ where $u \in \mathcal{M}$ and $v \in \mathcal{N}$ (instead of $x = (u, 0) \oplus (0, v) = \Phi^{-1}(u) \oplus \Phi^{-1}(v)$). Outcome: If $\mathcal{M}$ and $\mathcal{N}$ are disjoint linear manifolds of a linear space $\mathcal{X}$, then every x in $\mathcal{M} \oplus \mathcal{N}$ has a *unique decomposition with respect to $\mathcal{M}$ and $\mathcal{N}$*, denoted by $x = u \oplus v$, which is referred to as the *direct sum* of u in $\mathcal{M}$ and v in

$\mathcal{N}$. It should be noticed that $u \oplus v$ is just another notation for (u, v) that reminds us of the algebraic structure of the linear space $\mathcal{M} \oplus \mathcal{N}$. What really is being added in $\mathcal{M} \oplus \mathcal{N}$ is $(u, 0) \oplus (0, v)$.

If $\mathcal{M}$ and $\mathcal{N}$ are disjoint linear manifolds of a linear space $\mathcal{X}$, and if their (ordinary) sum is $\mathcal{X}$, then we say that $\mathcal{M}$ and $\mathcal{N}$ are *algebraic complements* of each other. In other words, two linear manifolds $\mathcal{M}$ and $\mathcal{N}$ of a linear space $\mathcal{X}$ form a pair of algebraic complements in $\mathcal{X}$ if

$$\mathcal{X} = \mathcal{M} + \mathcal{N} \quad \text{and} \quad \mathcal{M} \cap \mathcal{N} = \{0\}.$$

Accordingly, this can be written as

$$\mathcal{X} = \mathcal{M} \oplus \mathcal{N} \quad \text{and} \quad \mathcal{M} \cap \mathcal{N} = \{0\}$$

once we have identified the direct sum $\mathcal{M} \oplus \mathcal{N}$ with its isomorphic image $\Phi(\mathcal{M} \oplus \mathcal{N}) = \mathcal{M} + \mathcal{N} = \mathcal{X}$ through the natural isomorphism Φ.

Proposition 2.15. *Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of a linear space $\mathcal{X}$, and let B_M and B_N be Hamel bases for $\mathcal{M}$ and $\mathcal{N}$, respectively.*

(a) *$\mathcal{M} \cap \mathcal{N} = \{0\}$ if and only if $B_M \cap B_N = \varnothing$ and $B_M \cup B_N$ is linearly independent.*

(b) *$\mathcal{M} + \mathcal{N} = \mathcal{X}$ and $B_M \cup B_N$ is linearly independent if and only if $B_M \cup B_N$ is a Hamel basis for $\mathcal{X}$.*

In particular, if $B_M \cup B_N \subseteq B$, where B is a Hamel basis for $\mathcal{X}$, then

(a′) *$\mathcal{M} \cap \mathcal{N} = \{0\}$ if and only if $B_M \cap B_N = \varnothing$,*

(b′) *$\mathcal{M} + \mathcal{N} = \mathcal{X}$ if and only if $B_M \cup B_N = B$.*

Proof. (a) Recall that

$$\{0\} \subseteq \operatorname{span}(B_M \cap B_N) \subseteq \operatorname{span}(\mathcal{M} \cap \mathcal{N}) = \mathcal{M} \cap \mathcal{N}.$$

Thus $\mathcal{M} \cap \mathcal{N} = \{0\}$ implies $\operatorname{span}(B_M \cap B_N) = \{0\}$, which in turn implies $B_M \cap B_N = \varnothing$ (for $0 \notin B_M \cup B_N$). Moreover, if $\mathcal{M} \cap \mathcal{N} = \{0\}$, then the union of the linearly independent sets B_M and B_N is again linearly independent (see Problem 2.3). On the other hand, recall that

$$\{0\} \subseteq \mathcal{M} \cap \mathcal{N} = \operatorname{span} B_M \cap \operatorname{span} B_N = \operatorname{span}(B_M \cap B_N)$$

if $B_M \cup B_N$ is linearly independent (see Problem 2.4). Thus $B_M \cap B_N = \varnothing$ implies $\operatorname{span}(B_M \cap B_N) = \{0\}$, and hence $\mathcal{M} \cap \mathcal{N} = \{0\}$.

(b) Next recall that

$$\operatorname{span}(B_M \cup B_N) = \operatorname{span}(\mathcal{M} \cup \mathcal{N}) = \mathcal{M} + \mathcal{N} \subseteq \mathcal{X}$$

whenever B_M and B_N are Hamel bases for $\mathcal{M}$ and $\mathcal{N}$, respectively. Moreover, if $B_M \cup B_N$ is a Hamel basis for $\mathcal{X}$, then $B_M \cup B_N$ is linearly independent and $\mathcal{X} = \operatorname{span}(B_M \cup B_N)$ so that $\mathcal{M} + \mathcal{N} = \mathcal{X}$. On the other hand, if $\mathcal{M} + \mathcal{N} = \mathcal{X}$, then $\operatorname{span}(B_M \cup B_N) = \mathcal{X}$. Thus, according to Theorem 2.6, there exists a Hamel basis B' for $\mathcal{X}$ such that $B' \subseteq B_M \cup B_N$. If $B_M \cup B_N$ is linearly independent, then Theorem 2.5 ensures that there exists a Hamel basis B for $\mathcal{X}$ such that $B_M \cup B_N \subseteq B$. Therefore $B' \subseteq B$. But a Hamel basis is maximal (see Claim 2 in the proof of Theorem 2.5) so that $B' = B$. Hence $B_M \cup B_N = B$. □

Theorem 2.16. *Every linear manifold has an algebraic complement.*

Proof. Let $\mathcal{M}$ be a linear manifold of a linear space $\mathcal{X}$, let B_M be a Hamel basis for $\mathcal{M}$, and let B be a Hamel basis for $\mathcal{X}$ such that $B_M \subseteq B$ (see Theorem 2.5). Set $B_N = B \backslash B_M$ (which, as a subset of a linearly independent set B, is linearly independent itself), and set $\mathcal{N} = \operatorname{span} B_N$ (a linear manifold of $\mathcal{X}$). Thus B_M and B_N are Hamel basis for $\mathcal{M}$ and $\mathcal{N}$, respectively, both included in the Hamel basis B for $\mathcal{X}$. Since $B_M \cap B_N = \varnothing$ and $B_M \cup B_N = B$, it follows by Proposition 2.15 that $\mathcal{N}$ is an algebraic complement of $\mathcal{M}$. □

Lemma 2.17. *Let $\mathcal{M}$ be a linear manifold of a linear space $\mathcal{X}$. Every algebraic complement of $\mathcal{M}$ is isomorphic to the quotient space $\mathcal{X}/\mathcal{M}$.*

Proof. Let $\mathcal{M}$ be a linear manifold of a linear space $\mathcal{X}$ over a field $\mathbb{F}$, and let $\mathcal{X}/\mathcal{M}$ be the quotient space of $\mathcal{X}$ modulo $\mathcal{M}$, which is again a linear space over $\mathbb{F}$ (see Example 2.H). The natural mapping $\pi\colon \mathcal{X} \to \mathcal{X}/\mathcal{M}$, which assigns each vector x in $\mathcal{X}$ the equivalence class

$$\pi(x) \;=\; [x] \;=\; x + \mathcal{M}$$

in $\mathcal{X}/\mathcal{M}$, is a linear transformation (cf. Example 2.H). It is plain that π is surjective. Let $\mathcal{K}$ be a linear manifold of $\mathcal{X}$ and consider the restriction of π to $\mathcal{K}$, $\pi|_{\mathcal{K}}\colon \mathcal{K} \to \mathcal{X}/\mathcal{M}$, which is again a linear transformation (Problem 2.14).

Claim 1. If $\mathcal{M} \cap \mathcal{K} = \{0\}$, then $\pi|_{\mathcal{K}}$ is injective.

Proof. Problem 2.14 also says that $\mathcal{N}(\pi|_{\mathcal{K}}) = \mathcal{K} \cap \mathcal{N}(\pi)$. Since $\mathcal{N}(\pi) = \mathcal{M}$ (see Example 2.H), it follows that, if $\mathcal{M} \cap \mathcal{K} = \{0\}$, then $\mathcal{N}(\pi|_{\mathcal{K}}) = \{0\}$, and so the linear transformation $\pi|_{\mathcal{K}}$ is injective (Theorem 2.8). □

Claim 2. If $\mathcal{X} = \mathcal{M} + \mathcal{K}$, then $\pi|_{\mathcal{K}}$ is surjective.

Proof. Take an arbitrary $[x]$ in $\mathcal{X}/\mathcal{M}$ so that $[x] = x + \mathcal{M}$ for some x in $\mathcal{X}$. If $\mathcal{X} = \mathcal{M} + \mathcal{K}$, then $x = u + v$ with u in $\mathcal{M}$ and v in $\mathcal{K}$. Thus, as π is linear, $[x] = \pi(x) = \pi(u) + \pi(v)$. But $u \in \mathcal{M} = \mathcal{N}(\pi)$ so that $\pi(u) = [0]$, and hence $\pi(u) + \pi(v) = [0] + [v] = [0 + v] = [v] = \pi(v)$. Therefore, $[x] = \pi(v) = \pi|_{\mathcal{K}}(v)$, which lies in $\mathcal{R}(\pi|_{\mathcal{K}})$. Then $\mathcal{X}/\mathcal{M} \subseteq \mathcal{R}(\pi|_{\mathcal{K}})$, and so $\pi|_{\mathcal{K}}$ is surjective. □

Thus, if $\mathcal{K}$ is an algebraic complement of $\mathcal{M}$, then $\pi|_{\mathcal{K}}$ is invertible by Claims 1 and 2 and, since $\pi|_{\mathcal{K}}$ is linear, it is an isomorphism of $\mathcal{K}$ onto $\mathcal{X}/\mathcal{M}$. □

Theorem 2.18. *Let $\mathcal{M}$ be a linear manifold of a linear space $\mathcal{X}$. Every algebraic complement of $\mathcal{M}$ has the same dimension.*

Proof. According to Theorem 2.12 the above statement can be rewritten as follows. If $\mathcal{N}$ and $\mathcal{K}$ are algebraic complements of $\mathcal{M}$, then $\mathcal{K}$ and $\mathcal{N}$ are isomorphic. But this is a straightforward consequence of the previous lemma: $\mathcal{N}$ and $\mathcal{K}$ are both isomorphic to $\mathcal{X}/\mathcal{M}$, and hence isomorphic to each other. □

The dimension of an algebraic complement of $\mathcal{M}$ is therefore a property of $\mathcal{M}$ (i.e., it is an invariant for $\mathcal{M}$). We refer to this invariant as the *codimension* of $\mathcal{M}$: the codimension of a linear manifold $\mathcal{M}$, denoted by $\operatorname{codim}\mathcal{M}$, is the (constant) dimension of any algebraic complement of $\mathcal{M}$.

2.9 Projections

A *projection* is an idempotent linear transformation of a linear space into itself. Thus, if $\mathcal{X}$ is a linear space, then $P \in \mathcal{L}[\mathcal{X}]$ is a projection if and only if $P = P^2$. Briefly, projections are the idempotent elements of $\mathcal{L}[\mathcal{X}]$. It is plain that the null transformation O and the identity I in $\mathcal{L}[\mathcal{X}]$ are projections. A *nontrivial projection* in $\mathcal{L}[\mathcal{X}]$ is a projection P such that $O \neq P \neq I$. It is easy to verify that, if P is a projection, then so is $I - P$. Moreover, the null spaces and ranges of P and $I - P$ are related as follows (cf. Problem 1.4).

$$\mathcal{R}(P) = \mathcal{N}(I - P) \quad \text{and} \quad \mathcal{N}(P) = \mathcal{R}(I - P).$$

Projections are singularly useful linear transformations. One of their main properties is that the range and the null space of a projection form a pair of algebraic complements.

Theorem 2.19. *If $P \in \mathcal{L}[\mathcal{X}]$ is a projection, then $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are algebraic complements of each other.*

Proof. Let $\mathcal{X}$ be a linear space and let $P\colon \mathcal{X} \to \mathcal{X}$ be a projection. Recall that both the range $\mathcal{R}(P)$ and the null space $\mathcal{N}(P)$ are linear manifolds of $\mathcal{X}$ (because P is linear). Since P is idempotent, it follows that

$$\mathcal{R}(P) \;=\; \big\{x \in \mathcal{X}\colon\; Px = x\big\}$$

(the range of an idempotent mapping is the set of all its fixed points — Problem 1.4). If $x \in \mathcal{R}(P) \cap \mathcal{N}(P)$, then $x = Px = 0$, and hence

$$\mathcal{R}(P) \cap \mathcal{N}(P) \;=\; \{0\}.$$

Moreover, write any vector x in $\mathcal{X}$ as $x = Px + (x - Px)$. Since P is linear and idempotent, $P(x - Px) = Px - P^2x = 0$, and so $(x - Px)$ lies in $\mathcal{N}(P)$. Hence $x = u + v$ with $u = Px$ in $\mathcal{R}(P)$ and $v = (x - Px)$ in $\mathcal{N}(P)$. Therefore,

$$\mathcal{X} \;=\; \mathcal{R}(P) + \mathcal{N}(P). \qquad \square$$

On the other hand, for any pair of algebraic complements there exists a unique projection whose range and null space coincide with them.

Theorem 2.20. *Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of a linear space $\mathcal{X}$. If $\mathcal{M}$ and $\mathcal{N}$ are algebraic complements of each other, then there exists a unique projection $P\colon \mathcal{X} \to \mathcal{X}$ such that $\mathcal{R}(P) = \mathcal{M}$ and $\mathcal{N}(P) = \mathcal{N}$.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be algebraic complements in a linear space $\mathcal{X}$ so that

$$\mathcal{M} + \mathcal{N} = \mathcal{X} \quad \text{and} \quad \mathcal{M} \cap \mathcal{N} = \{0\}.$$

According to Theorem 2.14, for each $x \in \mathcal{X}$ there exists a unique $u \in \mathcal{M}$ and a unique $v \in \mathcal{N}$ such that $x = u + v$. Let $P\colon \mathcal{X} \to \mathcal{X}$ be the function that assigns to each x in $\mathcal{X}$ its unique summand u in $\mathcal{M}$ (i.e., $Px = u$). It is easy to verify that P is linear. Moreover, for each vector x in $\mathcal{X}$, $P^2x = P(Px) = Pu = u = Px$ (reason: u is itself its unique summand in $\mathcal{M}$), so that P is idempotent. By the very definition of P we get $\mathcal{R}(P) = \mathcal{M}$ and $\mathcal{N}(P) = \mathcal{N}$. Conclusion: $P\colon \mathcal{X} \to \mathcal{X}$ is a projection with $\mathcal{R}(P) = \mathcal{M}$ and $\mathcal{N}(P) = \mathcal{N}$. Now let $P'\colon \mathcal{X} \to \mathcal{X}$ be any projection with $\mathcal{R}(P') = \mathcal{M}$ and $\mathcal{N}(P') = \mathcal{N}$. Take an arbitrary $x \in \mathcal{X}$ and consider again its unique representation as $x = u + v$ with $u \in \mathcal{M} = \mathcal{R}(P')$ and $v \in \mathcal{N} = \mathcal{N}(P')$. Since P' is linear and idempotent, it follows that $P'x = P'u + P'v = u = Px$. Therefore, $P' = P$. □

Remark: An immediate corollary of Theorems 2.16 and 2.20 says that any linear manifold of a linear space is the range of some projection. That is, *if $\mathcal{M}$ is a linear manifold of a linear space $\mathcal{X}$, then there exists a projection $P\colon \mathcal{X} \to \mathcal{X}$ such that $\mathcal{R}(P) = \mathcal{M}$.*

If $\mathcal{M}$ and $\mathcal{N}$ are algebraic complements in a linear space $\mathcal{X}$, then the unique projection P in $\mathcal{L}[\mathcal{X}]$ with range $\mathcal{R}(P) = \mathcal{M}$ and null space $\mathcal{N}(P) = \mathcal{N}$ is called *the projection on $\mathcal{M}$ along $\mathcal{N}$*. If P is the projection on $\mathcal{M}$ along $\mathcal{N}$, then the projection on $\mathcal{N}$ along $\mathcal{M}$ is precisely the projection $E = I - P$ in $\mathcal{L}[\mathcal{X}]$, referred to as the *complementary projection* of P. Note that $EP = PE = O$.

Proposition 2.21. *Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of a linear space $\mathcal{X}$. If $\mathcal{M}$ and $\mathcal{N}$ are algebraic complements of each other, then the unique decomposition of each x in $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$ as a direct sum*

$$x = u \oplus v$$

of u in $\mathcal{M}$ and v in $\mathcal{N}$ is such that

$$u = Px \quad \text{and} \quad v = (I - P)x,$$

where $P\colon \mathcal{X} \to \mathcal{X}$ is the unique projection on $\mathcal{M}$ along $\mathcal{N}$.

Proof. Take an arbitrary x in $\mathcal{X}$ and consider its unique decomposition $x = u \oplus v$ in $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$. Note that the identification of $\mathcal{M} \oplus \mathcal{N}$ with

$\mathcal{M}+\mathcal{N}=\mathcal{X}$ is implicitly assumed in the proposition statement. Now write $x=(u,v)$ and set $Px=(u,0)$. The very same argument used in the proof of Theorem 2.20 can be applied here to verify that this actually defines a unique projection $P\colon \mathcal{M}\oplus\mathcal{N}\to\mathcal{M}\oplus\mathcal{N}$ such that $\mathcal{R}(P)=\mathcal{M}\oplus\{0\}$ and $\mathcal{N}(P)=\{0\}\oplus\mathcal{N}$. Finally, identify $\mathcal{M}\oplus\{0\}$ and $\{0\}\oplus\mathcal{N}$ with $\mathcal{M}$ and $\mathcal{N}$ (and hence $(u,0)$ and $(0,v)$ with u and v), respectively. □

According to Theorem 2.16 every linear space $\mathcal{X}$ can be represented as the sum $\mathcal{X}=\mathcal{M}+\mathcal{N}$ of a pair $\{\mathcal{M},\mathcal{N}\}$ of algebraic complements in $\mathcal{X}$. If $\mathcal{M}\oplus\mathcal{N}$ is identified with $\mathcal{M}+\mathcal{N}$, then this means that *every linear space $\mathcal{X}$ has a decomposition $\mathcal{X}=\mathcal{M}\oplus\mathcal{N}$ as a direct sum of disjoint linear manifolds of $\mathcal{X}$.*

Proposition 2.22. *Let $\mathcal{X}$ be a linear space and consider its decomposition*

$$\mathcal{X} \;=\; \mathcal{M}\oplus\mathcal{N}$$

as a direct sum of disjoint linear manifolds $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{X}$. Let $P\colon\mathcal{X}\to\mathcal{X}$ be the projection on $\mathcal{M}$ along $\mathcal{N}$, and let $E=I-P$ be the projection on $\mathcal{N}$ along $\mathcal{M}$. Every linear transformation $L\colon\mathcal{X}\to\mathcal{X}$ can be written as a 2×2 matrix with linear transformation entries

$$L \;=\; \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A=PL|_{\mathcal{M}}\colon\mathcal{M}\to\mathcal{M}$, $B=PL|_{\mathcal{N}}\colon\mathcal{N}\to\mathcal{M}$, $C=EL|_{\mathcal{M}}\colon\mathcal{M}\to\mathcal{N}$, and $D=EL|_{\mathcal{N}}\colon\mathcal{N}\to\mathcal{N}$.

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of a linear space $\mathcal{X}$. Suppose $\mathcal{M}$ and $\mathcal{N}$ are algebraic complements of each other, identify $\mathcal{M}\oplus\mathcal{N}$ with $\mathcal{M}+\mathcal{N}$, and consider the decomposition $\mathcal{X}=\mathcal{M}\oplus\mathcal{N}$. Let L be a linear transformation on $\mathcal{M}\oplus\mathcal{N}$ so that $L\in\mathcal{L}[\mathcal{X}]$. Take an arbitrary $x\in\mathcal{X}$ and consider its unique decomposition $x=u\oplus v$ in $\mathcal{X}=\mathcal{M}\oplus\mathcal{N}$ with u in $\mathcal{M}$ and v in $\mathcal{N}$. Now write $x=(u,v)$ so that $Lx=L(u,v)=L((u,0)\oplus(0,v))=L(u,0)\oplus L(0,v)=L|_{\mathcal{M}\oplus\{0\}}(u,0)\oplus L|_{\{0\}\oplus\mathcal{N}}(0,v)$. Identifying $\mathcal{M}\oplus\{0\}$ and $\{0\}\oplus\mathcal{N}$ with $\mathcal{M}$ and $\mathcal{N}$ (and so $(u,0)$ and $(0,v)$ with u and v), respectively, it follows that

$$Lx \;=\; L|_{\mathcal{M}}\,u\oplus L|_{\mathcal{N}}\,v,$$

where $L|_{\mathcal{M}}\,u$ and $L|_{\mathcal{N}}\,v$ lie in $\mathcal{X}=\mathcal{M}\oplus\mathcal{N}$. By Proposition 2.21 we may write

$$L|_{\mathcal{M}}\,u \;=\; PL|_{\mathcal{M}}\,u\oplus EL|_{\mathcal{M}}\,u,$$

$$L|_{\mathcal{N}}\,v \;=\; PL|_{\mathcal{N}}\,v\oplus EL|_{\mathcal{N}}\,v,$$

where P is the unique projection on $\mathcal{M}$ along $\mathcal{N}$ and $E=I-P$. Therefore,

$$Lx \;=\; (PL|_{\mathcal{M}}\,u+PL|_{\mathcal{N}}\,v)\oplus(EL|_{\mathcal{M}}\,u+EL|_{\mathcal{N}}\,v),$$

where $PL|_{\mathcal{M}}\,u+PL|_{\mathcal{N}}\,v$ is in $\mathcal{M}$ and $EL|_{\mathcal{M}}\,u+EL|_{\mathcal{N}}\,v$ is in $\mathcal{N}$. Since the ranges of $PL|_{\mathcal{M}}$ and $PL|_{\mathcal{N}}$ are included in $\mathcal{R}(P)=\mathcal{M}$, we may think of them

as linear transformations into $\mathcal{M}$. Similarly, $EL|_{\mathcal{M}}$ and $EL|_{\mathcal{N}}$ can be thought of as linear transformations into $\mathcal{N}$. Thus set $A = PL|_{\mathcal{M}}$ in $\mathcal{L}[\mathcal{M}]$, $B = PL|_{\mathcal{N}}$ in $\mathcal{L}[\mathcal{N}, \mathcal{M}]$, $C = EL|_{\mathcal{M}}$ in $\mathcal{L}[\mathcal{M}, \mathcal{N}]$, and $D = EL|_{\mathcal{N}}$ in $\mathcal{L}[\mathcal{N}]$ so that

$$Lx \,=\, (Au + Bv, Cu + Dv) \,\in\, \mathcal{M} \oplus \mathcal{N}$$

for every $x = (u, v) \in \mathcal{M} \oplus \mathcal{N}$. In terms of standard matrix notation, the vector Lx in $\mathcal{M} \oplus \mathcal{N}$ can be viewed as a 2×1 matrix with the first entry in $\mathcal{M}$ and the other in $\mathcal{N}$, namely, $\left(\begin{smallmatrix} Au+Bv \\ Cu+Dv \end{smallmatrix}\right)$. This is precisely the action of the 2×2 matrix with linear transformation entries, $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, on the 2×1 matrix with entries in $\mathcal{M}$ and $\mathcal{N}$ representing x, namely, $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$. Thus $Lx = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$, and hence we write $L = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. □

Example 2.O. Consider the setup of Proposition 2.22. Note that the projection on $\mathcal{M}$ along $\mathcal{N}$ can be written as

$$P \,=\, \begin{pmatrix} I & O \\ O & O \end{pmatrix}$$

with respect to the decomposition $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$, where I denotes the identity on $\mathcal{M}$. Thus $LP = \left(\begin{smallmatrix} A & O \\ C & O \end{smallmatrix}\right)$ and $PLP = \left(\begin{smallmatrix} A & O \\ O & O \end{smallmatrix}\right)$, so that $LP = PLP$ if and only if $C = O$. Note that $\mathcal{M}$ is L-invariant (i.e., $L(\mathcal{M}) \subseteq \mathcal{M}$) if and only if $PL|_{\mathcal{M}} = L|_{\mathcal{M}}$ (equivalently, if and only if $EL|_{\mathcal{M}} = O$ with $E = I - P$). Thus

$$L(\mathcal{M}) \subseteq \mathcal{M} \quad\Longleftrightarrow\quad A = L|_{\mathcal{M}} \quad\Longleftrightarrow\quad C = O \quad\Longleftrightarrow\quad LP = PLP.$$

Conclusion 1: The following assertions are pairwise equivalent.

(a) $\mathcal{M}$ is L-invariant.

(b) $L = \left(\begin{smallmatrix} L|_{\mathcal{M}} & B \\ O & D \end{smallmatrix}\right)$.

(c) $LP = PLP$.

Similarly, if we apply the same argument to $\mathcal{N}$, then

$$L(\mathcal{N}) \subseteq \mathcal{N} \quad\Longleftrightarrow\quad D = L|_{\mathcal{N}} \quad\Longleftrightarrow\quad B = O \quad\Longleftrightarrow\quad PL = PLP.$$

Conclusion 2: The following assertions are pairwise equivalent as well.

(a′) $\mathcal{M}$ and $\mathcal{N}$ are both L-invariant.

(b′) $L = \left(\begin{smallmatrix} L|_{\mathcal{M}} & O \\ O & L|_{\mathcal{N}} \end{smallmatrix}\right)$.

(c′) L and P commute (i.e., $PL = LP$).

Let $\mathcal{M}$ and $\mathcal{N}$ be algebraic complements in a linear space $\mathcal{X}$. If a linear transformation L in $\mathcal{L}[\mathcal{X}]$ is represented as $L = \left(\begin{smallmatrix} A & O \\ O & D \end{smallmatrix}\right)$ in terms of the decomposition $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$ (as in (a′) above), where $A \in \mathcal{L}[\mathcal{M}]$ and $D \in \mathcal{L}[\mathcal{N}]$, then it is usual to write $L = A \oplus D$. For instance, the projection on $\mathcal{M}$ along $\mathcal{N}$,

which is represented as $P = \left(\begin{smallmatrix} I & O \\ O & O \end{smallmatrix}\right)$ with respect to the same decomposition $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$, is usually written as $P = I \oplus O$. These are examples of the following concept.

Let $\{\mathcal{X}_i\}_{i=1}^n$ be a finite family of linear spaces over the same scalar field and consider their direct sum $\bigoplus_{i=1}^n \mathcal{X}_i$. Let $\{L_i\}_{i=1}^n$ be a family of linear transformations such that each L_i lies in each $\mathcal{L}[\mathcal{X}_i]$. The *direct sum* of $\{L_i\}_{i=1}^n$, denoted by $\bigoplus_{i=1}^n L_i$, is the mapping of $\bigoplus_{i=1}^n \mathcal{X}_i$ into itself defined by

$$\bigoplus_{i=1}^n L_i(x_1, \ldots, x_n) \;=\; (L_1 x_1, \ldots, L_n x_n)$$

for every $(x_1, \ldots, x_n)$ in $\bigoplus_{i=1}^n \mathcal{X}_i$. It is readily verified that $\bigoplus_{i=1}^n L_i$ is linear (i.e., $\bigoplus_{i=1}^n L_i \in \mathcal{L}[\bigoplus_{i=1}^n \mathcal{X}_i]$) and also that, for every index i,

$$\Big(\bigoplus_{i=1}^n L_i\Big)\Big|_{\mathcal{X}_i} \;=\; L_i.$$

The above identity is a short notation for the following assertion. "If 0_i is the origin of each $\mathcal{X}_i$ and O_i is the unique (linear) transformation of $\{0_i\}$ onto itself, then each linear manifold $\{0_1\} \oplus \cdots \oplus \{0_{i-1}\} \oplus \mathcal{X}_i \oplus \{0_{i+1}\} \oplus \cdots \oplus \{0_n\}$ of $\bigoplus_{i=1}^n \mathcal{X}_i$ is invariant for $\bigoplus_{i=1}^n L_i$ and the restriction of $\bigoplus_{i=1}^n L_i$ to that invariant linear manifold is the direct sum $O_1 \oplus \cdots \oplus O_{i-1} \oplus L_i \oplus O_{i+1} \oplus \cdots \oplus O_n$". Of course, we shall always use the short notation. Conversely, if $L \in \mathcal{L}[\bigoplus_{i=1}^n \mathcal{X}_i]$ is such that each restriction $L|_{\mathcal{X}_i}$ lies in $\mathcal{L}[\mathcal{X}_i]$, then L is the direct sum of $\{L|_{\mathcal{X}_i}\}_{i=1}^n$. That is, if each $\mathcal{X}_i$ in $\bigoplus_{i=1}^n \mathcal{X}_i$ is invariant for $L \in \mathcal{L}[\bigoplus_{i=1}^n \mathcal{X}_i]$, then

$$L \;=\; \bigoplus_{i=1}^n L|_{\mathcal{X}_i}.$$

Summing up: Set $\mathcal{X} = \bigoplus_{i=1}^n \mathcal{X}_i$ and consider linear transformations L_i in $\mathcal{L}[\mathcal{X}_i]$ for each i and L in $\mathcal{L}[\mathcal{X}]$.

$$L = \bigoplus_{i=1}^n L_i \quad \text{ if and only if } \quad L_i = L|_{\mathcal{X}_i}$$

for every index i (so that each $\mathcal{X}_i$, viewed as a linear manifold of the linear space $\bigoplus_{i=1}^n \mathcal{X}_i$, is invariant for L). The linear transformations $\{L_i\}$ are referred to as the *direct summands* of L.

In particular, consider the decomposition $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$ of a linear space $\mathcal{X}$ into the direct sum of a pair of algebraic complements $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{X}$, and take linear transformations $L \in \mathcal{L}[\mathcal{X}]$, $A \in \mathcal{L}[\mathcal{M}]$, and $D \in \mathcal{L}[\mathcal{N}]$. Then

$$L = \begin{pmatrix} A & O \\ O & D \end{pmatrix} = A \oplus D \quad \text{ if and only if } \quad A = L|_{\mathcal{M}} \text{ and } D = L|_{\mathcal{N}}$$

(so that $\mathcal{M}$ and $\mathcal{N}$ are both L-invariant), where A and D are the direct summands of L with respect to the decomposition $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$.

Suggested Reading

Birkhoff and MacLane [1]
Brown and Pearcy [3]
Halmos [2], [5]
Herstein [1], [2]
Hoffman and Kunze [1]
Kaplansky [1]
Lax [2]
MacLane and Birkhoff [1]
Naylor and Sell [1]
Roman [1]
Simmons [1]
Taylor and Lay [1]

Problems

Problem 2.1. Let $\mathcal{X}$ be a linear space over a field $\mathbb{F}$. Take arbitrary α and β in $\mathbb{F}$ and arbitrary x, y, and z in $\mathcal{X}$. Verify the following propositions.

(a) $(-\alpha)x = -(\alpha x)$.

(b) $0x = 0 = \alpha 0$.

(c) $\alpha x = 0 \implies \alpha = 0$ or $x = 0$.

(d) $x + y = x + z \implies y = z$.

(e) $\alpha x = \alpha y \implies x = y$ if $\alpha \neq 0$.

(f) $\alpha x = \beta x \implies \alpha = \beta$ if $x \neq 0$.

Problem 2.2. Let $\mathcal{X}$ be a real or complex linear space. A subset C of $\mathcal{X}$ is *convex* if $\alpha x + (1-\alpha)y$ is in C for every x, y in C and every α in $[0,1]$. A vector $x \in \mathcal{X}$ is a *convex linear combination* of vectors in $\mathcal{X}$ if there exist a finite set $\{x_i\}_{i=1}^n$ of vectors in $\mathcal{X}$ and a finite family of nonnegative scalars $\{\alpha_i\}_{i=1}^n$ such that $x = \sum_{i=1}^n \alpha_i x_i$ and $\sum_{i=1}^n \alpha_i = 1$. If A is a subset of $\mathcal{X}$, then the intersection of all convex sets containing A is called the *convex hull* of A, denoted by $\mathrm{co}(A)$.

(a) Show that the intersection of an arbitrary nonempty collection of convex sets is convex.

(b) Show that $\mathrm{co}(A)$ is the smallest (in the inclusion ordering) convex set that includes A.

(c) Show that C is convex if and only if every convex linear combination of vectors in C belongs to C.

Hint: To verify that *every convex linear combination of vectors in a convex set C belongs to C*, proceed as follows. Note that the italicized result holds for any convex linear combination of two vectors in C (by definition of convex set). Suppose it holds for every convex linear combination of n vectors in C, for some $n \in \mathbb{N}$. This implies that $\alpha \sum_{i=1}^n \alpha^{-1}\alpha_i x_i + \alpha_{n+1}x_{n+1}$ lies in C whenever $\{x_i\}_{i=1}^{n+1} \subset C$ and $\sum_{i=1}^{n+1}\alpha_i = 1$ with $0 < \alpha_{n+1}$, where $\alpha = \sum_{i=1}^n \alpha_i$ (reason: $\sum_{i=1}^n \alpha^{-1}\alpha_i x_i \in C$). Conclude the proof by induction.

(d) Show that $\mathrm{co}(A)$ coincides with the set of all convex linear combinations of vectors in A.

Hint: Let $\mathrm{clc}(A)$ denote the set of all convex linear combinations of vectors in A. Verify that $\mathrm{clc}(A)$ is a convex set. Now use (b) and (c) to show that $\mathrm{co}(A) \subseteq \mathrm{clc}(A) \subseteq \mathrm{clc}(\mathrm{co}(A)) = \mathrm{co}(A)$.

Problem 2.3. Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of a linear space $\mathcal{X}$, and let A and B be linearly independent subsets of $\mathcal{M}$ and $\mathcal{N}$, respectively. If $\mathcal{M} \cap \mathcal{N} = \{0\}$, then $A \cup B$ is linearly independent.

Hint: If $a \in A$ is a linear combination of vectors in $A \cup B$, then $a = b + a'$ for some $a' \in \mathcal{M}$ and some $b \in \mathcal{N}$.

Problem 2.4. Let A be a linearly independent subset of a linear space $\mathcal{X}$. If B and C are subsets of A, then

$$\mathrm{span}\,(B \cap C) \;=\; \mathrm{span}\,B \cap \mathrm{span}\,C.$$

Hint: Show that $\mathrm{span}\,B \cap \mathrm{span}\,C \subseteq \mathrm{span}\,(B \cup C)$ by Proposition 2.3.

Problem 2.5. The cardinality of a linearly independent subset of a linear space $\mathcal{X}$ is less than or equal to the cardinality of a subset of $\mathcal{X}$ that spans $\mathcal{X}$.

Hint: That is, *if* $A \subseteq \mathcal{X}$ *is such that* $\mathrm{span}\,A = \mathcal{X}$, *then* $\#C \leq \#A$ *for every linearly independent* $C \subseteq \mathcal{X}$. Indeed, if $B \subseteq \mathcal{X}$ is a Hamel basis for $\mathcal{X}$, then show that $\#C \leq \#B \leq \#A$. (Apply Theorems 2.5, 2.6, and 2.7 — see Problems 1.21(a) and 1.22.) Note that this generalizes Claim 0 in the proof of Theorem 2.7 for subsets of arbitrary cardinality.

Problem 2.6. Let $\mathcal{X}$ be a linear space, and let $\mathcal{M}$ be a linear manifold of $\mathcal{X}$. Verify the following propositions.

(a) $\dim \mathcal{M} = 0$ if and only if $\mathcal{M} = \{0\}$.

(b) $\dim \mathcal{M} \leq \dim \mathcal{X}$. (*Hint*: Problem 2.5.)

Problem 2.7. If $\mathcal{M}$ is a proper linear manifold of a *finite-dimensional* linear space $\mathcal{X}$, then $\dim \mathcal{M} < \dim \mathcal{X}$. Prove the above statement and show that it does not hold for infinite-dimensional linear spaces.

Hint: Show that $\dim \mathcal{X}_0 = \dim \mathcal{X}$, where $\mathcal{X}$ is the linear space of Example 2.J and $\mathcal{X}_0 = \{x = (\xi_1, \xi_2, \xi_3, \ldots) \in \mathcal{X}\colon\ \xi_1 = 0\}$.

Problem 2.8. Let $\mathcal{X}$ be a nonzero linear space over an *infinite* field, and let B be a Hamel basis for $\mathcal{X}$. Recall that every nonzero vector x in $\mathcal{X}$ has a unique representation in terms of B. That is, for each $x \neq 0$ in $\mathcal{X}$ there exists

a unique nonempty finite subset B_x of B and a unique finite family of nonzero scalars $\{\alpha_b\}_{b\in B_x} \subset \mathbb{F}$ such that

$$x = \sum_{b\in B_x} \alpha_b b.$$

For each positive integer $n \in \mathbb{N}$ let X_n be the set of all *nonzero* vectors in $\mathcal{X}$ whose representations as a (finite) linear combination of vectors in B have exactly n (nonzero) summands. That is, for each $n \in \mathbb{N}$, set

$$X_n = \big\{x \in \mathcal{X}\colon\ \#B_x = n\big\}.$$

(a) Prove that $\#X_n = \#(\mathbb{F}\times B)$ for all $n \in \mathbb{N}$.

Hint: Show that $\#X_n = \#(\mathbb{F}^n\times B)$ and recall: if $\mathbb{F}$ is an infinite set, then $\#\mathbb{F}^n = \#\mathbb{F}$ (Problems 1.23 and 1.28).

(b) Apply Theorem 1.10 to show that $\#\big(\bigcup_{n\in\mathbb{N}} X_n\big) \le \#(\mathbb{F}\times B)$.

(c) Verify that $\{X_n\}_{n\in\mathbb{N}}$ is a partition of $\mathcal{X}\backslash\{0\}$.

Thus conclude from (b) and (c) (see Problem 1.28(a)) that

$$\#\mathcal{X} = \#(\mathbb{F}\times B) = \max\{\#\mathbb{F}, \dim\mathcal{X}\}.$$

Problem 2.9. Prove the following proposition, which is known as the *Principle of Superposition*. A mapping $L\colon \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are linear spaces over the same scalar field, is a linear transformation if and only if

$$L\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) = \sum_{i=1}^{n} \alpha_i L x_i$$

for all *finite* sets $\{x_i\}_{i=1}^n$ of vectors in $\mathcal{X}$ and all *finite* sets of scalars $\{\alpha_i\}_{i=1}^n$.

Problem 2.10. Let $L\colon \mathcal{X} \to \mathcal{Y}$ be a linear transformation.

(a) Show that the null space $\mathcal{N}(L)$ and the range $\mathcal{R}(L)$ of L are linear manifolds of the linear spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Moreover, show that they both are L-invariant. That is, $L(\mathcal{N}(L)) \subseteq \mathcal{N}(L)$ and $L(\mathcal{R}(L)) \subseteq \mathcal{R}(L)$.

Now set $\mathcal{X} = \mathcal{Y}$ and show that the positive integral powers of L are linear transformations. That is, $L^n \in \mathcal{L}[\mathcal{X}]$ for every $n \ge 1$ whenever $L \in \mathcal{L}[\mathcal{X}]$.

(b) Show that the linear manifolds $\mathcal{N}(L^n)$ and $\mathcal{R}(L^n)$ are L-invariant. That is, $L(\mathcal{N}(L^n)) \subseteq \mathcal{N}(L^n)$ and $L(\mathcal{R}(L^n)) \subseteq \mathcal{R}(L^n)$ for every $n \ge 1$.

(c) Show that L^n is injective or surjective if L is. That is, $\mathcal{N}(L) = \{0\}$ implies $\mathcal{N}(L^n) = \{0\}$, and $\mathcal{R}(L) = \mathcal{X}$ implies $\mathcal{R}(L^n) = \mathcal{X}$, for every $n \ge 1$.

Problem 2.11. Let $L\colon \mathcal{X} \to \mathcal{Y}$ be a linear transformation of a linear space $\mathcal{X}$ into a linear space $\mathcal{Y}$. Prove the following propositions.

(a) If $\mathcal{M}$ is a linear manifold of $\mathcal{X}$, then $L(\mathcal{M})$ is a linear manifold of $\mathcal{Y}$ (i.e., the linear image of a linear manifold is a linear manifold).

(b) If $\mathcal{N}$ is a linear manifold of $\mathcal{Y}$, then $L^{-1}(\mathcal{N})$ is a linear manifold of $\mathcal{X}$ (i.e., the inverse image of a linear manifold under a linear transformation is again a linear manifold).

Problem 2.12. Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces, and let $L\colon \mathcal{X} \to \mathcal{Y}$ be a linear transformation. Show that the following assertions are equivalent.

(a) $A \subseteq \mathcal{X}$ is a linear manifold whenever $L(A) \subseteq \mathcal{Y}$ is a linear manifold.

(b) $\mathcal{N}(L) = \{0\}$.

Hint: Give a direct proof for (b)⇒(a) by using Problems 1.3(d) and 2.11(b). Give a contrapositive proof for (a)⇒(b) — recall: if x is a nonzero vector in $\mathcal{X}$, then $\{x\}$ is not a linear manifold of $\mathcal{X}$.

Problem 2.13. Prove that the set $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$ of all linear transformations of a linear space $\mathcal{X}$ into a linear space $\mathcal{Y}$ is itself a linear space (over the same common field of $\mathcal{X}$ and $\mathcal{Y}$) when vector addition and scalar multiplication in $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$ are defined pointwise as in Example 2.F.

Problem 2.14. Show that the restriction $L|_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{Y}$ of a linear transformation $L\colon \mathcal{X} \to \mathcal{Y}$ to a linear manifold $\mathcal{M}$ of $\mathcal{X}$ is itself a linear transformation. Moreover, also show that $\mathcal{N}(L|_{\mathcal{M}}) = \mathcal{M} \cap \mathcal{N}(T)$.

Problem 2.15. Show that the composition of two linear transformations is again a linear transformation. That is, if $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are linear spaces over the same scalar field, and if $L \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ and $T \in \mathcal{L}[\mathcal{Y}, \mathcal{Z}]$, then $TL \in \mathcal{L}[\mathcal{X}, \mathcal{Z}]$. Moreover, also show that $\mathcal{R}(TL) = T(\mathcal{R}(L))$.

Problem 2.16. Let $L\colon \mathcal{X} \to \mathcal{Y}$ be a linear transformation. It is trivially verified that, if L is surjective, then $\dim \mathcal{R}(L) = \dim \mathcal{Y}$. Now verify that, if L is injective, then $\dim \mathcal{R}(L) = \dim \mathcal{X}$.

Problem 2.17. Let $L\colon \mathcal{X} \to \mathcal{Y}$ be a linear transformation of a linear space $\mathcal{X}$ into a linear space $\mathcal{Y}$. The dimension of the range of L is the *rank* of L, and the dimension of the null space of L is the *nullity* of L. Show that rank and nullity are related as follows.

$$\dim \mathcal{N}(L) + \dim \mathcal{R}(L) \;=\; \dim \mathcal{X}.$$

Hint: Suppose $L \neq O$. Let B_N be a Hamel basis for $\mathcal{N}(L)$ and let B_X be a Hamel basis for $\mathcal{X}$ that properly includes B_N. (Theorem 2.5 — why is the inclusion proper?) Set $B_M = B_X \backslash B_N$ and $\mathcal{M} = \operatorname{span} B_M$. Show that B_M is a Hamel basis for $\mathcal{M}$, the restriction of L to $\mathcal{M}$ is injective (since $\mathcal{N}(L)$ and $\mathcal{M}$

are algebraic complements by Proposition 2.15), and $L|_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{R}(L)$ is an isomorphism (Problem 2.14). Thus $\dim \mathcal{R}(L) = \dim \mathcal{M}$ (Theorem 2.12), and $\dim \mathcal{X} = \#B_X = \#B_N + \#B_M = \dim \mathcal{N}(L) + \dim \mathcal{M}$ (Problem 1.30).

Problem 2.18. If $\dim \mathcal{R}(L)$ is finite, then L is called a *finite-dimensional* (or a *finite-rank*) *linear transformation*. Clearly, if $\mathcal{Y}$ is a finite-dimensional linear space, then every $L \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is finite dimensional. Verify that, if $\mathcal{X}$ is a finite-dimensional linear space, then every $L \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is finite dimensional. Moreover, if $L\colon \mathcal{X} \to \mathcal{Y}$ is a finite-dimensional linear transformation (so that $\mathcal{R}(L)$ is a finite-dimensional linear manifold of $\mathcal{Y}$), then show that

(a) L is injective if and only if $\dim \mathcal{R}(L) = \dim \mathcal{X}$,

(b) L is surjective if and only if $\dim \mathcal{R}(L) = \dim \mathcal{Y}$,

(c) L is injective if and only if it is surjective, whenever $\dim \mathcal{X} = \dim \mathcal{Y}$.

Problem 2.19. Let $\mathcal{X}$ be a linear space over a field $\mathbb{F}$ and let $\mathcal{X}^{\mathbb{N}_0}$ be the linear space (over the same field $\mathbb{F}$) of all $\mathcal{X}$-valued sequences $\{x_n\}_{n\in\mathbb{N}_0}$. Take a linear transformation of $\mathcal{X}$ into itself, $A \in \mathcal{L}[\mathcal{X}]$, and an arbitrary sequence $u = \{u_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{X}^{\mathbb{N}_0}$. Consider the (unique) sequence $x = \{x_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{X}^{\mathbb{N}_0}$ which is recursively defined as follows. Set $x_0 = u_0$ and, for each $n \in \mathbb{N}_0$, set

$$x_{n+1} \,=\, Ax_n + u_{n+1}.$$

As usual, let A^n denote the composition of A with itself n times for each integer $n \geq 0$, with $A^0 = I$, the identity in $\mathcal{L}[\mathcal{X}]$. Prove by induction that

$$x_n \,=\, \sum_{i=0}^{n} A^{n-i} u_i$$

for every $n \in \mathbb{N}_0$. Now let $L\colon \mathcal{X}^{\mathbb{N}_0} \to \mathcal{X}^{\mathbb{N}_0}$ be the map that assigns to each sequence u in $\mathcal{X}^{\mathbb{N}_0}$ this unique sequence x in $\mathcal{X}^{\mathbb{N}_0}$, so that

$$x \,=\, Lu.$$

Show that L is a linear transformation of $\mathcal{X}^{\mathbb{N}_0}$ into itself. The recursive equation (or the *difference equation*) $x_{n+1} = Ax_n + u_{n+1}$ is called a *discrete linear dynamical system* because L is *linear*. Its unique solution is given by $x = Lu$ (i.e., $x_n = \sum_{i=0}^{n} A^{n-i} u_i$ for every $n \in \mathbb{N}_0$).

Problem 2.20. Let $\mathbb{F}$ denote either the real or complex field, and let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces over $\mathbb{F}$. For any polynomial p (in one variable in $\mathbb{F}$ with coefficients α_i in $\mathbb{F}$ and of finite order n; i.e., $p(z) = \sum_{i=0}^{n} \alpha_i z^i$ for $z \in \mathbb{F}$), set

$$p(L) \,=\, \sum_{i=0}^{n} \alpha_i L^i,$$

where $L \in \mathcal{L}[\mathcal{X}]$ and $\{\alpha_i\}_{i=0}^n$ is a finite set of coefficients in $\mathbb{F}$ (note: $L^0 = I$). Show that $p(L) \in \mathcal{L}[\mathcal{X}]$ (in particular, $L^n \in \mathcal{L}[\mathcal{X}]$ for each $n \geq 0$) for every $L \in \mathcal{L}[\mathcal{X}]$. Take $L \in \mathcal{L}[\mathcal{X}]$, $K \in \mathcal{L}[\mathcal{Y}]$, and $M \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$. Prove the implication.

(a) If $ML = KM$, then $Mp(L) = p(K)M$ for any polynomial p.

Thus conclude: $p(L)$ *is similar to* $p(K)$ *whenever* L *is similar to* K. A linear transformation L in $\mathcal{L}[\mathcal{X}]$ is called *nilpotent* if $L^n = O$ for some integer $n \in \mathbb{N}$, and *algebraic* if $p(L) = O$ for some polynomial p. It is clear that every nilpotent linear transformation is algebraic. Prove the following propositions.

(b) A linear transformation is similar to an algebraic (nilpotent) linear transformation if and only if it is itself algebraic (nilpotent).

(c) Sum and composition of nilpotent linear transformations are not necessarily nilpotent.

Hint: The matrices $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $L = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ in $\mathcal{L}[\mathbb{C}^2]$ are both nilpotent. $L + T$ is an involution. LT and TL are idempotent.

Problem 2.21. Let $\mathbb{F}$ denote either the real or complex field, and let $\mathcal{X}$ be a linear space over $\mathbb{F}$. A subset K of $\mathcal{X}$ is a *cone* (with vertex at the origin) if $\alpha x \in K$ whenever $x \in K$ and $\alpha \geq 0$. Recall the definition of a convex set in Problem 2.2 and verify the following assertions.

(a) Every linear manifold is a convex cone.

(b) The union of nonzero disjoint linear manifolds is a nonconvex cone.

Let S be a nonempty set and consider the linear space $\mathbb{F}^S$. Show that

(c) $\{x \in \mathbb{F}^S\colon\ x(s) \geq 0 \text{ for all } s \in S\}$ is a convex cone in $\mathbb{F}^S$.

Problem 2.22. Show that the implication (a)⇒(b) in Theorem 2.14 does not generalize to three linear manifolds, say $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{R}$, if we simply assume that they are pairwise disjoint. (*Hint*: $\mathbb{R}^3$.)

Problem 2.23. Let $\{\mathcal{M}_i\}_{i=1}^n$ be a finite collection of linear manifolds of a linear space $\mathcal{X}$. Show that the following assertions are equivalent.

(a) $\mathcal{M}_i \cap \sum_{j=1, j\neq i}^n \mathcal{M}_j = \{0\}$ for every $i = 1, \ldots, n$.

(b) For each x in $\sum_{i=1}^n \mathcal{M}_i$ there exists a unique n-tuple $(x_1, \ldots, x_n)$ in $\prod_{i=1}^n \mathcal{M}_i$ such that $x = \sum_{i=1}^n x_i$.

Hint: (a)⇒(b) for $n = 2$ by Theorem 2.14. Take any integer $n > 2$ and suppose (a)⇒(b) for every $2 \leq m < n$. Show that, if (a) holds true for $m + 1$, then (b) holds true for $m + 1$. Now conclude the proof of (a)⇒(b) by induction in n. Next show that (b)⇒(a) by Theorem 2.14.

Problem 2.24. Let $\{\mathcal{M}_i\}_{i=1}^n$ be a finite collection of linear manifolds of a linear space $\mathcal{X}$, and let B_i be a Hamel basis for each $\mathcal{M}_i$. If $\mathcal{M}_i \cap \sum_{j=1,j\neq i}^n \mathcal{M}_j = \{0\}$ for every $i = 1, \ldots, n$, then $\bigcup_{i=1}^n B_i$ is a Hamel basis for $\sum_{i=1}^n \mathcal{M}_i$. Prove.

Hint: Apply Proposition 2.15 for $n = 2$. Now use the hint to Problem 2.23.

Problem 2.25. Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of a linear space.

(a) If $\mathcal{M}$ and $\mathcal{N}$ are disjoint, then

$$\dim(\mathcal{M} \oplus \mathcal{N}) = \dim(\mathcal{M} + \mathcal{N}) = \dim \mathcal{M} + \dim \mathcal{N}.$$

Hint: Problem 1.30, Theorem 2.14, and Proposition 2.15.

(b) If $\mathcal{M}$ and $\mathcal{N}$ are finite-dimensional, then

$$\dim(\mathcal{M} + \mathcal{N}) = \dim \mathcal{M} + \dim \mathcal{N} - \dim(\mathcal{M} \cap \mathcal{N}).$$

Problem 2.26. Let $\mathcal{M}$ be a proper linear manifold of a linear space $\mathcal{X}$ so that $\mathcal{M} \in \mathcal{L}at(\mathcal{X})\backslash\{\mathcal{X}\}$. Consider the inclusion ordering of $\mathcal{L}at(\mathcal{X})$. Show that

$$\mathcal{M} \text{ is maximal in } \mathcal{L}at(\mathcal{X})\backslash\{\mathcal{X}\} \iff \operatorname{codim} \mathcal{M} = 1.$$

Problem 2.27. Let φ be a nonzero linear functional on a linear space $\mathcal{X}$ (i.e., a nonzero element of $\mathcal{X}'$, the algebraic dual of $\mathcal{X}$). Prove the following results.

(a) $\mathcal{N}(\varphi)$ is maximal in $\mathcal{L}at(\mathcal{X})\backslash\{\mathcal{X}\}$.

That is, the null space of every nonzero linear functional in $\mathcal{X}'$ is a maximal proper linear manifold of $\mathcal{X}$. Conversely, if $\mathcal{M}$ is a maximal linear manifold in $\mathcal{L}at(\mathcal{X})\backslash\{\mathcal{X}\}$, then there exists a nonzero φ in $\mathcal{X}'$ such that $\mathcal{M} = \mathcal{N}(\varphi)$.

(b) Every maximal element of $\mathcal{L}at(\mathcal{X})\backslash\{\mathcal{X}\}$ is the null space of some nonzero φ in $\mathcal{X}'$.

Problem 2.28. Let $\mathcal{X}$ be a linear space over a field $\mathbb{F}$. The set

$$H_{\varphi,\alpha} = \{x \in \mathcal{X}: \ \varphi(x) = \alpha\},$$

determined by a *nonzero* φ in $\mathcal{X}'$ and a scalar α in $\mathbb{F}$, is called a *hyperplane* in $\mathcal{X}$. It is clear that $H_{\varphi,0}$ coincides with $\mathcal{N}(\varphi)$ but $H_{\varphi,\alpha}$ is not a linear manifold of $\mathcal{X}$ if α is a nonzero scalar. A *linear variety* is a translation of a proper linear manifold. That is, a linear variety V is a subset of $\mathcal{X}$ that coincides with the coset of x modulo $\mathcal{M}$,

$$V = \mathcal{M} + x = \{y \in \mathcal{X}: \ y = z + x \text{ for some } z \in \mathcal{M}\},$$

for some $x \in \mathcal{X}$ and some $\mathcal{M} \in \mathcal{L}at(\mathcal{X})\backslash\{\mathcal{X}\}$. If $\mathcal{M}$ is maximal in $\mathcal{L}at(\mathcal{X})\backslash\{\mathcal{X}\}$, then $\mathcal{M} + x$ is called a *maximal linear variety*. Show that a hyperplane is precisely a maximal linear variety.

Problem 2.29. Let $\mathcal{X}$ be a linear space over a field $\mathbb{F}$, and let P and E be projections in $\mathcal{L}[\mathcal{X}]$. Suppose $E \neq O$, and let α be an arbitrary *nonzero* scalar in $\mathbb{F}$. Prove the following proposition.

(a) $P + \alpha E$ is a projection if and only if $PE + EP = (1 - \alpha)E$.

Moreover, if $P + \alpha E$ is a projection, then show that

(b) P and E commute (i.e., $PE = EP$), and so PE is a projection;

(c) $PE = O$ if and only if $\alpha = 1$ and $PE \neq O$ if and only if $\alpha = -1$.

Therefore,

(d) $P + \alpha E$ is a projection implies $\alpha = 1$ or $\alpha = -1$.

Thus conclude:

(e) $P + E$ is a projection if and only if $PE = EP = O$,

(f) $P - E$ is a projection if and only if $PE = EP = E$.

Next prove that, for arbitrary projections P and E in $\mathcal{L}[\mathcal{X}]$,

(g) $\mathcal{R}(P) \cap \mathcal{R}(E) \subseteq \mathcal{R}(PE) \cap \mathcal{R}(EP)$.

Furthermore, if P and E commute, then show that

(h) PE is a projection and $\mathcal{R}(P) \cap \mathcal{R}(E) = \mathcal{R}(PE)$,

and so (still under the assumption that E and P commute),

(i) $PE = O$ if and only if $\mathcal{R}(P) \cap \mathcal{R}(E) = \{0\}$.

Problem 2.30. An *algebra* (or a *linear algebra*) is a linear space $\mathcal{A}$ that is also a ring with respect to a second binary operation on $\mathcal{A}$ called *product* (notation: $xy \in \mathcal{A}$ is the product of $x \in \mathcal{A}$ and $y \in \mathcal{A}$). The product is related to scalar multiplication by the property

$$\alpha(xy) \;=\; (\alpha x)y \;=\; x(\alpha y)$$

for every $x, y \in \mathcal{A}$ and every scalar α. We shall refer to a *real* or *complex algebra* if $\mathcal{A}$ is a real or complex linear space. Recall that this new binary operation on $\mathcal{A}$ (i.e., the product in the ring $\mathcal{A}$) is associative,

$$x(yz) \;=\; (xy)z,$$

and distributive with respect to vector addition,

$$x(y + z) = xy + xz \quad \text{and} \quad (y + z)x = yx + zx,$$

for every x, y, and z in $\mathcal{A}$. If $\mathcal{A}$ possesses a neutral element 1 under the product operation (i.e., if there exists $1 \in \mathcal{A}$ such that $x1 = 1x = x$ for every $x \in \mathcal{A}$), then $\mathcal{A}$ is said to be an *algebra with identity* (or a *unital algebra*). Such a

neutral element 1 is called the *identity* (or *unit*) of $\mathcal{A}$. If $\mathcal{A}$ is an algebra with identity, and if $x \in \mathcal{A}$ has an *inverse* (denoted by x^{-1}) with respect to the product operation (i.e., if there exists $x^{-1} \in \mathcal{A}$ such that $xx^{-1} = x^{-1}x = 1$), then x is an *invertible* element of $\mathcal{A}$. Recall that the identity is unique if it exists, and so is the inverse of an invertible element of $\mathcal{A}$. If the product operation is commutative, then $\mathcal{A}$ is said to be a *commutative algebra*.

(a) Let $\mathcal{X}$ be a linear space of dimension greater than 1. Show that $\mathcal{L}[\mathcal{X}]$ is a noncommutative algebra with identity when the product in $\mathcal{L}[\mathcal{X}]$ is interpreted as composition (i.e., $LT = L \circ T$ for every $L, T \in \mathcal{L}[\mathcal{X}]$). The identity I in $\mathcal{L}[\mathcal{X}]$ is precisely the neutral element under the product operation. L is an invertible of $\mathcal{L}[\mathcal{X}]$ if and only if L is injective and surjective.

A *subalgebra* of $\mathcal{A}$ is a linear manifold $\mathcal{M}$ of $\mathcal{A}$ (when $\mathcal{A}$ is viewed as a linear space) which is an algebra in its own right with respect to the product operation of $\mathcal{A}$ (i.e., $uv \in \mathcal{M}$ whenever $u \in \mathcal{M}$ and $v \in \mathcal{M}$). A subalgebra $\mathcal{M}$ of $\mathcal{A}$ is a *left ideal* of $\mathcal{A}$ if $ux \in \mathcal{M}$ whenever $u \in \mathcal{M}$ and $x \in \mathcal{A}$. A *right ideal* of $\mathcal{A}$ is a subalgebra $\mathcal{M}$ of $\mathcal{A}$ such that $xu \in \mathcal{M}$ whenever $x \in \mathcal{A}$ and $u \in \mathcal{M}$. An *ideal* (or a *two-sided ideal* or a *bilateral ideal*) of $\mathcal{A}$ is a subalgebra $\mathcal{I}$ of $\mathcal{A}$ that is both a left ideal and a right ideal.

(b) Let $\mathcal{X}$ be an infinite-dimensional linear space. Show that the set of all finite-dimensional linear transformations in $\mathcal{L}[\mathcal{X}]$ is a proper left ideal of $\mathcal{L}[\mathcal{X}]$ with no identity. (*Hint*: Problem 2.25(b).)

(c) Show that, if $\mathcal{A}$ is an algebra and $\mathcal{I}$ is a proper ideal of $\mathcal{A}$, then the quotient space $\mathcal{A}/\mathcal{I}$ of $\mathcal{A}$ modulo $\mathcal{I}$ is an algebra. This is called the *quotient algebra* of $\mathcal{A}$ with respect to $\mathcal{I}$. If $\mathcal{A}$ has an identity 1, then the coset $1 + \mathcal{I}$ is the identity of $\mathcal{A}/\mathcal{I}$.

Hint: Recall that vector addition and scalar multiplication in the linear space $\mathcal{A}/\mathcal{I}$ are defined by

$$(x + \mathcal{I}) + (y + \mathcal{I}) = (x + y) + \mathcal{I},$$
$$\alpha(x + \mathcal{I}) = \alpha x + \mathcal{I},$$

for every $x, y \in \mathcal{A}$ and every scalar α (see Example 2.H). Now show that the product of cosets in $\mathcal{A}/\mathcal{I}$ can be likewise defined by

$$(x + \mathcal{I})(y + \mathcal{I}) = xy + \mathcal{I}$$

for every $x, y \in \mathcal{A}$ (i.e., if $x' = x + u$ and $y' = y + v$, with $x, y \in \mathcal{A}$ and $u, v \in \mathcal{I}$, then there exists $z \in \mathcal{I}$ such that $x'y' + w = xy + z$ for any $w \in \mathcal{I}$, whenever $\mathcal{I}$ is a two-sided ideal of $\mathcal{A}$).

Problem 2.31. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over the same scalar field. A linear transformation $\Phi\colon \mathcal{A} \to \mathcal{B}$ (of the linear spaces $\mathcal{A}$ into the linear space $\mathcal{B}$) that preserves products — i.e., such that $\Phi(xy) = \Phi(x)\Phi(y)$ for every x, y in $\mathcal{A}$

— is called a *homomorphism* (or an *algebra homomorphism*) of $\mathcal{A}$ into $\mathcal{B}$. A *unital homomorphism* between unital algebras is one that takes the identity of $\mathcal{A}$ to the identity of $\mathcal{B}$. If Φ is an isomorphism (of the linear spaces $\mathcal{A}$ onto the linear space $\mathcal{B}$) and also a homomorphism (of the algebra $\mathcal{A}$ onto the algebra $\mathcal{B}$), then it is an *algebra isomorphism* of $\mathcal{A}$ onto $\mathcal{B}$. In this case $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic algebras*.

(a) Let $\{e_\gamma\}$ be a Hamel basis for the linear space $\mathcal{A}$. Show that a linear transformation $\Phi\colon \mathcal{A} \to \mathcal{B}$ is an algebra isomorphism if and only if $\Phi(e_\alpha e_\beta) = \Phi(e_\alpha)\Phi(e_\alpha)$ for every pair $\{e_\alpha, e_\beta\}$ of elements of the basis $\{e_\gamma\}$.

(b) Let $\mathcal{I}$ be an ideal of $\mathcal{A}$ and let $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the natural mapping of $\mathcal{A}$ onto the quotient algebra $\mathcal{A}/\mathcal{I}$. Show that π is a homomorphism such that $\mathcal{N}(\pi) = \mathcal{I}$. (*Hint*: Example 2.H.)

(c) Let $\mathcal{X}$ and $\mathcal{Y}$ be isomorphic linear spaces and let $W\colon \mathcal{X} \to \mathcal{Y}$ be an isomorphism between them. Consider the mapping $\Phi\colon \mathcal{L}[\mathcal{X}] \to \mathcal{L}[\mathcal{Y}]$ defined by $\Phi(L) = WLW^{-1}$ for every $L \in \mathcal{L}[\mathcal{X}]$. Show that Φ is an algebra isomorphism of the algebra $\mathcal{L}[\mathcal{X}]$ onto the algebra $\mathcal{L}[\mathcal{Y}]$.

Problem 2.32. Here is a useful result, which holds in any ring with identity (sometimes referred to as the *Matrix Inversion Lemma*). Take $A, B \in \mathcal{L}[\mathcal{X}]$ on a linear space $\mathcal{X}$. If $I - AB$ is invertible, then so is $I - BA$, and

$$(I - BA)^{-1} \;=\; I + B(I - AB)^{-1}A.$$

Hint: For every $A, B, C \in \mathcal{L}[\mathcal{X}]$ verify that

(a) $(I + BCA)(I - BA) = I - BA + BCA - BCABA$,

(b) $(I - BA)(I + BCA) = I - BA + BCA - BABCA$,

(c) $I - BA + BCA - B(C - I)A = I$.

Now set $C = (I - AB)^{-1}$ so that $C(I - AB) = I = (I - AB)C$, and hence

(d) $CAB = C - I = ABC$.

Thus conclude that

(e) $(I + BCA)(I - BA) = I = (I - BA)(I + BCA)$.

Problem 2.33. Take a linear transformation $L \in \mathcal{L}[\mathcal{X}]$ on a linear space $\mathcal{X}$ and consider its nonnegative integral powers L^n. Verify that, for every $n \geq 0$,

$$\mathcal{N}(L^n) \subseteq \mathcal{N}(L^{n+1}) \quad \text{and} \quad \mathcal{R}(L^{n+1}) \subseteq \mathcal{R}(L^n).$$

Let n_0 be an arbitrary nonnegative integer. Prove the following propositions.

(a) If $\mathcal{N}(L^{n_0+1}) = \mathcal{N}(L^{n_0})$, then $\mathcal{N}(L^{n+1}) = \mathcal{N}(L^n)$ for every integer $n \geq n_0$.

(b) If $\mathcal{R}(L^{n_0+1}) = \mathcal{R}(L^{n_0})$, then $\mathcal{R}(L^{n+1}) = \mathcal{R}(L^n)$ for every integer $n \geq n_0$.

Hint: Rewrite the statements in (a) and (b) as follows.

(a) If $\mathcal{N}(L^{n_0+1}) = \mathcal{N}(L^{n_0})$, then $\mathcal{N}(L^{n_0+k+1}) = \mathcal{N}(L^{n_0+k})$ for every $k \geq 1$.

(b) If $L^{n_0+1}(\mathcal{X}) = L^{n_0}(\mathcal{X})$, then $L^{n_0+k+1}(\mathcal{X}) = L^{n_0+k}(\mathcal{X})$ for every $k \geq 1$.

Show that (a) holds for $k = 1$. Now show that the conclusion in (a) holds for $k+1$ whenever it holds for k. Similarly, show that (b) holds for $k = 1$, then show that the conclusion in (b) holds for $k+1$ whenever it holds for k. Thus conclude the proof of (a) and (b) by induction.

Problem 2.34. Set $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \infty$, the set of all *extended nonnegative integers* with its natural (extended) ordering. The previous problem suggests the following definitions. The *ascent* of $L \in \mathcal{L}[\mathcal{X}]$ (notation: $\operatorname{asc}(L)$) is the least nonnegative integer such that $\mathcal{N}(L^{n+1}) = \mathcal{N}(L^n)$, and the *descent* of L (notation: $\operatorname{dsc}(L)$) is the least nonnegative integer such that $\mathcal{R}(L^{n+1}) = \mathcal{R}(L^n)$:

$$\operatorname{asc}(L) = \min\left\{n \in \overline{\mathbb{N}}_0\colon\ \mathcal{N}(L^{n+1}) = \mathcal{N}(L^n)\right\},$$
$$\operatorname{dsc}(L) = \min\left\{n \in \overline{\mathbb{N}}_0\colon\ \mathcal{R}(L^{n+1}) = \mathcal{R}(L^n)\right\}.$$

It is plain that

$$\operatorname{asc}(L) = 0 \iff \mathcal{N}(L) = \{0\},$$
$$\operatorname{dsc}(L) = 0 \iff \mathcal{R}(L) = \mathcal{X}.$$

Now prove the following propositions.

(a) $\operatorname{asc}(L) < \infty$ and $\operatorname{dsc}(L) = 0$ implies $\operatorname{asc}(L) = 0$.

Hint: Suppose $\operatorname{dsc}(L) = 0$ (i.e., suppose $\mathcal{R}(L) = \mathcal{X}$). If $\operatorname{asc}(L) \neq 0$ (i.e., if $\mathcal{N}(L) \neq \{0\}$), then take $0 \neq x_1 \in \mathcal{N}(L) \cap \mathcal{R}(L)$ and x_2, x_3 in $\mathcal{R}(L) = \mathcal{X}$ such that $x_1 = Lx_2$ and $x_2 = Lx_3$, and so $x_1 = L^2x_3$. Proceed by induction to construct a sequence $\{x_n\}_{n\geq 1}$ of vectors in $\mathcal{X} = \mathcal{R}(L)$ such that $x_n = Lx_{n+1}$ and $0 \neq x_1 = L^n x_{n+1} \in \mathcal{N}(L)$, and so $L^{n+1}x_{n+1} = 0$. Then $x_{n+1} \in \mathcal{N}(L^{n+1}) \backslash N(T^n)$ for each $n \geq 1$, and $\operatorname{asc}(L) = \infty$ by Problem 2.33.

(b) $\operatorname{asc}(L) < \infty$ and $\operatorname{dsc}(L) < \infty$ implies $\operatorname{asc}(L) = \operatorname{dsc}(L)$.

Hint: Set $m = \operatorname{dsc}(L)$, so that $\mathcal{R}(L^m) = \mathcal{R}(L^{m+1})$, and set $T = L|_{\mathcal{R}(L^m)}$. Since $\mathcal{R}(L^m)$ is L-invariant, $T \in \mathcal{L}[\mathcal{R}(L^m)]$ (Problem 2.10(b)). Verify that $\mathcal{R}(T) = T(\mathcal{R}(L^m)) = \mathcal{R}(TL^m) = \mathcal{R}(L^{m+1}) = \mathcal{R}(L^m)$ (see Problem 2.15). Thus conclude that $\operatorname{dsc}(T) = 0$. Since $\operatorname{asc}(T) < \infty$ (because $\operatorname{asc}(L) < \infty$), it follows by (a) that $\operatorname{asc}(T) = 0$. That is, $\mathcal{N}(T) = \{0\}$. Take $x \in \mathcal{N}(L^{m+1})$ and set $y = L^m x$ in $\mathcal{R}(L^m)$. Show that $Ty = L^{m+1}x = 0$, so $y = 0$, and hence $x \in \mathcal{N}(L^m)$. Therefore, $N(L^{m+1}) \subseteq \mathcal{N}(L^m)$. Use Problem 2.33 to conclude that $\operatorname{asc}(L) \leq m$. On the other hand, suppose $m \neq 0$ (otherwise apply (a)) and take z in $\mathcal{R}(L^{m-1})\backslash\mathcal{R}(L^m)$ so that $Lz = L(L^{m-1}u) = L^m u$ is in $\mathcal{R}(L^m)$ for $u \in \mathcal{X}$. Since $L^m(\mathcal{R}(L^m)) = \mathcal{R}(L^{2m}) = \mathcal{R}(L^m)$, infer that $Lz = L^m v$ for $v \in \mathcal{R}(L^m)$. Verify that $L^m(u - v) = 0$ and $L^{m-1}(u - v) = z - L^{m-1}v \neq 0$ (reason: since $v \in \mathcal{R}(L^m)$, $L^{m-1}v \in \mathcal{R}(L^{2m-1}) = \mathcal{R}(L^m)$ and $z \notin \mathcal{R}(L^m)$). Thus $(u - v) \in \mathcal{N}(L^m)\backslash\mathcal{N}(L^{m-1})$, and so $\operatorname{asc}(L) \geq m$.

Problem 2.35. Consider the setup of the previous problem. If $\operatorname{asc}(L)$ and $\operatorname{dsc}(L)$ are both finite, then they are equal by Problem 2.34(b). Set $m = \operatorname{asc}(L) = \operatorname{dsc}(L)$ in $\mathbb{N}$. Show that the linear manifolds $\mathcal{R}(L^m)$ and $\mathcal{N}(T^m)$ of the linear space $\mathcal{X}$ are algebraic complements of each other. That is,

$$\mathcal{R}(L^m) \cap \mathcal{N}(T^m) = \{0\} \quad \text{and} \quad \mathcal{X} = \mathcal{R}(L^m) \oplus \mathcal{N}(T^m).$$

Hint: If y is in $\mathcal{R}(T^m) \cap \mathcal{N}(L^m)$, then $y = L^m x$ for some $x \in \mathcal{X}$ and $L^m y = 0$. Verify that $x \in \mathcal{N}(L^{2m}) = \mathcal{N}(L^m)$, and infer that $y = 0$. Now consider the hint to Problem 2.34(b) with $T = L|_{\mathcal{R}(L^m)} \in \mathcal{L}[\mathcal{R}(L^m)]$. Since $\mathcal{R}(T) = \mathcal{R}(L^m)$, it follows that $\mathcal{R}(T^m) = \mathcal{R}(L^m)$ (Problem 2.10(c)). Take any $x \in \mathcal{X}$. Verify that there exists $u \in \mathcal{R}(L^m)$ such that $T^m u = L^m u = L^m x$, and so $v = x - u$ is in $\mathcal{N}(L^m)$. Thus $x = u + v \in \mathcal{R}(T^m) + \mathcal{N}(L^m)$. Finally, use Theorem 2.14.

3

Topological Structures

The basic concept behind the subject of point-set topology is the notion of "closeness" between two points in a set X. In order to get a numerical gauge of how close together two points in X may be, we shall provide an extra structure to X, viz., a topological structure, that again goes beyond its purely set-theoretic structure. For most of our purposes the notion of closeness associated with a metric will be sufficient, and this leads to the concept of "metric space": a set upon which a "metric" is defined. The metric-space structure that a set acquires when a metric is defined on it is a special kind of topological structure. Metric spaces comprise the kernel of this chapter, but general topological spaces are also introduced.

3.1 Metric Spaces

A *metric* (or *metric function*, or *distance function*) is a real-valued function on the Cartesian product of an arbitrary set with itself that has the following four properties, called the *metric axioms*.

Definition 3.1. Let X be an arbitrary set. A real-valued function d on the Cartesian product $X \times X$,

$$d : X \times X \to \mathbb{R},$$

is a *metric* on X if the following conditions are satisfied for all x, y, z in X.

$$\begin{array}{lll} \text{(i)} & d(x,y) \geq 0 \quad \text{and} \quad d(x,x) = 0 & (\textit{nonnegativeness}), \\ \text{(ii)} & d(x,y) = 0 \quad \text{only if} \quad x = y & (\textit{positiveness}), \\ \text{(iii)} & d(x,y) = d(y,x) & (\textit{symmetry}), \\ \text{(iv)} & d(x,y) \leq d(x,z) + d(z,y) & (\textit{triangle inequality}). \end{array}$$

A set X equipped with a metric on it is a *metric space*.

A word on notation and terminology. The value of the metric d on a pair of points of X is called the *distance* between those points. According to the above definition a metric space actually is an ordered pair (X, d) where X is

C.S. Kubrusly, *The Elements of Operator Theory*, DOI 10.1007/978-0-8176-4998-2_3,

© Springer Science+Business Media, LLC 2011

an arbitrary set, called the *underlying set* of the metric space (X, d), and d is a metric function defined on it. We shall often refer to a metric space in several ways. Sometimes we shall speak of X itself as a metric space when the metric d is either clear in the context or is immaterial. In this case we shall simply say "X is a metric space". On the other hand, in order to avoid confusion among different metric spaces, we may occasionally insert a subscript on the metrics. For instance, (X, d_X) and (Y, d_Y) will stand for metric spaces where X and Y are the respective underlying sets, d_X denotes the metric on X, and d_Y the metric on Y. Moreover, if a set X can be equipped with more than one metric, say d_1 and d_2, then (X, d_1) and (X, d_2) will represent *different* metric spaces with the same underlying set X. In brief, a metric space is an arbitrary set with an additional structure defined by means of a metric d. Such an additional structure is *the topological structure induced by the metric d*.

If (X, d) is a metric space, and if A is a subset of X, then it is easy to show that the restriction $d|_{A\times A}: A\times A \to \mathbb{R}$ of the metric d to $A\times A$ is a metric on A — called the *relative metric*. Equipped with the relative metric, A is a *subspace* of X. We shall drop the subscript $A\times A$ from $d|_{A\times A}$ and say that (A, d) is a subspace of (X, d). Thus a subspace of a metric space (X, d) is a subset A of the underlying set X equipped with the *relative metric*, which is itself a metric space. Roughly speaking, A inherits the metric of (X, d). If (A, d) is a subspace of (X, d) and A is a proper subset of X, then (A, d) is said to be a *proper subspace* of the metric space (X, d).

Example 3.A. The function $d: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by

$$d(\alpha, \beta) \;=\; |\alpha - \beta|$$

for every $\alpha, \beta \in \mathbb{R}$ is a metric on $\mathbb{R}$. That is, it satisfies all the metric axioms in Definition 3.1, where $|\alpha| = (\alpha^2)^{\frac{1}{2}}$ stands for the *absolute value* of $\alpha \in \mathbb{R}$. This is the *usual metric* on $\mathbb{R}$. The real line $\mathbb{R}$ equipped with its usual metric is the most important concrete metric space. If we refer to $\mathbb{R}$ as a metric space without specifying a metric on it, then it is understood that $\mathbb{R}$ has been equipped with its usual metric. Similarly, the function $d: \mathbb{C}\times\mathbb{C} \to \mathbb{R}$ given by $d(\xi, \upsilon) = |\xi - \upsilon|$ for every $\xi, \upsilon \in \mathbb{C}$ is a metric on $\mathbb{C}$. Again, $|\xi| = (\overline{\xi}\xi)^{\frac{1}{2}}$ stands for the absolute value (or *modulus*) of $\xi \in \mathbb{C}$, with the upper bar denoting *complex conjugate* of a complex number. This is the *usual metric* on $\mathbb{C}$. More generally, let $\mathbb{F}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and let $\mathbb{F}^n$ be the set of all ordered n-tuples of scalars in $\mathbb{F}$. For each real number $p \geq 1$ consider the function $d_p: \mathbb{F}^n\times\mathbb{F}^n \to \mathbb{R}$ defined by

$$d_p(x, y) \;=\; \Big(\sum_{i=1}^{n} |\xi_i - \upsilon_i|^p\Big)^{\frac{1}{p}},$$

and also the function $d_\infty: \mathbb{F}^n\times\mathbb{F}^n \to \mathbb{R}$ given by

$$d_\infty(x, y) \;=\; \max_{1\leq i\leq n} |\xi_i - \upsilon_i|,$$

for every $x = (\xi_1, \ldots, \xi_n)$ and $y = (\upsilon_1, \ldots, \upsilon_n)$ in $\mathbb{F}^n$. These are metrics on $\mathbb{F}^n$. Indeed, all the metric axioms up to the triangle inequality are trivially verified. The triangle inequality follows from the Minkowski inequality (see Problem 3.4(a)). Note that $(\mathbb{Q}^n, d_p)$ is a subspace of $(\mathbb{R}^n, d_p)$ and $(\mathbb{Q}^n, d_\infty)$ is a subspace of $(\mathbb{R}^n, d_\infty)$. The special (very special, really) metric space $(\mathbb{R}^n, d_2)$ is called n-dimensional *Euclidean space* and d_2 is the *Euclidean metric* on $\mathbb{R}^n$. The metric space $(\mathbb{C}^n, d_2)$ is called n-dimensional *unitary space*. The singular role played by the metric d_2 will become clear in due course.

Recall that the notion of a bounded subset was defined for partially ordered sets in Section 1.5. In particular, boundedness is well defined for subsets of the simply ordered set $(\mathbb{R}, \leq)$, the set of all real numbers $\mathbb{R}$ equipped with its natural ordering $\leq$ (see Section 1.6). Let us introduce a suitable and common notation for a subset of $\mathbb{R}$ that is bounded above. Since the simply ordered set $\mathbb{R}$ is a boundedly complete lattice (Example 1.C), it follows that a subset R of $\mathbb{R}$ is bounded above if and only if it has a supremum, $\sup R$, in $\mathbb{R}$. In such a case we shall write $\sup R < \infty$. Thus the notation $\sup R < \infty$ simply means that R is a subset of $\mathbb{R}$ which is bounded above. Otherwise (i.e., if $R \subseteq \mathbb{R}$ is not bounded above) we write $\sup R = \infty$. With this in mind we shall extend the notion of boundedness from $(\mathbb{R}, \leq)$ to a metric space (X, d) as follows. A nonempty subset A of X is a *bounded set* in the metric space (X, d) if

$$\sup_{x,y \in A} d(x, y) < \infty.$$

That is, A is bounded in (X, d) if $\{d(x, y) \in \mathbb{R}\colon\ x, y \in A\}$ is a bounded subset of $(\mathbb{R}, \leq)$ or, equivalently, if the set $\{d(x, y) \in \mathbb{R}\colon\ x, y \in A\}$ is bounded above in $\mathbb{R}$ (because $0 \leq d(x, y)$ for every $x, y \in X$). An *unbounded set* is, of course, a set A that is not bounded in (X, d). The *diameter* of a nonempty bounded subset A of X (notation: $\operatorname{diam}(A)$) is defined by

$$\operatorname{diam}(A) = \sup_{x,y \in A} d(x, y)$$

so that $\operatorname{diam}(A) < \infty$ whenever a nonempty set A is bounded in (X, d). By convention the empty set $\varnothing$ is bounded and $\operatorname{diam}(\varnothing) = 0$. If A is unbounded we write $\operatorname{diam}(A) = \infty$. Let F be a function of a set S to a metric space (Y, d). F is a *bounded function* if its range, $\mathcal{R}(F) = F(S)$, is a bounded subset in (Y, d); that is, if

$$\sup_{s,t \in S} d(F(s), F(t)) < \infty.$$

Note that R is bounded as a subset of the metric space $\mathbb{R}$ equipped with its usual metric if and only if R is bounded as a subset of the simply ordered set $\mathbb{R}$ equipped with its natural ordering. Thus the notion of a *bounded subset of* $\mathbb{R}$ and the notion of a *bounded real-valued function* on an arbitrary set S are both unambiguously defined.

Proposition 3.2. *Let S be a set and let $\mathbb{F}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Equip $\mathbb{F}$ with its usual metric. A function $\varphi \in \mathbb{F}^S$* (i.e., *a function $\varphi: S \to \mathbb{F}$*) *is bounded if and only if*

$$\sup_{s\in S} |\varphi(s)| < \infty.$$

Proof. Consider a function φ from a set S to the field $\mathbb{F}$. Take s and t arbitrary in S, and let d be the usual metric on $\mathbb{F}$ (see Example 3.A). Since $\varphi(s), \varphi(t) \in \mathbb{F}$, if follows by Problem 3.1(a) that

$$\begin{aligned} |\varphi(s)| - |\varphi(t)| \le \big||\varphi(s)| - |\varphi(t)|\big| &= \big|d(\varphi(s),0) - d(0,\varphi(t))\big| \\ &\le d(\varphi(s),\varphi(t)) = |\varphi(s) - \varphi(t)| \le |\varphi(s)| + |\varphi(t)|. \end{aligned}$$

If $\sup_{s\in S}|\varphi(s)| < \infty$ (i.e., if $\{|\varphi(s)| \in \mathbb{R}\colon\ s \in S\}$ is bounded above), then

$$d(\varphi(s),\varphi(t)) \le 2 \sup_{s\in S} |\varphi(s)|,$$

and hence $\sup_{s,t\in S} d(\varphi(s),\varphi(t)) \le 2\sup_{s\in S}|\varphi(s)|$ so that the function φ is bounded. On the other hand, if $\sup_{s,t\in S} d(\varphi(s),\varphi(t)) < \infty$, then

$$|\varphi(s)| \le \sup_{s,t\in S} d(\varphi(s),\varphi(t)) + |\varphi(t)| \quad \text{for every} \quad s,t \in S,$$

and so the real number $\sup_{s,t\in S} d(\varphi(s),\varphi(t)) + |\varphi(t)|$ is an upper bound for $\{|\varphi(s)| \in \mathbb{R}\colon\ s \in S\}$ for every $t \in S$. Thus $\sup_{s\in S}|\varphi(s)| < \infty$. □

Example 3.B. For each real number $p \ge 1$, let ℓ^p_+ denote the set of all scalar-valued (real or complex) infinite sequences $\{\xi_k\}_{k\in\mathbb{N}}$ in $\mathbb{C}^{\mathbb{N}}$ (or in $\mathbb{C}^{\mathbb{N}_0}$) such that $\sum_{k=1}^{\infty}|\xi_k|^p < \infty$. We shall refer to this condition by saying that the elements of ℓ^p_+ are *p-summable sequences*. Notation: $\sum_{k=1}^{\infty}|\xi_k|^p = \sup_{n\in\mathbb{N}}\sum_{k=1}^{n}|\xi_k|^p$. Thus, according to Proposition 3.2, $\sum_{k=1}^{\infty}|\xi_k|^p < \infty$ means that the nonnegative sequence $\{\sum_{k=1}^{n}|\xi_k|^p\}_{n\in\mathbb{N}}$ is bounded as a real-valued function on $\mathbb{N}$. Note that, if $\{\xi_k\}_{k\in\mathbb{N}}$ and $\{\upsilon_k\}_{k\in\mathbb{N}}$ are arbitrary sequences in ℓ^p_+, then the Minkowski inequality (Problem 3.4(b)) ensures that $\sum_{k=1}^{\infty}|\xi_k - \upsilon_k|^p < \infty$. Hence we may consider the function $d_p\colon \ell^p_+ \times \ell^p_+ \to \mathbb{R}$ given by

$$d_p(x,y) = \Big(\sum_{k=1}^{\infty}|\xi_k - \upsilon_k|^p\Big)^{\frac{1}{p}}$$

for every $x = \{\xi_k\}_{k\in\mathbb{N}}$ and $y = \{\upsilon_k\}_{k\in\mathbb{N}}$ in ℓ^p_+. We claim that d_p is a metric on ℓ^p_+. Indeed, as in Example 3.A, all the metric axioms up to the triangle inequality are readily verified, and the triangle inequality follows from the Minkowski inequality (Problem 3.4(b)). Therefore (ℓ^p_+, d_p) is a metric space for each $p \ge 1$, and the metric d_p is referred to as the *usual metric* on ℓ^p_+. Now let ℓ^∞_+ denote the set of all scalar-valued *bounded sequences*; that is, the set of all real or complex-valued sequences $\{\xi_k\}_{k\in\mathbb{N}}$ such that $\sup_{k\in\mathbb{N}}|\xi_k| < \infty$. Again,

the Minkowski inequality (Problem 3.4(b)) ensures that $\sup_{k\in\mathbb{N}}|\xi_k - \upsilon_k| < \infty$ whenever $\{\xi_k\}_{k\in\mathbb{N}}$ and $\{\upsilon_k\}_{k\in\mathbb{N}}$ lie in ℓ_+^∞, and hence we may consider the function $d_\infty\colon \ell_+^\infty \times \ell_+^\infty \to \mathbb{R}$ defined by

$$d_\infty(x,y) \;=\; \sup_{k\in\mathbb{N}} |\xi_k - \upsilon_k|$$

for every $x = \{\xi_k\}_{k\in\mathbb{N}}$ and $y = \{\upsilon_k\}_{k\in\mathbb{N}}$ in ℓ_+^∞. Proceeding as before (using the Minkowski inequality to verify the triangle inequality), it follows that $(\ell_+^\infty, d_\infty)$ is a metric space, and the metric d_∞ is referred to as the *usual metric* on ℓ_+^∞. These metric spaces are the natural generalizations (for infinite sequences) of the metric spaces considered in Example 3.A, and again the metric space (ℓ_+^2, d_2) will play a central role in the forthcoming chapters. There are counterparts of ℓ_+^p and ℓ_+^∞ for nets in $\mathbb{C}^{\mathbb{Z}}$. In fact, for each $p \geq 1$ let ℓ^p denote the set of all scalar-valued (real or complex) nets $\{\xi_k\}_{k\in\mathbb{Z}}$ such that $\sum_{k=-\infty}^{\infty}|\xi_k|^p < \infty$ (i.e., such that the nonnegative sequence $\{\sum_{k=-n}^{n}|\xi_k|^p\}_{n\in\mathbb{N}_0}$ is bounded), and let ℓ^∞ denote the set of all bounded nets in $\mathbb{C}^{\mathbb{Z}}$ (i.e., the set of all scalar-valued nets $\{\xi_k\}_{k\in\mathbb{Z}}$ such that $\sup_{k\in\mathbb{Z}}|\xi_k| < \infty$). The functions $d_p\colon \ell^p \times \ell^p \to \mathbb{R}$ and $d_\infty\colon \ell^\infty \times \ell^\infty \to \mathbb{R}$, given by

$$d_p(x,y) \;=\; \Big(\sum_{k=-\infty}^{\infty} |\xi_k - \upsilon_k|^p\Big)^{\frac{1}{p}}$$

for every $x = \{\xi_k\}_{k\in\mathbb{Z}}$ and $y = \{\upsilon_k\}_{k\in\mathbb{Z}}$ in ℓ^p, and

$$d_\infty(x,y) \;=\; \sup_{k\in\mathbb{Z}} |\xi_k - \upsilon_k|$$

for every $x = \{\xi_k\}_{k\in\mathbb{Z}}$ and $y = \{\upsilon_k\}_{k\in\mathbb{Z}}$ in ℓ^∞, are metrics on ℓ^p (for each $p \geq 1$) and on ℓ^∞, respectively.

Let (X,d) be a metric space. If x is an arbitrary point in X, and A is an arbitrary nonempty subset of X, then the *distance from x to A* is the number

$$d(x,A) \;=\; \inf_{a\in A} d(x,a)$$

in $\mathbb{R}$. If A and B are nonempty subsets of X, then the *distance between A and B* is the real number

$$d(A,B) \;=\; \inf_{a\in A,\, b\in B} d(a,b).$$

Example 3.C. Let S be a set and let (Y,d) be a metric space. Let $B[S,Y]$ denote the subset of Y^S consisting of all bounded mappings of S into (Y,d). According to Problem 3.6,

$$\sup_{s\in S} d(f(s),g(s)) \;\leq\; \operatorname{diam}(\mathcal{R}(f)) + \operatorname{diam}(\mathcal{R}(g)) + d(\mathcal{R}(f),\mathcal{R}(g))$$

so that $\sup_{s\in S} d(f(s), g(s)) \in \mathbb{R}$ for every $f, g \in B[S, Y]$. Thus we may consider the function $d_\infty : B[S, Y] \times B[S, Y] \to \mathbb{R}$ defined by

$$d_\infty(f, g) = \sup_{s\in S} d(f(s), g(s))$$

for each pair of mappings $f, g \in B[S, Y]$. This is a metric on $B[S, Y]$. Indeed, d_∞ clearly satisfies conditions (i), (ii), and (iii) in Definition 3.1. To verify the triangle inequality (condition (iv)), proceed as follows. Take an arbitrary $s \in S$ and note that, if f, g, and h are mappings in $B[S, Y]$, then (by the triangle inequality in (Y, d))

$$d(f(s), g(s)) \leq d(f(s), h(s)) + d(h(s), g(s)) \leq d_\infty(f, h) + d_\infty(h, g).$$

Hence $d_\infty(f, g) \leq d_\infty(f, h) + d_\infty(f, g)$, and therefore $(B[S, Y], d_\infty)$ is a metric space. The metric d_∞ is referred to as the *sup-metric* on $B[S, Y]$. Note that the metric spaces $(\ell^\infty_+, d_\infty)$ and (ℓ^∞, d_∞) of the previous example are particular cases of $(B[S, Y], d_\infty)$. Indeed, $\ell^\infty_+ = B[\mathbb{N}, \mathbb{C}]$ and $\ell^\infty = B[\mathbb{Z}, \mathbb{C}]$.

Example 3.D. The general concept of a continuous mapping between metric spaces will be defined in the next section. However, assuming that the reader is familiar with the particular notion of a scalar-valued continuous function of a real variable, we shall now consider the following example. Let $C[0, 1]$ denote the set of all scalar-valued (real or complex) continuous functions defined on the interval $[0, 1]$. For every $x, y \in C[0, 1]$ set

$$d_p(x, y) = \Big(\int_0^1 |x(t) - y(t)|^p \, dt \Big)^{\frac{1}{p}},$$

where p is a real number such that $p \geq 1$, and

$$d_\infty(x, y) = \sup_{t\in[0,1]} |x(t) - y(t)|.$$

These are metrics on the set $C[0, 1]$. That is, $d_p : C[0, 1] \times C[0, 1] \to \mathbb{R}$ and $d_\infty : C[0, 1] \times C[0, 1] \to \mathbb{R}$ are well-defined functions that satisfy all the conditions in Definition 3.1. Indeed, nonnegativeness and symmetry are trivially verified, positiveness for d_p is ensured by the continuity of the elements in $C[0, 1]$, and the triangle inequality comes by the Minkowski inequality (Problem 3.4(c)) as follows. For every $x, y, z \in C[0, 1]$,

$$\begin{aligned} d_p(x, y) &= \Big(\int_0^1 |x(t) - z(t) + z(t) - y(t)|^p \, dt \Big)^{\frac{1}{p}} \\ &\leq \Big(\int_0^1 |x(t) - z(t)|^p \, dt \Big)^{\frac{1}{p}} + \Big(\int_0^1 |z(t) - y(t)|^p \, dt \Big)^{\frac{1}{p}} \\ &= d_p(x, z) + d_p(z, y), \end{aligned}$$

and

$$\begin{aligned} d_\infty(x,y) &= \sup_{t\in[0,1]} |x(t)-z(t)+z(t)-y(t)| \\ &\le \sup_{t\in[0,1]} |x(t)-z(t)| + \sup_{t\in[0,1]} |z(t)-y(t)| \\ &= d_\infty(x,z)+d_\infty(z,y). \end{aligned}$$

Let $B[0,1]$ denote the set $B[S,Y]$ of Example 3.C when $S=[0,1]$ and $Y=\mathbb{F}$ (with $\mathbb{F}$ standing either for the real field $\mathbb{R}$ or for the complex field $\mathbb{C}$). Since $C[0,1]$ is a subset of $B[0,1]$ (reason: every scalar-valued continuous function defined on $[0,1]$ is bounded), it follows that $(C[0,1],d_\infty)$ is a subspace of the metric space $(B[0,1],d_\infty)$. The metric d_∞ is called the *sup-metric* on $C[0,1]$ and, as we shall see later, the "sup" in its definition in fact is a "max".

Let X be an arbitrary set. A real-valued function $d\colon X\times X\to\mathbb{R}$ on the Cartesian product $X\times X$ is a *pseudometric* on X if it satisfies the axioms (i), (iii), and (iv) of Definition 3.1. A *pseudometric space* (X,d) is a set X equipped with a pseudometric d. The difference between a metric space and a pseudometric space is that a pseudometric does not necessarily satisfy the axiom (ii) in Definition 3.1 (i.e., it is possible for a pseudometric to vanish at a pair (x,y) even though $x\ne y$). However, given a pseudometric space (X,d), there exists a natural way to obtain a metric space $(\widetilde{X},\widetilde{d})$ associated with (X,d), where $\widetilde{d}$ is a metric on $\widetilde{X}$ associated with the pseudometric d on X. Indeed, as we shall see next, a pseudometric d induces an equivalence relation $\sim$ on X, and $\widetilde{X}$ is precisely the quotient space $X/\!\sim$ (i.e., the collection of all equivalence classes $[x]$ with respect to $\sim$ for every x in X).

Proposition 3.3. *Let d be a pseudometric on a set X and consider the relation $\sim$ on X defined as follows. If x and x' are elements of X, then*

$$x'\sim x \quad \text{if} \quad d(x',x)=0.$$

The relation $\sim$ is an equivalence relation on X with the following property. For every x, x', y, and y' in X,

$$x'\sim x \ \text{ and } \ y'\sim y \quad \text{imply} \quad d(x',y')=d(x,y).$$

Let $X/\!\sim$ be the quotient space of X modulo $\sim$. For each pair $([x],[y])$ in $X/\!\sim\times X/\!\sim$ set

$$\widetilde{d}([x],[y]) = d(x,y)$$

for an arbitrary pair (x,y) in $[x]\times[y]$. This defines a function

$$\widetilde{d}\colon X/\!\sim\times X/\!\sim\ \to\mathbb{R},$$

which is a metric on the quotient space $X/\!\sim$.

Proof. It is clear that the relation $\sim$ on X is reflexive and symmetric because a pseudometric is nonnegative and symmetric. Transitivity comes from the

triangle inequality: $0 \le d(x,x'') \le d(x,x') + d(x',x'')$ for every $x, x', x'' \in X$. Thus $\sim$ is an equivalence relation on X. Moreover, if $x' \sim x$ and $y' \sim y$ (i.e., if $x' \in [x]$ and $y' \in [y]$), then the triangle inequality in the pseudometric space (X,d) ensures that

$$d(x,y) \;\le\; d(x,x') + d(x',y') + d(y',y) \;=\; d(x',y')$$

and, similarly, $d(x',y') \le d(x,y)$. Therefore

$$d(x',y') = d(x,y) \quad \text{whenever} \quad x' \sim x \;\text{ and }\; y' \sim y.$$

That is, given a pair of equivalence classes $[x] \subseteq X$ and $[y] \subseteq X$, the restriction of d to $[x]\times[y] \subseteq X \times X$, $d|_{[x]\times[y]}\colon [x]\times[y] \to \mathbb{R}$, is a constant function. Thus, for each pair $([x],[y])$ in $X/\!\sim \times X/\!\sim$, set

$$\widetilde{d}([x],[y]) \;=\; d|_{[x]\times[y]}(x,y) \;=\; d(x,y)$$

for any $x \in [x]$ and $y \in [y]$. This defines a function $\widetilde{d}\colon X/\!\sim \times X/\!\sim \to \mathbb{R}$ which is nonnegative, symmetric, and satisfies the triangle inequality (along with d). The reason for defining equivalence classes is to ensure positiveness for $\widetilde{d}$ from the nonnegativeness of the pseudometric d: if $\widetilde{d}([x],[y]) = 0$, then $d(x,y) = 0$ so that $x \sim y$, and hence $[x] = [y]$. □

Example 3.E. The previous example exhibited different metric spaces with the same underlying set of all scalar-valued continuous functions on the interval $[0,1]$. Here we allow discontinuous functions as well. Let S be a nondegenerate interval of the real line $\mathbb{R}$ (typical examples: $S = [0,1]$ or $S = \mathbb{R}$). For each real number $p \ge 1$ let $r^p(S)$ denote the set of all scalar-valued (real or complex) *p-integrable functions* on S. In this context, "p-integrable" means that a scalar-valued function x on S is Riemann integrable and $\int_S |x(s)|^p\, ds < \infty$ (i.e., the Riemann integral $\int_S |x(s)|^p\, ds$ exists as a number in $\mathbb{R}$). Consider the function $\delta_p\colon r^p(S)\times r^p(S) \to \mathbb{R}$ given by

$$\delta_p(x,y) \;=\; \Big(\int_S |x(s) - y(s)|^p\, ds \Big)^{\frac{1}{p}}$$

for every $x, y \in r^p(S)$. The Minkowski inequality (see Problem 3.4(c)) ensures that the function δ_p is well defined, and also that it satisfies the triangle inequality. Moreover, nonnegativeness and symmetry are readily verified, but positiveness fails. For instance, if 0 denotes the null function on $S = [0,1]$ (i.e., $0(s) = 0$ for all $s \in S$), and if $x(s) = 1$ for $s = \frac{1}{2}$ and zero elsewhere, then $\delta_p(x,0) = 0$ although $x \ne 0$ (since $x(\frac{1}{2}) \ne 0(\frac{1}{2})$). Thus δ_p actually is a pseudometric on $r^p(S)$ rather than a metric, so that $(r^p(S), \delta_p)$ is a pseudometric space. However, if we "redefine" $r^p(S)$ by endowing it with a new notion of equality, different from the usual pointwise equality for functions, then perhaps we might make δ_p a metric on such a "redefinition" of $r^p(S)$. This in fact is the idea behind Proposition 3.3. Consider the equivalence relation $\sim$ on $r^p(S)$

defined as in Proposition 3.3: if x and x' are functions in $r^p(S)$, then $x' \sim x$ if $\delta_p(x', x) = 0$. Now set $R^p(S) = r^p(S)/\sim$, the collection of all equivalence classes $[x] = \{x' \in r^p(S)\colon\ \delta_p(x', x) = 0\}$ for every $x \in r^p(S)$. Thus, by Proposition 3.3, $(R^p(S), d_p)$ is a metric space with the metric $d_p\colon R^p(S) \times R^p(S) \to \mathbb{R}$ defined by $d_p([x], [y]) = \delta_p(x, y)$ for arbitrary $x \in [x]$ and $y \in [y]$ for every $[x], [y]$ in $R^p(S)$. Note that equality in $R^p(S)$ is interpreted in the following way: if $[x]$ and $[y]$ are equivalence classes in $R^p(S)$, and if x and y are arbitrary functions in $[x]$ and $[y]$, respectively, then $[x] = [y]$ if and only if $\delta_p(x, y) = 0$. If x is any element of $[x]$, then, in this context, it is usual to write x for $[x]$ and hence $d_p(x, y)$ for $d_p([x], [y])$. Thus, following the common usage, we shall write $x \in R^p(S)$ instead of $[x] \in R^p(S)$, and also

$$d_p(x, y) \;=\; \Big(\int_S |x(s) - y(s)|^p \, ds \Big)^{\frac{1}{p}}$$

for every $x, y \in R^p(S)$ to represent the metric d_p on $R^p(S)$. This is referred to as the *usual metric* on $R^p(S)$. Note that, according to this convention, $x = y$ in $R^p(S)$ if and only if $d_p(x, y) = 0$.

3.2 Convergence and Continuity

The notion of convergence, together with the notion of continuity, plays a central role in the theory of metric spaces.

Definition 3.4. Let (X, d) be a metric space. An X-valued sequence $\{x_n\}$ (or a sequence in X indexed by $\mathbb{N}$ or by $\mathbb{N}_0$) *converges* to a point x in X if for each real number $\varepsilon > 0$ there exists a positive integer n_ε such that

$$n \geq n_\varepsilon \quad \text{implies} \quad d(x_n, x) < \varepsilon.$$

If $\{x_n\}$ converges to $x \in X$, then $\{x_n\}$ is said to be a *convergent sequence* and x is said to be the *limit* of $\{x_n\}$ (usual notation: $\lim x_n = x$, $\lim_n x_n = x$, or $x_n \to x$; also $\lim_{n\to\infty} x_n = x$, $x_n \to x$ as $n \to \infty$, or even $x_n \xrightarrow[n\to\infty]{} x$).

As defined above, convergence depends on the metric d that equips the metric space (X, d). To emphasize the role played by the metric d, it is usual to refer to an X-valued convergent sequence $\{x_n\}$ by saying that $\{x_n\}$ *converges in* (X, d). If an X-valued sequence $\{x_n\}$ does not converge in (X, d) to the point $x \in X$, then we shall write $x_n \not\to x$. Clearly, if $x_n \not\to x$, then the sequence $\{x_n\}$ either converges in (X, d) to another point different from x or does not converge in (X, d) to any x in X. The notion of convergence in a metric space (X, d) is a natural extension of the ordinary notion of convergence in the real line $\mathbb{R}$ (equipped with its usual metric). Indeed, let (X, d) be a metric space, and consider an X-valued sequence $\{x_n\}$. Let x be an arbitrary point in X and consider the real-valued sequence $\{d(x_n, x)\}$. According to Definition 3.4,

$$x_n \to x \quad \text{if and only if} \quad d(x_n, x) \to 0.$$

This shows at once that *a convergent sequence in a metric space has a unique limit* (as we had anticipated in Definition 3.4 by referring to *the* limit of a convergent sequence). In fact, if a and b are points in X, then the triangle inequality says that $0 \le d(a,b) \le d(a,x_n) + d(x_n,b)$ for every index n. Thus, if $x_n \to a$ and $x_n \to b$ (i.e., $d(a,x_n) \to 0$ and $d(x_n,b) \to 0$), then $d(a,b) = 0$ (see Problems 3.10(c,e)). Hence $a = b$.

Example 3.F. Let $C[0,1]$ denote the set of all scalar-valued continuous functions on the interval $[0,1]$, and let $\{x_n\}$ be a $C[0,1]$-valued sequence such that, for each integer $n \ge 1$, $x_n\colon [0,1] \to \mathbb{R}$ is defined by

$$x_n(t) = \begin{cases} 1 - nt & t \in [0, \frac{1}{n}], \\ 0, & t \in (\frac{1}{n}, 1]. \end{cases}$$

Consider the metric spaces $(C[0,1], d_p)$ for $p \ge 1$ and $(C[0,1], d_\infty)$ which were introduced in Example 3.D. It is readily verified that the sequence $\{x_n\}$ converges in $(C[0,1], d_p)$ to the null function $0 \in C[0,1]$ for every $p \ge 1$. Indeed, take an arbitrary $p \ge 1$ and note that

$$d_p(x_n, 0) = \Big(\int_0^1 |x_n(t)|^p \, dt \Big)^{\frac{1}{p}} < \big(\tfrac{1}{n}\big)^{\frac{1}{p}}$$

for each $n \ge 1$. Since the sequence of real numbers $\big\{(\frac{1}{n})^{\frac{1}{p}}\big\}$ converges to zero (when the real line $\mathbb{R}$ is equipped with its usual metric — apply Definition 3.4), it follows that $d_p(x_n, 0) \to 0$ as $n \to \infty$ (Problem 3.10(c)). That is,

$$x_n \to 0 \quad \text{in} \quad (C[0,1], d_p).$$

However, $\{x_n\}$ does not converge in the metric space $(C[0,1], d_\infty)$. Indeed, if there exists $x \in C[0,1]$ such that $d_\infty(x_n, x) \to 0$, then it is easy to show that $x(0) = 1$ and $x(\varepsilon) = 0$ for all $\varepsilon \in (0,1]$. Hence $x \notin C[0,1]$, which is a contradiction. Conclusion: There is no $x \in C[0,1]$ such that $x_n \to x$ in $(C[0,1], d_\infty)$. Equivalently,

$$\{x_n\} \text{ does not converge in } (C[0,1], d_\infty).$$

Example 3.G. Consider the metric space $(B[S,Y], d_\infty)$ introduced in Example 3.C, where $B[S,Y]$ denotes the set of all bounded functions of a set S into a metric space (Y,d), and d_∞ is the sup-metric. Let $\{f_n\}$ be a $B[S,Y]$-valued sequence (i.e., a sequence of functions in $B[S,Y]$), and let f be an arbitrary function in $B[S,Y]$. Since

$$0 \le d(f_n(s), f(s)) \le \sup_{s \in S} d(f_n(s), f(s)) = d_\infty(f_n, f)$$

for each index n and all $s \in S$, it follows by Problem 3.10(c) that

$$f_n \to f \text{ in } (B[S,Y], d_\infty) \quad \text{implies} \quad f_n(s) \to f(s) \text{ in } (Y,d)$$

for every $s \in S$. If $f_n \to f$ in $(B[S,Y], d_\infty)$, then we say that the sequence $\{f_n\}$ of functions in $B[S,Y]$ *converges uniformly* to the function f in $B[S,Y]$. If $f_n(s) \to f(s)$ in (Y,d) for every $s \in S$, then we say that $\{f_n\}$ *converges pointwise* to f. Thus *uniform convergence implies pointwise convergence* (to the same limit), but the converse fails. For instance, set $S = [0,1]$, $Y = \mathbb{F}$ (either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ equipped with their usual metric d), and set $B[0,1] = B[[0,1], \mathbb{F}]$. Recall that the metric space $(C[0,1], d_\infty)$ of Example 3.D is a subspace of $(B[0,1], d_\infty)$. (Indeed, every scalar-valued continuous function defined on a bounded closed interval is a bounded function — we shall consider a generalized version of this well-known result later in this chapter.) If $\{g_n\}$ is a sequence of functions in $C[0,1]$ given by

$$g_n(s) \;=\; \frac{s^2}{s^2 + (1 - ns)^2}$$

for each integer $n \geq 1$ and every $s \in S = [0,1]$, then it is easy to show that

$$g_n(s) \to 0 \quad \text{in} \quad (\mathbb{R}, d)$$

for every $s \in [0,1]$ (cf. Definition 3.4), so that the sequence $\{g_n\}$ of functions in $C[0,1]$ converges pointwise to the null function $0 \in C[0,1]$. However, since $0 \leq g_n(s) \leq 1$ for all $s \in [0,1]$ and $g_n(\frac{1}{n}) = 1$, for each $n \geq 1$, it follows that

$$d_\infty(g_n, 0) \;=\; \sup_{s \in [0,1]} |g_n(s)| \;=\; 1$$

for every $n \geq 1$. Hence $\{g_n\}$ does not converge uniformly to the null function, and so it does not converge uniformly to any limit (for, if it converges uniformly, then it converges pointwise to the same limit). Thus the $C[0,1]$-valued sequence $\{g_n\}$ does not converge in the metric space $(C[0,1], d_\infty)$. Briefly,

$$\{g_n\} \text{ does not converge in } (C[0,1], d_\infty).$$

But it converges to the null function $0 \in C[0,1]$ in the metric spaces $(C[0,1], d_p)$ of Example 3.D. That is, for every $p \geq 1$,

$$g_n \to 0 \quad \text{in} \quad (C[0,1], d_p).$$

Indeed, $g_n(0) = 0$, $g_n(s) = (1 + f_n(s))^{-1}$ with $f_n(s) = (n - \frac{1}{s})^2 \geq 0$ for every $s \in (0,1]$, and $g_n(\frac{1}{n}) = 1$. Note that $f_n(\frac{1}{n}) = 0$, $f_n(1) = (n-1)^2 \geq 0$, and $f_n'(s) = 0$ only at $s = \frac{1}{n}$. Thus each f_n is decreasing on $(0, \frac{1}{n}]$ and increasing on $[\frac{1}{n}, 1]$, and hence each g_n is increasing on $[0, \frac{1}{n}]$ and decreasing on $[\frac{1}{n}, 1]$. Therefore, for an arbitrary $\varepsilon \in (0, \frac{1}{2}]$, and for every $n \geq 2$ and every $p \geq 1$,

$$\int_0^{\frac{1}{n}} |g_n(s)|^p \, ds + \int_{\frac{1}{n}}^{\frac{1}{n}+\varepsilon} |g_n(s)|^p \, ds + \int_{\frac{1}{n}+\varepsilon}^{1} |g_n(s)|^p \, ds \;\leq\; \tfrac{1}{n} + \varepsilon + g_n(\tfrac{1}{n} + \varepsilon)^p.$$

Since $f_n(\frac{1}{n}+\varepsilon) = \varepsilon^2 n^4(1+\varepsilon n)^{-2}$, it follows that $g_n(\frac{1}{n}+\varepsilon) \to 0$ as $n \to \infty$. This and the above inequality ensure that $\int_0^1 |g_n(s)|^p\, ds \to 0$, and so

$$d_p(g_n, 0) \to 0 \quad \text{as} \quad n \to \infty \quad \text{for every} \quad p \geq 1.$$

Proposition 3.5. *An X-valued sequence converges in a metric space (X, d) to $x \in X$ if and only if every subsequence of it converges in (X, d) to x.*

Proof. Let $\{x_n\}$ be an X-valued sequence. If every subsequence of it converges to a fixed limit, then, in particular, the sequence itself converges to the same limit. On the other hand, suppose $x_n \to x$ in (X, d). That is, for every $\varepsilon > 0$ there exists a positive integer n_ε such that $n \geq n_\varepsilon$ implies $d(x_n, x) < \varepsilon$. Take an arbitrary subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$. Since $k \leq n_k$ (reason: $\{n_k\}_{k\in\mathbb{N}}$ is a strictly increasing subsequence of the sequence $\{n\}_{n\in\mathbb{N}}$ — see Section 1.7), it follows that $k \geq n_\varepsilon$ implies $n_k \geq n_\varepsilon$ which in turn implies $d(x_{n_k}, x) < \varepsilon$. Therefore $x_{n_k} \to x$ in (X, d) as $k \to \infty$. □

As we saw in Section 1.7, nets constitute a natural generalization of (infinite) sequences. Thus it comes as no surprise that the concept of convergence can be generalized from sequences to nets in a metric space (X, d). In fact, an X-valued net $\{x_\gamma\}_{\gamma\in\Gamma}$ (or a net in X) indexed by a directed set Γ *converges* to $x \in X$ if for each real number $\varepsilon > 0$ there exists an index γ_ε in Γ such that

$$\gamma \geq \gamma_\varepsilon \quad \text{implies} \quad d(x_\gamma, x) < \varepsilon.$$

If $\{x_\gamma\}_{\gamma\in\Gamma}$ converges to a point x in X, then $\{x_\gamma\}_{\gamma\in\Gamma}$ is said to be a *convergent net* and x is said to be the *limit* of $\{x_\gamma\}_{\gamma\in\Gamma}$ (usual notation: $\lim x_\gamma = x$, $\lim_\gamma x_\gamma = x$, or $x_\gamma \to x$). Just as in the particular case of sequences, a convergent net in a metric space has a *unique* limit.

The notion of a real-valued continuous function on $\mathbb{R}$ is essential in classical analysis. One of the main reasons for investigating metric spaces is the generalization of the idea of continuity for maps between abstract metric spaces: *a map between metric spaces is continuous if it preserves closeness*.

Definition 3.6. Let $F\colon X \to Y$ be a function from a set X to a set Y. Equip X and Y with metrics d_X and d_Y, respectively, so that (X, d_X) and (Y, d_Y) are metric spaces. $F\colon (X, d_X) \to (Y, d_Y)$ is *continuous at the point* x_0 in X if for each real number $\varepsilon > 0$ there exists a real number $\delta > 0$ (which certainly depends on ε and may depend on x_0 as well) such that

$$d_X(x, x_0) < \delta \quad \text{implies} \quad d_Y(F(x), F(x_0)) < \varepsilon.$$

F is *continuous* (or *continuous on* X) if it is continuous at every point of X; and *uniformly continuous* (on X) if for each real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that

$$d_X(x, x') < \delta \quad \text{implies} \quad d_Y(F(x), F(x')) < \varepsilon$$

for *all* x and x' in X.

It is plain that *a uniformly continuous mapping is continuous*, but the converse fails. The difference between continuity and uniform continuity is that if a function F is uniformly continuous, then for each $\varepsilon > 0$ it is possible to take a $\delta > 0$ (which depends only on ε) so as to ensure that the implication $\{d_X(x, x_0) < \delta \implies d_Y(F(x), F(x_0)) < \varepsilon\}$ holds for *all* points x_0 of X. We say that a mapping $F\colon (X, d_X) \to (Y, d_Y)$ is *Lipschitzian* if there exists a real number $\gamma > 0$ (called a *Lipschitz constant*) such that

$$d_Y(F(x), F(x')) \le \gamma\, d_X(x, x')$$

for all $x, x' \in X$ (which is referred to as the *Lipschitz condition*). It is readily verified that *every Lipschitzian mapping is uniformly continuous*, but, again, the converse fails (see Problem 3.16). A *contraction* is a Lipschitzian mapping $F\colon (X, d_X) \to (Y, d_Y)$ with a Lipschitz constant $\gamma \le 1$. That is, F is a contraction if $d_Y(F(x), F(x')) \le d_X(x, x')$ for all $x, x' \in X$ or, equivalently, if

$$\sup_{x \ne x'} \frac{d_Y(F(x), F(x'))}{d_X(x, x')} \le 1.$$

A function F is said to be a *strict contraction* if it is a Lipschitzian mapping with a Lipschitz constant $\gamma < 1$, which means that

$$\sup_{x \ne x'} \frac{d_Y(F(x), F(x'))}{d_X(x, x')} < 1.$$

Note that, if $d_Y(F(x), F(x')) < d_X(x, x')$ for all $x, x' \in X$, then F is a contraction but not necessarily a strict contraction.

Consider a function F from a metric space (X, d_X) to a metric space (Y, d_Y). If F is continuous at a point $x_0 \in X$, then x_0 is said to be a *point of continuity* of F. Otherwise, if F is not continuous at a point $x_0 \in X$, then x_0 is said to be a *point of discontinuity* of F, and F is said to be *discontinuous at* x_0. F is not continuous if there exists at least one point $x_0 \in X$ such that F is discontinuous at x_0. According to Definition 3.6, a function F is discontinuous at $x_0 \in X$ if and only if the following assertion holds true: there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exists an $x_\delta \in X$ with the property that

$$d_X(x_\delta, x_0) < \delta \quad \text{and} \quad d_Y(F(x_\delta), F(x_0)) \ge \varepsilon.$$

Example 3.H. (a) Consider the set $R^2(\mathbb{R})$ defined in Example 3.E. Set $Y = R^2(\mathbb{R})$ and let X be the subset of Y made up of all functions x in $R^2(\mathbb{R})$ for which the formula

$$y(t) = \int_{-\infty}^{t} x(s)\, ds \quad \text{for each} \quad t \in \mathbb{R}$$

defines a function in $R^2(\mathbb{R})$. Briefly,

$$X = \{x \in Y\colon \int_{-\infty}^{\infty} \left|\int_{-\infty}^{t} x(s)\,ds\right|^2 dt < \infty\}.$$

Recall that a "function" in Y is, in fact, an equivalence class of functions as discussed in Example 3.E. Thus consider the mapping $F\colon X \to Y$ that assigns to each function x in X the function $y = F(x)$ in Y defined by the above formula. Now equip $R^2(\mathbb{R})$ with its usual metric d_2 (cf. Example 3.E) so that (X, d_2) is a subspace of the metric space (Y, d_2). We claim that $F\colon (X, d_2) \to (Y, d_2)$ is *nowhere continuous*; that is, the mapping F is discontinuous at every $x_0 \in X$ (see Problem 3.17(a)).

(b) Now let S be a (nondegenerate) closed and bounded interval of the real line $\mathbb{R}$ (typical example: $S = [0, 1]$) and consider the set $R^2(S)$ defined in Example 3.E. If x is a function in $R^2(S)$ (so that it is Riemann integrable), then set

$$y(t) = \int_{\min S}^{t} x(s)\,ds \quad \text{for each} \quad t \in S.$$

According to the Hölder inequality in Problem 3.3(c), we get $\int_S |x(s)|\,ds \le \left(\int_S ds\right)^{\frac{1}{2}} \left(\int_S |x(s)|^2\,ds\right)^{\frac{1}{2}}$ for every $x \in R^2(S)$. Then $|y(t)|^2 = |\int_0^t x(s)\,ds|^2 \le \left(\int_S |x(s)|\,ds\right)^2 \le \operatorname{diam}(S)\left(\int_S |x(s)|^2\,ds\right)$ for each $t \in S$, and so

$$\int_S |y(t)|^2\,dt \le \operatorname{diam}(S)^2 \int_S |x(s)|^2\,ds < \infty$$

for every $x \in R^2(S)$. Thus the previous identity defines a function y in $R^2(S)$. Let F be a mapping of $R^2(S)$ into itself that assigns to each function x in $R^2(S)$ this function y in $R^2(S)$, so that $y = F(x)$. Equip $R^2(S)$ with its usual metric d_2 (Example 3.E). It is easy to show that $F\colon (R^2(S), d_2) \to (R^2(S), d_2)$ is uniformly continuous. As a matter of fact, the mapping F is Lipschitzian (cf. Problem 3.17(b)). Comparing the example in item (a) with the present one, we observe how different the metric spaces $R^2(\mathbb{R})$ and $R^2(S)$, both equipped with the usual metric d_2, can be: the "same" integral transformation F that is nowhere continuous when defined on an appropriate subspace of $(R^2(\mathbb{R}), d_2)$ becomes Lipschitzian when defined on $(R^2(S), d_2)$.

The concepts of convergence and continuity are tightly intertwined. A particularly important result on the connection of these central concepts says that *a function is continuous if and only if it preserves convergence*. This leads to a necessary and sufficient condition for continuity in terms of convergence.

Theorem 3.7. *Consider a mapping $F\colon (X, d_X) \to (Y, d_Y)$ of a metric space (X, d_X) into a metric space (Y, d_Y) and let x_0 be a point in X. The following assertions are equivalent.*

(a) *F is continuous at x_0.*

(b) *The Y-valued sequence $\{F(x_n)\}$ converges in (Y, d_Y) to $F(x_0) \in Y$ whenever $\{x_n\}$ is an X-valued sequence that converges in (X, d_X) to $x_0 \in X$.*

Proof. If $\{x_n\}$ is an X-valued sequence such that $x_n \to x_0$ in (X, d_X) for some x_0 in X, then (Definition 3.4) for every $\delta > 0$ there exists a positive integer n_δ such that

$$n \geq n_\delta \quad \text{implies} \quad d_X(x_n, x_0) < \delta.$$

If $F\colon (X, d_X) \to (Y, d_Y)$ is continuous at x_0, then (Definition 3.6) for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x_n, x_0) < \delta \quad \text{implies} \quad d_Y(F(x), F(x_0)) < \varepsilon.$$

Therefore, if $x_n \to x_0$ and F is continuous at x_0, then for each $\varepsilon > 0$ there exists a positive integer n_ε (e.g., $n_\varepsilon = n_\delta$) such that

$$n \geq n_\varepsilon \quad \text{implies} \quad d_Y(F(x_n), F(x_0)) < \varepsilon,$$

which means that (a)⇒(b). On the other hand, if F is not continuous at x_0, then there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists $x_\delta \in X$ with the property that

$$d_X(x_\delta, x_0) < \delta \quad \text{and} \quad d_Y(F(x_\delta), F(x_0)) \geq \varepsilon.$$

In particular, for each positive integer n there exists $x_n \in X$ such that

$$d_X(x_n, x_0) < \tfrac{1}{n} \quad \text{and} \quad d_Y(F(x_n), F(x_0)) \geq \varepsilon.$$

Thus $x_n \to x_0$ in (X, d_X) (since $d_X(x_n, x_0) \to 0$) and $F(x_n) \not\to F(x_0)$ in (Y, d_Y) (since $d_Y(F(x_n), F(x_0)) \not\to 0$). That is, the Y-valued sequence $\{F(x_n)\}$ does not converge to $F(x_0)$ (it may not converge in (Y, d_Y) or, if it converges in (Y, d_Y), then it does not converge to $F(x_0)$). Therefore, the denial of (a) implies the denial of (b). Equivalently, (b)⇒(a). □

Note that the proof of (a)⇒(b) can be rewritten in terms of nets so that, *if the mapping* $F\colon (X, d_X) \to (Y, d_Y)$ *is continuous at* $x_0 \in X$, *and if* $\{x_\gamma\}_{\gamma\in\Gamma}$ *is an* X*-valued net that converges to* x_0, *then* $\{F(x_\gamma)\}_{\gamma\in\Gamma}$ *is a* Y*-valued net that converges to* $F(x_0)$. That is, if (b′) is the statement obtained from (b) by changing "sequence" to "net" in (b), then (a)⇒(b′). Conversely, since (b′)⇒(b) trivially (in fact, (b) is a particular case of (b′)), it also follows that (b′)⇒(a) (because (b)⇒(a)).

We shall say that a mapping $F\colon (X, d_X) \to (Y, d_Y)$ of a metric space (X, d_X) into a metric space (Y, d_Y) *preserves convergence* if the Y-valued sequence $\{F(x_n)\}$ converges in (Y, d_Y) whenever the X-valued sequence $\{x_n\}$ converges in (X, d_X), and

$$F(\lim x_n) \;=\; \lim F(x_n).$$

Corollary 3.8. *A map between metric spaces is continuous if and only if it preserves convergence.*

Proof. Combine the above definition and the definition of a continuous function with Theorem 3.7. □

Example 3.I. Let $C[0,1]$ denote the set of all scalar-valued (real or complex) continuous functions defined on the interval $[0,1]$. Consider the map $\varphi\colon C[0,1] \to \mathbb{C}$ defined by

$$\varphi(x) \;=\; \int_0^1 x(t)\,dt$$

for every x in $C[0,1]$. Equip $C[0,1]$ with the sup-metric d_∞ and $\mathbb{C}$ with its usual metric d. Take an arbitrary convergent sequence $\{x_n\}$ in $(C[0,1], d_\infty)$ (i.e., an arbitrary $C[0,1]$-valued sequence that converges in the metric space $(C[0,1], d_\infty)$) and set $x_0 = \lim x_n \in C[0,1]$. Note that

$$\begin{aligned} 0 \;\le\; d\big(\varphi(x_n), \varphi(x_0)\big) \;&=\; \Big| \int_0^1 x_n(t)\,dt - \int_0^1 x_0(t)\,dt \Big| \\ &= \Big| \int_0^1 \big(x_n(t) - x_0(t)\big)\,dt \Big| \;\le\; \int_0^1 \big|x_n(t) - x_0(t)\big|\,dt \\ &\le \sup_{t\in[0,1]} \big|x_n(t) - x_0(t)\big| \int_0^1 dt \;=\; d_\infty(x_n, x_0) \end{aligned}$$

for each positive integer n (by the Hölder inequality: Problem 3.3(c)). Since $d_\infty(x_n, x_0) \to 0$, it follows that $d(\varphi(x_n), \varphi(x_0)) \to 0$. Therefore $\varphi(x_n) \to \varphi(x_0)$ in $(\mathbb{C}, d)$ whenever $x_n \to x_0$ in $(C[0,1], d_\infty)$ so that $\varphi\colon (C[0,1], d_\infty) \to (\mathbb{C}, d)$ is continuous by Corollary 3.8. (In fact, φ is a contraction, thus Lipschitzian, and hence uniformly continuous.)

3.3 Open Sets and Topology

Let (X, d) be a metric space. For each point x_0 in X and each nonnegative real number ρ, the set

$$B_\rho(x_0) \;=\; \big\{x \in X\colon\; d(x, x_0) < \rho\big\}$$

is the *open ball* with *center* x_0 and *radius* ρ (or the open ball centered at x_0 with radius ρ). If $\rho = 0$, then $B_0(x_0)$ is empty; otherwise, $B_\rho(x_0)$ always contains at least its center. The set

$$B_\rho[x_0] \;=\; \big\{x \in X\colon\; d(x, x_0) \le \rho\big\}$$

is the *closed ball* with center x_0 and radius ρ. It is clear that $B_\rho(x_0) \subseteq B_\rho[x_0]$. If $\rho = 0$, then $B_0[x_0] = \{x_0\}$ for every $x_0 \in X$.

Definition 3.9. A subset U of a metric space X is an *open set* in X if it includes a nonempty open ball centered at each one of its points.

That is, U is an open set in X if and only if for every u in U there exists a positive number ρ such that $B_\rho(u) \subseteq U$. Equivalently, $U \subseteq X$ is open in the metric space (X, d) if and only if for *every* $u \in U$ there exists $\rho > 0$ such that

$$d(x,u) < \rho \quad \text{implies} \quad x \in U.$$

Thus, according to Definition 3.9, a subset A of a metric space (X,d) is not open if and only if there exists at least one point a in A such that every open ball with positive radius ρ centered at a contains a point of X not in A. In other words, $A \subset X$ is not open in the metric space (X,d) if and only if there exists at least one point $a \in A$ with the following property: for *every* $\rho > 0$ there exists $x \in X$ such that

$$d(x,a) < \rho \quad \text{and} \quad x \in X \backslash A.$$

This shows at once that the empty set $\varnothing$ is open in X (reason: if a set is not open, then it has at least one point); and also that the underlying set X is always open in the metric space (X,d) (reason: there is no point in $X \backslash X$).

Proposition 3.10. *An open ball is an open set.*

Proof. Let $B_\rho(x_0)$ be an open ball in a metric space (X,d) with center at an arbitrary $x_0 \in X$ and with an arbitrary radius $\rho \geq 0$. Suppose $\rho > 0$ so that $B_\rho(x_0) \neq \varnothing$ (otherwise $B_\rho(x_0)$ is empty and hence trivially open). Take an arbitrary $u \in B_\rho(x_0)$, which means that $u \in X$ and $d(u,x_0) < \rho$. Set $\beta = \rho - d(u,x_0)$ so that $0 < \beta \leq \rho$, and let x be a point in X. If $d(x,u) < \beta$, then the triangle inequality ensures that

$$d(x,x_0) \;\leq\; d(x,u) + d(u,x_0) \;<\; \beta + d(u,x_0) \;=\; \rho,$$

and so $x \in B_\rho(x_0)$. Conclusion: For each $u \in B_\rho(x_0)$ there is a $\beta > 0$ such that

$$d(x,u) < \beta \quad \text{implies} \quad x \in B_\rho(x_0).$$

That is, $B_\rho(x_0)$ is an open set. □

An *open neighborhood* of a point x in a metric space is an open set containing x. In particular (see Proposition 3.10), every open ball with positive radius centered at a point x in a metric space is an open neighborhood of x. A *neighborhood* of a point x in a metric space X is any subset of X that includes an open neighborhood of x. Clearly, every open neighborhood of x is a neighborhood of x.

Open sets give an alternative definition of continuity and convergence.

Lemma 3.11. *Let $F\colon X \to Y$ be a mapping of a metric space X into a metric space Y, and let x_0 be a point in X. The following assertions are equivalent.*

(a) *F is continuous at x_0.*

(b) *The inverse image of every neighborhood of $F(x_0)$ is a neighborhood of x_0.*

Proof. Consider the image $F(x_0) \in Y$ of $x_0 \in X$. Take an arbitrary neighborhood $N \subseteq Y$ of $F(x_0)$. Since N includes an open neighborhood U of $F(x_0)$,

it follows that there exists an open ball $B_\varepsilon(F(x_0)) \subseteq U \subseteq N$ with center at $F(x_0)$ and radius $\varepsilon > 0$. If the mapping $F: X \to Y$ is continuous at x_0 (cf. Definition 3.6), then there exists $\delta > 0$ such that

$$d_Y(F(x), F(x_0)) < \varepsilon \quad \text{whenever} \quad d_X(x, x_0) < \delta,$$

where d_X and d_Y are the metrics on X and Y, respectively. In other words, there exists $\delta > 0$ such that

$$x \in B_\delta(x_0) \quad \text{implies} \quad F(x) \in B_\varepsilon(F(x_0)).$$

Thus $B_\delta(x_0) \subseteq F^{-1}(B_\varepsilon(F(x_0))) \subseteq F^{-1}(U) \subseteq F^{-1}(N)$. Since the open ball $B_\delta(x_0)$ is an open neighborhood of x_0, and since $B_\delta(x_0) \subseteq F^{-1}(N)$, it follows that $F^{-1}(N)$ is a neighborhood of x_0. Hence (a) implies (b). Now suppose (b) holds true. Then, in particular, the inverse image $F^{-1}(B_\varepsilon(F(x_0)))$ of *each* open ball $B_\varepsilon(F(x_0))$ with center $F(x_0)$ and radius $\varepsilon > 0$ includes a neighborhood $N \subseteq X$ of x_0. This neighborhood N includes an open neighborhood U of x_0, which in turn includes an open ball $B_\delta(x_0)$ with center x_0 and radius $\delta > 0$ (cf. Definition 3.9). Therefore, for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$B_\delta(x_0) \subseteq U \subseteq N \subseteq F^{-1}(B_\varepsilon(F(x_0))).$$

Hence (see Problems 1.2(c,j))

$$F(B_\delta(x_0)) \subseteq B_\varepsilon(F(x_0)).$$

Equivalently, if $x \in B_\delta(x_0)$, then $F(x) \in B_\varepsilon(F(x_0))$. Thus, for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, x_0) < \delta \quad \text{implies} \quad d_Y(F(x), F(x_0)) < \varepsilon,$$

where d_X and d_Y denote the metrics on X and Y, respectively. That is, (a) holds true (Definition 3.6). □

Theorem 3.12. *A map between metric spaces is continuous if and only if the inverse image of each open set is an open set.*

Proof. Let $F: X \to Y$ be a mapping of a metric space X into a metric space Y.

(a) Take any neighborhood $N \subseteq Y$ of $F(x) \in Y$ (for an arbitrary $x \in X$). Since N includes an open neighborhood of $F(x)$, say U, it follows that $F(x) \in U \subseteq N$, which implies

$$x \in F^{-1}(U) \subseteq F^{-1}(N).$$

If the inverse image (under F) of each open set in Y is an open set in X, then $F^{-1}(U)$ is open in X, and so $F^{-1}(U)$ is an open neighborhood of x. Hence the inverse image $F^{-1}(N)$ is a neighborhood of x. Therefore, the inverse image of every neighborhood of $F(x)$ (for any $x \in X$) is a neighborhood of x. Thus F is continuous by Lemma 3.11.

(b) Conversely, take an arbitrary open subset U of Y. Suppose $\mathcal{R}(F) \cap U \neq \varnothing$ and take $x \in F^{-1}(U) \subseteq X$ arbitrary. Thus $F(x) \in U$ so that U is an open neighborhood of $F(x)$. If F is continuous, then it is continuous at x. Therefore, according to Lemma 3.11, $F^{-1}(U)$ is a neighborhood of x, and so it includes a nonempty open ball $B_\delta(x)$ centered at x. Thus $B_\delta(x) \subseteq F^{-1}(U)$ so that $F^{-1}(U)$ is open in X (reason: it includes a nonempty open ball of an arbitrary point of it). If $\mathcal{R}(F) \cap U = \varnothing$, then $F^{-1}(U) = \varnothing$ which is open. Conclusion: $F^{-1}(U)$ is open in X for every open subset U of Y. □

Corollary 3.13. *The composition of two continuous functions is again a continuous function.*

Proof. Let X, Y, and Z be metric spaces, and let $F\colon X \to Y$ and $G\colon Y \to Z$ be continuous functions. Take an arbitrary open set U in Z. Theorem 3.12 says that $G^{-1}(U)$ is an open set in Y, and so $(GF)^{-1}(U) = F^{-1}(G^{-1}(U))$ is an open set in X. Thus, using Theorem 3.12 again, $GF\colon X \to Z$ is continuous. □

An X-valued sequence $\{x_n\}$ is said to be *eventually in* a subset A of X if there exists a positive integer n_0 such that

$$n \geq n_0 \quad \text{implies} \quad x_n \in A.$$

Theorem 3.14. *Let $\{x_n\}$ be a sequence in a metric space X and let x be a point in X. The following assertions are equivalent.*

(a) *$x_n \to x$ in X.*

(b) *$\{x_n\}$ is eventually in every neighborhood of x.*

Proof. If $x_n \to x$, then (definition of convergence) $\{x_n\}$ is eventually in every nonempty open ball centered at x. Hence it is eventually in every neighborhood of x (cf. definitions of neighborhood and of open set). Conversely, if $\{x_n\}$ is eventually in every neighborhood of x, then, in particular, it is eventually in every nonempty open ball centered at x, which means that $x_n \to x$. □

Theorem 3.14 is naturally extended from sequences to nets. *A net $\{x_\gamma\}_{\gamma \in \Gamma}$ in a metric space X converges to $x \in X$ if and only if for every neighborhood N of x there exists an index $\gamma_0 \in \Gamma$ such that $x_\gamma \in N$ for every $\gamma \geq \gamma_0$.*

Given a metric space X, the collection of all open sets in X is of paramount importance. Its fundamental properties are stated in the next theorem.

Theorem 3.15. *If X is a metric space, then*

(a) *the whole set X and the empty set $\varnothing$ are open,*

(b) *the intersection of a finite collection of open sets is open,*

(c) *the union of an arbitrary collection of open sets is open.*

Proof. We have already verified that assertion (a) holds true. Let $\{U_n\}$ be a *finite* collection of open subsets of X. Suppose $\bigcap_n U_n \neq \varnothing$ (otherwise $\bigcap_n U_n$ is a open set). Take an arbitrary $u \in \bigcap_n U_n$ so that $u \in U_n$ for every index n. As each U_n is an open subset of X, there are open balls $B_{\alpha_n}(u) \subseteq U_n$ (with center at u and radius $\alpha_n > 0$) for each n. Consider the set $\{\alpha_n\}$ consisting of the radius of each $B_{\alpha_n}(u)$. Since $\{\alpha_n\}$ is a *finite* set of positive numbers, it follows that it has a positive minimum. Set $\alpha = \min\{\alpha_n\} > 0$ so that $B_\alpha(u) \subseteq \bigcap_n U_n$. Thus $\bigcap_n U_n$ is open in X (i.e., for each $u \in \bigcap_n U_n$ there exists an open ball $B_\alpha(u) \subseteq \bigcap_n U_n$), which concludes the proof of (b). The proof of (c) goes as follows. Let $\mathcal{U}$ be an arbitrary collection of open subsets of X. Suppose $\mathcal{U}$ is nonempty (otherwise it is open by (a)) and take an arbitrary $u \in \bigcup \mathcal{U}$ so that $u \in U$ for some $U \in \mathcal{U}$. As U is an open subset of X, there exists a nonempty open ball $B_\rho(u) \subseteq U \subseteq \bigcup \mathcal{U}$, which means that $\bigcup \mathcal{U}$ is open in X. □

Corollary 3.16. *A subset of a metric space is open if and only if it is a union of open balls.*

Proof. The union of open balls in a metric space X is an open set in X because open balls are open sets (cf. Proposition 3.10 and Theorem 3.15). On the other hand, let U be an open set in a metric space X. If U is empty, then it coincides with the empty open ball. If U is a nonempty open subset of X, then each $u \in U$ is the center of an open ball, say $B_{\rho_u}(u)$, included in U. Thus $U = \bigcup_{u \in U}\{u\} \subseteq \bigcup_{u \in U} B_{\rho_u}(u) \subseteq U$, and hence $U = \bigcup_{u \in U} B_{\rho_u}(u)$. □

The collection $\mathcal{T}$ of all open sets in a metric space X (which is a subcollection of the power set $\wp(X)$) is called the *topology* (or the *metric topology*) on X. As the elements of $\mathcal{T}$ are the open sets in the metric space (X, d), and since the definition of an open set in X depends on the particular metric d that equips the metric space (X, d), the collection $\mathcal{T}$ is also referred to as the *topology induced* (or *generated*, or *determined*) by the metric d.

Our starting point in this chapter was the definition of a metric space. A metric has been defined on a set X as a real-valued function on $X \times X$ that satisfies the metric axioms of Definition 3.1. A possible and different approach is to define axiomatically an abstract notion of open sets (instead of an abstract notion of distance as we did in Definition 3.1), and then to build up a theory based on it. Such a "new" beginning goes as follows.

Definition 3.17. A subcollection $\mathcal{T}$ of the power set $\wp(X)$ of a set X is a *topology* on X if it satisfies the following three axioms.

(i) The whole set X and the empty set $\varnothing$ belong to $\mathcal{T}$.

(ii) The intersection of a finite collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

(iii) The union of an arbitrary collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

A set X equipped with a topology $\mathcal{T}$ is referred to as a *topological space* (denoted by $(X, \mathcal{T})$ or simply by X), and the elements of $\mathcal{T}$ are called the *open*

subsets of X with respect to $\mathcal{T}$. Thus a topology $\mathcal{T}$ on an underlying set X is always identified with the collection of all open subsets of X: U is open in X with respect to $\mathcal{T}$ if and only if $U \in \mathcal{T}$. It is clear (see Theorem 3.15) that every metric space (X, d) is a topological space, where the topology $\mathcal{T}$ (the metric topology, that is) is that induced by the metric. This topology $\mathcal{T}$ induced by the metric d, and the topological space $(X, \mathcal{T})$ obtained by equipping X with $\mathcal{T}$, are said to be *metrized* by d. If $(X, \mathcal{T})$ is a topological space and if there is a metric d on X that metrizes $\mathcal{T}$, then the topological space $(X, \mathcal{T})$ and the topology $\mathcal{T}$ are called *metrizable*. The notion of topological space is broader than the notion of metric space. Although every metric space is a topological space, the converse fails. There are topological spaces that are not metrizable.

Example 3.J. Let X be an arbitrary set. Define a function $d: X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y, \end{cases}$$

for every x and y in X. It is readily verified that d is a metric on X. This is the *discrete metric* on X. A set X equipped with the discrete metric is called a *discrete space*. In a discrete space every open ball with radius $\rho \in (0, 1)$ is a singleton in X: $B_\rho(x_0) = \{x_0\}$ for every $x_0 \in X$ and every $\rho \in (0, 1)$. Thus, according to Definition 3.9, every subset of X is an open set in the metric space (X, d) equipped with the discrete metric d. That is, the metric topology coincides with the power set of X. Conversely, if X is an arbitrary set, then the collection $\mathcal{T} = \wp(X)$ is a topology on X (since $\mathcal{T} = \wp(X)$ trivially satisfies the above three axioms), called the *discrete topology*, which is the *largest* topology on X (any other topology on X is a subcollection of the discrete topology). Summing up: The discrete topology

$$\mathcal{T} = \wp(X) \text{ is metrizable}$$

by the discrete metric. On the other extreme lies the topology $\mathcal{T} = \{\varnothing, X\}$, called the *indiscrete topology*, which is the *smallest* topology on X (it is a subcollection of any other topology on X). If X has more than one point, then the indiscrete topology

$$\mathcal{T} = \{\varnothing, X\} \text{ is not metrizable.}$$

Indeed, suppose there is a metric on d on X that induces the indiscrete topology. Take u in X arbitrary and consider the set $X \backslash \{u\}$. Since $\varnothing \neq X \backslash \{u\} \neq X$, it follows that this set is not open (with respect to the indiscrete topology). Thus there exists $v \in X \backslash \{u\}$ with the following property: for every $\rho > 0$ there exists $x \in X$ such that

$$d(x, v) < \rho \quad \text{and} \quad x \in X \backslash (X \backslash \{u\}) = \{u\}.$$

Hence $x = u$ so that $d(u, v) < \rho$ for every $\rho > 0$. Therefore $u = v$ (i.e., $d(u, v) = 0$), which is a contradiction (because $v \in X \backslash \{u\}$). Conclusion: There is no metric on X that induces the indiscrete topology.

Continuity and convergence in a topological space can be defined as follows. A mapping $F: X \to Y$ of a topological space $(X, \mathcal{T}_X)$ into a topological space $(Y, \mathcal{T}_Y)$ is *continuous* if $F^{-1}(U) \in \mathcal{T}_X$ for every $U \in \mathcal{T}_Y$. An X-valued sequence $\{x_n\}$ *converges* in a topological space $(X, \mathcal{T})$ to a limit $x \in X$ if it is eventually in every $U \in \mathcal{T}$ that contains x. Carefully note that, for the particular case of metric spaces (or of metrizable topological spaces), the above definitions of continuity and convergence agree with Definitions 3.6 and 3.4 when the topological spaces are equipped with their metric topology. Indeed, these definitions are the topological-space versions of Theorems 3.12 and 3.14.

Many (but not all) of the theorems in the following sections hold for general topological spaces (metrizable or not), and we shall prove them by using a topological-space style (based on open sets rather than on open balls) whenever this is possible and convenient. However, as we had anticipated at the introduction of this chapter, our attention will focus mainly on metric spaces.

3.4 Equivalent Metrics and Homeomorphisms

Let (X, d_1) and (X, d_2) be two metric spaces with the same underlying set X. The metrics d_1 and d_2 are said to be *equivalent* (or d_1 and d_2 are *equivalent metrics* on X — notation: $d_1 \sim d_2$) if they induce the same topology (i.e., a subset of X is open in (X, d_1) if and only if it is open in (X, d_2)). This notion of equivalence in fact is an equivalence relation on the collection of all metrics defined on a given set X. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are the metric topologies on X induced by the metrics d_1 and d_2, respectively, then

$$d_1 \sim d_2 \quad \text{if and only if} \quad \mathcal{T}_1 = \mathcal{T}_2.$$

If $\mathcal{T}_1 \subseteq \mathcal{T}_2$ (i.e., if every open set in (X, d_1) is open in (X, d_2)), then $\mathcal{T}_2$ is said to be *stronger* than $\mathcal{T}_1$. In this case we also say that $\mathcal{T}_1$ is *weaker* than $\mathcal{T}_2$. The terms *finer* and *coarser* are also used as synonyms for "stronger" and "weaker", respectively. If either $\mathcal{T}_1 \subseteq \mathcal{T}_2$ or $\mathcal{T}_2 \subseteq \mathcal{T}_1$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be *commensurable*. Otherwise (i.e., if neither $\mathcal{T}_1 \subseteq \mathcal{T}_2$ nor $\mathcal{T}_2 \subseteq \mathcal{T}_1$), the topologies are said to be *incommensurable*. As we shall see below, if $\mathcal{T}_2$ is stronger than $\mathcal{T}_1$, then continuity with respect to $\mathcal{T}_1$ implies continuity with respect to $\mathcal{T}_2$. On the other hand, if $\mathcal{T}_2$ is stronger than $\mathcal{T}_1$, then convergence with respect to $\mathcal{T}_2$ implies convergence with respect to $\mathcal{T}_1$. Briefly and roughly: "Strong convergence" implies "weak convergence" but "weak continuity" implies "strong continuity".

Theorem 3.18. *Let d_1 and d_2 be metrics on a set X, and consider the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ induced by d_1 and d_2, respectively. The following assertions are pairwise equivalent.*

(a) *$\mathcal{T}_2$ is stronger than $\mathcal{T}_1$* (i.e., $\mathcal{T}_1 \subseteq \mathcal{T}_2$).

(b) *Every mapping* $F: X \to Y$ *that is continuous at* $x_0 \in X$ *as a mapping of* (X, d_1) *into the metric space* (Y, d) *is continuous at* x_0 *as a mapping of* (X, d_2) *into* (Y, d).

(c) *Every* X*-valued sequence that converges in* (X, d_2) *to a limit* $x \in X$ *converges in* (X, d_1) *to the same limit* x.

(d) *The identity map of* (X, d_2) *onto* (X, d_1) *is continuous*.

Proof. Consider the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X induced by the metrics d_1 and d_2 on X. Let $\mathcal{T}$ denote the topology on a set Y induced by a metric d on Y.

Proof of (a)⇒(b). If $F: (X, d_1) \to (Y, d)$ is continuous at $x_0 \in X$, then (Lemma 3.11) for every $U \in \mathcal{T}$ that contains $F(x_0)$ there exists $U' \in \mathcal{T}_1$ containing x_0 such that $U' \subseteq F^{-1}(U)$. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $U' \in \mathcal{T}_2$: the inverse image (under F) of every open neighborhood of $F(x_0)$ in $\mathcal{T}$ includes an open neighborhood of x_0 in $\mathcal{T}_2$, which clearly implies that the inverse image (under F) of every neighborhood of $F(x_0)$ in $\mathcal{T}$ is a neighborhood of x_0 in $\mathcal{T}_2$. Thus, applying Lemma 3.11 again, $F: (X, d_2) \to (Y, d)$ is continuous at x_0.

Proof of (a)⇒(c). Let $\{x_n\}$ be an X-valued sequence. If $x_n \to x \in X$ in (X, d_2), then (Theorem 3.14) $\{x_n\}$ is eventually in *every* open neighborhood of x in $\mathcal{T}_2$. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$ then, in particular, $\{x_n\}$ is eventually in every neighborhood of x in $\mathcal{T}_1$. Therefore, applying Theorem 3.14 again, $x_n \to x$ in (X, d_1).

Proof of (b)⇒(d). The identity map $I: (X, d_1) \to (X, d_1)$ of a metric space onto itself is trivially continuous. Thus, by setting $(Y, d) = (X, d_1)$ in (b), it follows that (b) implies (d).

Proof of (c)⇒(d). Corollary 3.8 ensures that (c) implies (d).

Proof of (d)⇒(a). According to Theorem 3.12, (d) implies (a) (i.e., if the identity $I: (X, d_2) \to (X, d_1)$ is continuous, then $U = I^{-1}(U)$ is open in $\mathcal{T}_2$ whenever U is open in $\mathcal{T}_1$, and hence $\mathcal{T}_1 \subseteq \mathcal{T}_2$). □

As the discrete topology is the strongest topology on X, the above theorem ensures that any function $F: X \to Y$ that is continuous in some topology on X is continuous in the discrete topology. Actually, since every subset of X is open in the discrete topology, it follows that the inverse image of every subset of Y — no matter which topology equips the set Y — is an open subset of X when X is equipped with the discrete topology. Therefore, *every function defined on a discrete topological space is continuous*. On the other hand, if an X-valued (infinite) sequence converges in the discrete topology, then it is *eventually constant* (i.e., it has only a finite number of entries not equal to its limit), and hence it converges in any topology on X.

Corollary 3.19. *Let* (X, d_1) *and* (X, d_2) *be metric spaces with the same underlying set* X. *The following assertions are pairwise equivalent*.

(a) *d_2 and d_1 are equivalent metrics on X.*

(b) *A mapping of X into a set Y is continuous at $x_0 \in X$ as a mapping of (X, d_1) into the metric space (Y, d) if and only if it is continuous at x_0 as a mapping of (X, d_2) into (Y, d).*

(c) *An X-valued sequence converges in (X, d_1) to $x \in X$ if and only if it converges in (X, d_2) to x.*

(d) *The identity map of (X, d_1) onto (X, d_2) and its inverse* (i.e., *the identity map of (X, d_2) onto (X, d_1)*) *are both continuous.*

Proof. Recall that, by definition, two metrics d_1 and d_2 on a set X are equivalent if the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X, induced by d_1 and d_2 respectively, coincide (i.e., if $\mathcal{T}_1 = \mathcal{T}_2$). Now apply Theorem 3.18. □

A one-to-one mapping G of a metric space X onto a metric space Y is a *homeomorphism* if both $G: X \to Y$ and $G^{-1}: Y \to X$ are continuous. Equivalently, a homeomorphism between metric spaces is an invertible (i.e., injective and surjective) mapping that is continuous and has a continuous inverse. Thus G is a homeomorphism from X to Y if and only if G^{-1} is a homeomorphism from Y to X. Two metric spaces are *homeomorphic* if there exists a homeomorphism between them. A function $F: X \to Y$ of a metric space X into a metric space Y is an *open map* (or an *open mapping*) if the image of each open set in X is open in Y (i.e., $F(U)$ is open in Y whenever U is open in X).

Theorem 3.20. *Let X and Y be metric spaces. If $G: X \to Y$ is invertible, then*

(a) *G is open if and only if G^{-1} is continuous*,

(b) *G is continuous if and only if G^{-1} is open*,

(c) *G is a homeomorphism if and only if G and G^{-1} are both open.*

Proof. If G is invertible, then the inverse image of B ($B \subseteq Y$) under G coincides with the image of B under the inverse of G (tautologically: $G^{-1}(B) = G^{-1}(B)$). Applying the same argument to the inverse G^{-1} of G (which is clearly invertible), $(G^{-1})^{-1}(A) = G(A)$ for each $A \subseteq X$. Thus the theorem is a straightforward combination of the definitions of open map and homeomorphism by using the alternative definition of continuity in Theorem 3.12. □

Thus a homeomorphism provides *simultaneously* a one-to-one correspondence between the underlying sets X and Y (so that $X \leftrightarrow Y$, since a homeomorphism is injective and surjective) and between their topologies (so that $\mathcal{T}_X \leftrightarrow \mathcal{T}_Y$, since a homeomorphism puts the open sets of $\mathcal{T}_X$ into a one-to-one correspondence with the open sets of $\mathcal{T}_Y$). Indeed, if $\mathcal{T}_X$ and $\mathcal{T}_Y$ are the topologies on X and Y, respectively, then a homeomorphism $G: X \to Y$ induces a map $\mathcal{G}: \mathcal{T}_X \to \mathcal{T}_Y$, defined by $\mathcal{G}(U) = G(U)$ for every $U \in \mathcal{T}_X$, which is injective and surjective according to Theorem 3.20. Thus any property of a metric

space X expressed entirely in terms of set operations and open sets is also possessed by each metric space homeomorphic to X. We call a property of a metric space a *topological property* or a *topological invariant* if whenever it is true for one metric space, say X, it is true for every metric space homeomorphic to X (trivial examples: the cardinality of the underlying set and the cardinality of the topology). A map $F: X \to Y$ of a metric space X into a metric space Y is a *topological embedding* of X into Y if it establishes a homeomorphism of X onto its range $\mathcal{R}(F)$ (i.e., $F: X \to Y$ is a topological embedding of X into Y if it is such that $F: X \to F(X)$ is a homeomorphism of X onto the subspace $F(X)$ of Y).

Example 3.K. Suppose $G: X \to Y$ is a homeomorphism of a metric space X onto a metric space Y. Let A be a subspace of X and consider the subspace $G(A)$ of Y. According to Problem 3.30 the restriction $G|_A: A \to G(A)$ of G to A onto $G(A)$ is continuous. Similarly, the restriction $G^{-1}|_{G(A)}: G(A) \to A$ of the inverse of G to $G(A)$ onto $G^{-1}(G(A)) = A$ is continuous as well. Since $G^{-1}|_{G(A)} = (G|_A)^{-1}$ (Problem 1.8), it follows that

$$G|_A: A \to G(A) \text{ is a homeomorphism,}$$

and so A and $G(A)$ are homeomorphic metric spaces (as subspaces of X and Y, respectively). Therefore, the restriction $G|_A: A \to Y$ of a homeomorphism $G: X \to Y$ to any subset A of X is a topological embedding of A into Y.

The notions of homeomorphism, open map, topological invariant, and topological embedding are germane to topological spaces in general (and to metric spaces in particular). For instance, both Theorem 3.20 and Example 3.K can be likewise stated (and proved) in a topological-space setting. In other words, the metric has played no role in the above paragraph, and "metric space" can be replaced with "topological space" there. Next we shall consider a couple of concepts that only make sense in a metric space. A homeomorphism G of a metric space (X, d_X) onto a metric space (Y, d_Y) is a *uniform homeomorphism* if both G and G^{-1} are uniformly continuous. Two metric spaces are *uniformly homeomorphic* if there exists a uniform homeomorphism mapping one of them onto the other. An *isometry* between metric spaces is a map that preserves distance. Precisely, a mapping $J: (X, d_X) \to (Y, d_Y)$ of a metric space (X, d_X) *into* a metric space (Y, d_Y) is an *isometry* if

$$d_Y(J(x), J(x')) = d_X(x, x')$$

for every pair of points x, x' in X. It is clear that *every isometry is an injective contraction*, and hence an injective and uniformly continuous mapping. Thus *every surjective isometry is a uniform homeomorphism* (the inverse of a surjective isometry is again a surjective isometry — trivial example: the identity mapping of a metric space into itself is a surjective isometry on that space). Two metric spaces are *isometric* (or *isometrically equivalent*) if there exists a surjective isometry between them, so that two *isometrically equivalent metric*

spaces are uniformly homeomorphic. It is trivially verified that a composition of surjective isometries is a surjective isometry (transitivity), and this shows that the notion of isometrically equivalent metric spaces deserves its name: it is indeed an equivalence relation on any collection of metric spaces. If two metric spaces are isometrically equivalent, then they can be thought of as being essentially the same metric space — they may differ on the set-theoretic nature of their points but, as far as the metric space (topological) structure is concerned, they are indistinguishable. A surjective isometry not only preserves open sets (for it is a homeomorphism), but it also preserves distance.

Now consider two metric spaces (X, d_1) and (X, d_2) with the same underlying set X. According to Corollary 3.19 the metrics d_1 and d_2 are equivalent if and only if the identity map of (X, d_1) onto (X, d_2) is a homeomorphism (i.e., if and only if $I\colon (X, d_1) \to (X, d_2)$ and its inverse $I^{-1}\colon (X, d_2) \to (X, d_1)$ are both continuous). We say that the metrics d_1 and d_2 are *uniformly equivalent* if the identity map of (X, d_1) onto (X, d_2) is a uniform homeomorphism (i.e., if $I\colon (X, d_1) \to (X, d_2)$ and its inverse $I^{-1}\colon (X, d_2) \to (X, d_1)$ are both uniformly continuous). For instance, if I and I^{-1} are both Lipschitzian, which means that there exist real numbers $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\, d_1(x, x') \;\le\; d_2(x, x') \;\le\; \beta\, d_1(x, x')$$

for every x, x' in X, then the metrics d_1 and d_2 are uniformly equivalent, and hence equivalent. Thus, if d_1 and d_2 are equivalent metrics on X, then (X, d_1) and (X, d_2) are homeomorphic metric spaces. However, the converse fails: there exist uniformly homeomorphic metric spaces with the same underlying set for which the identity is not a homeomorphism.

Example 3.L. Take two metric spaces (X, d_1) and (X, d_2) with the same underlying set X. Consider the product spaces $(X\times X, d)$ and $(X\times X, d')$, where

$$\begin{aligned} d((x, y), (u, v)) &= d_1(x, u) + d_2(y, v), \\ d'((x, y), (u, v)) &= d_2(x, u) + d_1(y, v), \end{aligned}$$

for all ordered pairs (x, y) and (u, v) in $X\times X$. In other words, $(X\times X, d) = (X, d_1)\times(X, d_2)$ and $(X\times X, d') = (X, d_2)\times(X, d_1)$ — see Problem 3.9. Suppose the metrics d_1 and d_2 on X are not equivalent so that either the identity map of (X, d_1) onto (X, d_2) or the identity map of (X, d_2) onto (X, d_1) (or both) is not continuous. Let $I\colon (X, d_1) \to (X, d_2)$ be the one that is not continuous. The identity map

$$I\colon (X\times X, d) \to (X\times X, d') \text{ is not continuous.}$$

Indeed, if it is continuous, then the restriction of it to a subspace of $(X\times X, d)$ is continuous (Problem 3.30). In particular, the restriction of it to (X, d_1) — viewed as a subspace of $(X\times X, d) = (X, d_1)\times(X, d_2)$ — is continuous. But such a restriction is clearly identified with the identity map of (X, d_1)

onto (X, d_2), which is not continuous. Thus $I\colon (X\times X, d) \to (X\times X, d')$ is not continuous, and hence the metrics d and d' on $X\times X$ are not equivalent. Now let $J\colon X\times X \to X\times X$ be the involution (Problem 1.11) on $X\times X$ defined by

$$J((x, y)) = (y, x) \quad \text{for every} \quad (x, y) \in X\times X.$$

It is easy to show that $J\colon (X\times X, d) \to (X\times X, d')$ is a surjective isometry. Thus

$$J\colon (X\times X, d) \to (X\times X, d') \text{ is a uniform homeomorphism.}$$

Summing up: The metric spaces $(X\times X, d)$ and $(X\times X, d')$, with the same underlying set $X\times X$, are uniformly homeomorphic (more than that, they are isometrically equivalent), but the metrics d and d' on $X\times X$ are not equivalent.

Since two metric spaces with the same underlying set may be homeomorphic even if the identity between them is not a homeomorphism, it follows that a weaker version of Corollary 3.19 is obtained if we replace the homeomorphic identity with an arbitrary homeomorphism. This in fact can be formulated for arbitrary metric spaces (not necessarily with the same underlying set).

Theorem 3.21. *Let X and Y be metric spaces and let G be an invertible mapping of X onto Y. The following assertions are pairwise equivalent.*

(a) *G is a homeomorphism.*

(b) *A mapping F of X into a metric space Z is continuous if and only if the composition $FG^{-1}\colon Y \to Z$ is continuous.*

(c) *An X-valued sequence $\{x_n\}$ converges in X to a limit $x \in X$ if and only if the Y-valued sequence $\{G(x_n)\}$ converges in Y to $G(x)$.*

Proof. Let $G\colon X \to Y$ be an invertible mapping of a metric space X onto a metric space Y.

Proof of (a)⇒(b). Let $F\colon X \to Z$ be a mapping of X into a metric space Z, and consider the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{G^{-1}} & X \\ & \searrow^{H} & \downarrow{\scriptstyle F} \\ & & Z \end{array}$$

so that $H = FG^{-1}\colon Y \to Z$. Suppose (a) holds true, and consider the following assertions.

(b_1) $F\colon X \to Z$ is continuous.

(b_2) $F^{-1}(U)$ is an open set in X whenever U is an open set in Z.

(b_3) $G(F^{-1}(U))$ is an open set in Y whenever U is an open set in Z.

(b_4) $(FG^{-1})^{-1}(U)$ is an open set in Y whenever U is an open set in Z.

(b_5) $H = FG^{-1}: Y \to Z$ is continuous.

Theorem 3.12 says that (b_1) and (b_2) are equivalent. But (b_2) holds true if and only if (b_3) holds true by Theorem 3.20 (the homeomorphism $G: X \to Y$ puts the open sets of X into a one-to-one correspondence with the open sets of Y). Now note that, as G is invertible,

$$\begin{aligned} G(F^{-1}(A)) &= \big\{G(x) \in Y\colon\ F(x) \in A\big\} \\ &= \big\{y \in Y\colon\ F(G^{-1}(y)) \in A\big\} \ = \ (FG^{-1})^{-1}(A) \end{aligned}$$

for every subset A of Z. Thus (b_3) is equivalent to (b_4), which in turn is equivalent to (b_5) (cf. Theorem 3.12 again). Conclusion: (b_1)$\Leftrightarrow$(b_5) whenever (a) holds true.

Proof of (b)$\Rightarrow$(a). If (b) holds, then it holds in particular for $Z = X$ and for $Z = Y$. Thus (b) ensures that the following assertions hold true.

(b') If a mapping $F: X \to X$ of X into itself is continuous, then the mapping $H = FG^{-1}: Y \to X$ is continuous.

(b'') A mapping $F: X \to Y$ of X into Y is continuous whenever the mapping $H = FG^{-1}: Y \to Y$ is continuous.

Since the identity of X onto itself is continuous, (b') implies that $G^{-1}: Y \to X$ is continuous. By setting $F = G$ in (b'') it follows that $G: X \to Y$ is continuous (because the identity $I = GG^{-1}: Y \to Y$ is continuous). Summing up: (b) implies that both G and G^{-1} are continuous, which means that (a) holds true.

Proof of (a)$\Leftrightarrow$(c). According to Corollary 3.8 an invertible mapping G between metric spaces is continuous and has a continuous inverse if and only if both G and G^{-1} preserve convergence. □

3.5 Closed Sets and Closure

A subset V of a metric space X is *closed* in X if its complement $X \backslash V$ is an open set in X.

Theorem 3.22. *If X is a metric space, then*

(a) *the whole set X and the empty set $\varnothing$ are closed,*

(b) *the union of a finite collection of closed sets is closed,*

(c) *the intersection of an arbitrary collection of closed sets is closed.*

Proof. Apply the De Morgan laws to each item of Theorem 3.15. □

Thus the concepts "closed" and "open" are dual to each other (U is open in X if and only if its complement $X\backslash U$ is closed in X, and V is closed in X if and only if its complement $X\backslash V$ is open in X); but they are neither exclusive (a set in a metric space may be both open and closed) nor exhaustive (a set in a metric space may be neither open nor closed).

Theorem 3.23. *A map between metric spaces is continuous if and only if the inverse image of each closed set is a closed set.*

Proof. Let $F\colon X \to Y$ be a mapping of a metric space X into a metric space Y. Recall that $F^{-1}(Y\backslash B) = X\backslash F^{-1}(B)$ for every subset B of Y (Problem 1.2(b)). Suppose F is continuous and take an arbitrary closed set V in Y. Since $Y\backslash V$ is open in Y, it follows by Theorem 3.12 that $F^{-1}(Y\backslash V)$ is open in X. Thus $F^{-1}(V) = X\backslash F^{-1}(Y\backslash V)$ is closed in X. Therefore, the inverse image under F of an arbitrary closed set V in Y is closed in X. Conversely, suppose the inverse image under F of each closed set in Y is a closed set in X and take an arbitrary open set U in Y. Thus $F^{-1}(Y\backslash U)$ is closed in X (because $Y\backslash U$ is closed in Y) so that $F^{-1}(U) = X\backslash F^{-1}(Y\backslash U)$ is open in X. Conclusion: The inverse image under F of an arbitrary open set U in Y is open in X. Therefore F is continuous by Theorem 3.12. □

A function $F\colon X \to Y$ of a metric space X into a metric space Y is a *closed map* (or a *closed mapping*) if the image of each closed set in X is closed in Y (i.e., $F(V)$ is closed in Y whenever V is closed in X). In general, a map $F\colon X \to Y$ of a metric space X into a metric space Y may possess any combination of the attributes "continuous", "open", and "closed" (i.e., these are independent concepts). However, if $F\colon X \to Y$ is invertible (i.e., injective and surjective), then it is a closed map if and only if it is an open map.

Theorem 3.24. *Let X and Y be metric spaces. If a map $G\colon X \to Y$ is invertible, then*

(a) *G is closed if and only if G^{-1} is continuous,*

(b) *G is continuous if and only if G^{-1} is closed,*

(c) *G is a homeomorphism if and only if G and G^{-1} are both closed.*

Proof. Replace "open map" with "closed map" in the proof of Theorem 3.20 and use Theorem 3.23 instead of Theorem 3.12. □

Let A be a set in a metric space X and let $\mathcal{V}_A$ be the collection of all closed subsets of X that include A:

$$\mathcal{V}_A = \big\{V \in \wp(X)\colon\ V \text{ is closed in } X \text{ and } A \subseteq V\big\}.$$

The whole set X always belongs to $\mathcal{V}_A$ so that $\mathcal{V}_A$ is never empty. The intersection of all sets in $\mathcal{V}_A$ is called the *closure* of A in X, denoted by A^- (i.e., $A^- = \bigcap \mathcal{V}_A$). According to Theorem 3.22(c) it follows that

$$A^- \text{ is closed in } X \quad \text{and} \quad A \subseteq A^-.$$

If $V \in \mathcal{V}_A$, then it is plain that $A^- = \bigcap \mathcal{V}_A \subseteq V$. Thus, with respect to the inclusion ordering of $\wp(X)$,

$$A^- \text{ is the smallest closed subset of } X \text{ that includes } A,$$

and hence (since A^- is closed in X)

$$A \text{ is closed in } X \quad \text{if and only if} \quad A = A^-.$$

From the above displayed results it is readily verified that

$$\varnothing^- = \varnothing, \quad X^- = X, \quad (A^-)^- = A^-$$

and, if B also is a set in X,

$$A \subseteq B \quad \text{implies} \quad A^- \subseteq B^-.$$

Moreover, since both A and B are subsets of $A \cup B$, we get $A^- \subseteq (A \cup B)^-$ and $B^- \subseteq (A \cup B)^-$ so that $A^- \cup B^- \subseteq (A \cup B)^-$. On the other hand, since $(A \cup B)^-$ is the smallest closed subset of X that includes $A \cup B$, and since $A^- \cup B^-$ is closed (Theorem 3.22(b)) and includes $A \cup B$ (because $A \subseteq A^-$ and $B \subseteq B^-$ so that $A \cup B \subseteq A^- \cup B^-$), it follows that $(A \cup B)^- \subseteq A^- \cup B^-$. Therefore, if A and B are subsets of X, then

$$(A \cup B)^- = A^- \cup B^-.$$

It is easy to show by induction that the above identity holds for any finite collection of subsets of X. That is, *the closure of the union of a finite collection of subsets of X coincides with the union of their closures*. In general (i.e., by allowing infinite collections as well) one has inclusion rather than equality. Indeed, if $\{A_\gamma\}_{\gamma \in \Gamma}$ is an arbitrary indexed family of subsets of X, then

$$\bigcup_\gamma A_\gamma^- \subseteq \Big(\bigcup_\gamma A_\gamma\Big)^-$$

since $A_\alpha \subseteq \bigcup_\gamma A_\gamma$ and hence $A_\alpha^- \subseteq (\bigcup_\gamma A_\gamma)^-$ for each index $\alpha \in \Gamma$. Similarly,

$$\Big(\bigcap_\gamma A_\gamma\Big)^- \subseteq \bigcap_\gamma A_\gamma^-$$

since $\bigcap_\gamma A_\gamma \subseteq \bigcap_\gamma A_\gamma^-$ and $\bigcap_\gamma A_\gamma^-$ is closed in X by Theorem 3.22(c). However, these inclusions are not reversible in general, so that equality does not hold.

Example 3.M. Set $X = \mathbb{R}$ with its usual metric and consider the following subsets of $\mathbb{R}$: $A_n = [0, 1 - \frac{1}{n}]$, which is closed in $\mathbb{R}$ for each positive integer n, and $A = [0, 1)$, which is not closed in $\mathbb{R}$. Since

$$\bigcup_{n=1}^{\infty} A_n = A,$$

it follows that *the union of an infinite collection of closed sets is not necessarily closed* (cf. Theorem 3.22(b)). Moreover, as $A_n^- = A_n$ for each n and $A^- = [0,1]$,

$$[0,1) = \bigcup_{n=1}^{\infty} A_n^- \subset \Big(\bigcup_{n=1}^{\infty} A_n\Big)^- = [0,1],$$

which is a *proper* inclusion. If $B = [1,2]$ (so that $B^- = B$), then

$$\varnothing = (A \cap B)^- \subset A^- \cap B^- = \{1\},$$

so that the closure of any (even finite) intersection of sets may be a *proper* subset of the intersection of their closures.

A point x in X is *adherent* to A (or an *adherent point* of A, or a *point of adherence* of A) if it belongs to the closure A^- of A. It is clear that every point of A is an adherent point of A (i.e., $A \subseteq A^-$).

Proposition 3.25. *Let A be a subset of a metric space X and let x be a point in X. The following assertions are pairwise equivalent.*

(a) *x is a point of adherence of A.*

(b) *Every open set in X that contains x meets A* (i.e., *if U is open in X and $x \in U$, then $A \cap U \neq \varnothing$*).

(c) *Every neighborhood of x contains at least one point of A* (*which may be x itself*).

Proof. Suppose there is an open set U in X containing x for which $A \cap U = \varnothing$. Then $A \subseteq X\backslash U$, the set $X\backslash U$ is closed in X, and $x \notin X\backslash U$. Since A^- is the smallest closed subset of X that includes A, it follows that $A^- \subseteq X\backslash U$ so that $x \notin A^-$. Thus the denial of (b) implies the denial of (a), which means that (a) implies (b). Conversely, if $x \notin A^-$, then x lies in the open set $X\backslash A^-$, which does not meet A^- ($A^- \cap X\backslash A^- = \varnothing$). Therefore, the denial of (a) implies the denial of (b); that is, (b) implies (a). Finally note that (b) is equivalent to (c) as an obvious consequence of the definition of neighborhood. □

A point x in X is a *point of accumulation* (or an *accumulation point*, or a *cluster point*) of A if it is a point of adherence of $A\backslash\{x\}$. The set of all accumulation points of A is the *derived set* of A, denoted by $A^\star$. Thus $x \in A^\star$ if and only if $x \in (A\backslash\{x\})^-$. It is clear that *every point of accumulation of A is also a point of adherence of A*; that is, $A^\star \subseteq A^-$ (since $A\backslash\{x\} \subseteq A$ implies $(A\backslash\{x\})^- \subseteq A^-$). Actually,

$$A^- = A \cup A^\star.$$

Indeed, since $A^\star \subseteq A^-$ and $A \subseteq A^-$, it follows that $A \cup A^\star \subseteq A^-$. On the other hand, if $x \notin A \cup A^\star$, then $(A\backslash\{x\})^- = A^-$ (since $A\backslash\{x\} = A$ whenever $x \notin A$), and hence $x \notin A^-$ (because $x \notin A^\star$ so that $x \notin (A\backslash\{x\})^-$). Thus if $x \in A^-$, then $x \in A \cup A^\star$, which means that $A^- \subseteq A \cup A^\star$. Hence $A^- = A \cup A^\star$. So

$$A = A^- \quad \text{if and only if} \quad A^\star \subseteq A.$$

That is, *A is closed in X if and only if it contains all its accumulation points.* It is trivially verified that

$$A \subseteq B \quad \text{implies} \quad A^\star \subseteq B^\star$$

whenever A and B are subsets of X. Also note that $A^- = \varnothing$ if and only if $A = \varnothing$ (for $\varnothing^- = \varnothing$ and $\varnothing \subseteq A \subseteq A^-$), and $A^\star = \varnothing$ whenever $A = \varnothing$ (because $A^\star \subseteq A^-$), but the converse fails (e.g., the derived set of a singleton is empty).

Proposition 3.26. *Let A be a subset of a metric space X and let x be a point in X. The following assertions are pairwise equivalent.*

(a) *x is a point of accumulation of A.*

(b) *Every open set in X that contains x also contains at least one point of A other than x.*

(c) *Every neighborhood of x contains at least one point of A distinct from x.*

Proof. Since $x \in X$ is a point of accumulation of A if and only if it is a point of adherence of $A\backslash\{x\}$, it follows by Proposition 3.25 that the assertions (a), (b), and (c) are equivalent (replace A with $A\backslash\{x\}$ in Proposition 3.25). □

Everything that has been written so far in this section pertains to the realm of topological spaces (metrizable or not). However, the following results are typical of metric spaces.

Proposition 3.27. *Let A be a subset of a metric space (X, d) and let x be a point in X. The following assertions are pairwise equivalent.*

(a) *x is a point of adherence of A.*

(b) *Every nonempty open ball centered at x meets A.*

(c) *$A \neq \varnothing$ and $d(x, A) = 0$.*

(d) *There exists an A-valued sequence that converges to x in (X, d).*

Proof. The equivalence (a)⇔(b) follows by Proposition 3.25 (recall: every nonempty open ball centered at x is a neighborhood of x and, conversely, every neighborhood of x includes a nonempty open ball centered at x, so that every nonempty open ball centered at x meets A if and only if every neighborhood of x meets A). Clearly (b)⇔(c) (i.e., for each $\varepsilon > 0$ there exists $a \in A$ such that $d(x, a) < \varepsilon$ if and only if $A \neq \varnothing$ and $\inf_{a \in A} d(x, a) = 0$). Theorem 3.14

ensures that (d)⇒(b). On the other hand, if (b) holds true, then for each positive integer n the open ball $B_{1/n}(x)$ meets A (i.e., $B_{1/n}(x) \cap A \neq \varnothing$). Take $x_n \in B_{1/n}(x) \cap A$ so that $x_n \in A$ and $0 \leq d(x_n, x) < \frac{1}{n}$ for each n. Thus $\{x_n\}$ is an A-valued sequence such that $d(x_n, x) \to 0$. Therefore (b)⇒(d). □

Proposition 3.28. *Let A be a subset of a metric space (X, d) and let x be a point in X. The following assertions are pairwise equivalent.*

(a) *x is a point of accumulation of A.*

(b) *Every nonempty open ball centered at x contains a point of A distinct from x.*

(c) *Every nonempty open ball centered at x contains infinitely many points of A.*

(d) *There exists an $A\backslash\{x\}$-valued sequence of pairwise distinct points that converges to x in (X, d).*

Proof. (d)⇒(c) by Theorem 3.14, (c)⇒(b) trivially, and (d)⇒(a)⇒(b) by the previous proposition. To complete the proof, it remains to show that (b)⇒(d). Let $B_\varepsilon(x)$ be an open ball centered at $x \in X$ with radius $\varepsilon > 0$. We shall say that an A-valued sequence $\{x_k\}_{k\in\mathbb{N}}$ has Property P_n, for some integer $n \in \mathbb{N}$, if x_k is in $B_{1/k}(x)\backslash\{x\}$ for each $k = 1, \ldots, n+1$ and if $d(x_{k+1}, x) < d(x_k, x)$ for every $k = 1, \ldots, n$.

Claim. If assertion (b) holds true, then there exists an A-valued sequence that has Property P_n for every $n \in \mathbb{N}$.

Proof. Suppose assertion (b) holds true so that $(B_\varepsilon(x)\backslash\{x\}) \cap A \neq \varnothing$ for every $\varepsilon > 0$. Now take an arbitrary x_1 in $(B_1(x)\backslash\{x\}) \cap A$ and an arbitrary x_2 in $(B_{\varepsilon_2}(x)\backslash\{x\}) \cap A$ with $\varepsilon_2 = \min\{\frac{1}{2}, d(x_1, x)\}$. Every A-valued sequence whose first two entries coincide with x_1 and x_2 has Property P_1. Suppose there exists an A-valued sequence that has Property P_n for some integer $n \in \mathbb{N}$. Take any point from $(B_{\varepsilon_{n+2}}(x)\backslash\{x\}) \cap A$ where $\varepsilon_{n+2} = \min\{\frac{1}{n+2}, d(x_{n+1}, x)\}$, and replace the $(n+2)$th entry of that sequence with this point. The resulting sequence has Property P_{n+1}. Thus there exists an A-valued sequence that has Property P_{n+1} whenever there exists one that has Property P_n, and this concludes the proof by induction. □

However, an A-valued sequence $\{x_k\}_{k\in\mathbb{N}}$ that has Property P_n for every $n \in \mathbb{N}$ in fact is an $A\backslash\{x\}$-valued sequence of pairwise distinct points such that $0 < d(x_k, x) < \frac{1}{k}$ for every $k \in \mathbb{N}$. Therefore (b)⇒(d). □

Recall that "point of adherence" and "point of accumulation" are concepts defined for sets, while "limit of a convergent sequence" is, of course, a concept defined for sequences. But the range of a sequence is a set, and it can have (many) accumulation points. Let (X, d) be a metric space and let $\{x_n\}$ be an X-valued sequence. A point x in X is a *cluster point of the sequence* $\{x_n\}$ if

some subsequence of $\{x_n\}$ converges to x. The cluster points of a sequence are precisely the accumulation points of its range (Proposition 3.28). If a sequence is convergent, then (Proposition 3.5) its range has only one point of accumulation which coincides with the unique limit of the sequence.

Corollary 3.29. *The derived set $A^\star$ of every subset A of a metric space (X, d) is closed in (X, d).*

Proof. Let A be an arbitrary subset of a metric space (X, d). We want to show that $(A^\star)^- = A^\star$ (i.e., $A^\star$ is closed) or, equivalently, $(A^\star)^- \subseteq A^\star$ (recall: every set is included in its closure). If A is empty, then the result is trivially verified ($\varnothing^\star = \varnothing = \varnothing^-$). Thus suppose A is nonempty. Take an arbitrary x^- in $(A^\star)^-$ and an arbitrary $\varepsilon > 0$. Proposition 3.27 ensures that $B_\varepsilon(x^-) \cap A^\star \neq \varnothing$. Take $x^\star$ in $B_\varepsilon(x^-) \cap A^\star$ and set $\delta = \varepsilon - d(x^\star, x^-)$. Note that $0 < \delta \leq \varepsilon$ (because $0 \leq d(x^\star, x^-) < \varepsilon$). Since $x^\star \in A^\star$, we get by Proposition 3.28 that $B_\delta(x^\star) \cap A$ contains infinitely many points. Take x in $B_\delta(x^\star) \cap A$ distinct from x^- and from $x^\star$. Thus $0 < d(x, x^-) \leq d(x, x^\star) + d(x^\star, x^-) < \delta + d(x^\star, x^-) = \varepsilon$ by the triangle inequality. Therefore $x \in B_\varepsilon(x^-)$ and $x \neq x^-$. Conclusion: Every nonempty ball $B_\varepsilon(x^-)$ centered at x^- contains a point x of A other than x^-. Thus $x^- \in A^\star$ by Proposition 3.28, and so $(A^\star)^- \subseteq A^\star$. □

The preceding corollary does not hold in a general topological space. Indeed, if a set X containing more than one point is equipped with the indiscrete topology (where the only open sets are $\varnothing$ and X), then the derived set $\{x\}^\star$ of a singleton $\{x\}$ is $X \backslash \{x\}$ which is not closed in that topology.

Theorem 3.30. (The Closed Set Theorem). *A subset A of a set X is closed in the metric space (X, d) if and only if every A-valued sequence that converges in (X, d) has its limit in A.*

Proof. (a) Take an arbitrary A-valued sequence $\{x_n\}$ that converges to $x \in X$ in (X, d). By Theorem 3.14 $\{x_n\}$ is eventually in every neighborhood of x, and hence every neighborhood of x contains a point of A. Thus x is a point of adherence of A (Proposition 3.25); that is, $x \in A^-$. If $A = A^-$ (equivalently, if A is closed in (X, d)), then $x \in A$.

(b) Conversely, take an arbitrary point $x \in A^-$ (i.e., an arbitrary point of adherence of A). According to Proposition 3.27, there exists an A-valued sequence that converges to x in (X, d). If every A-valued sequence that converges in (X, d) has its limit in A, then $x \in A$. Thus $A^- \subseteq A$, and hence $A = A^-$ (since $A \subseteq A^-$ for every set A). That is, A is closed in (X, d). □

This is a particularly useful result that will often be applied throughout this book. Part (a) of the proof holds for general topological spaces but not part (b). The counterpart of the above theorem for general (not necessarily metrizable) topological spaces is stated in terms of nets (instead of sequences).

Example 3.N. Consider the set $B[X,Y]$ of all bounded mappings of a metric space (X,d_X) into a metric space (Y,d_Y), and let $BC[X,Y]$ denote the subset of $B[X,Y]$ consisting of all bounded continuous mappings of (X,d_X) into (Y,d_Y). Equip $B[X,Y]$ with the sup-metric d_∞ as in Example 3.C. We shall use the Closed Set Theorem to show that

$$BC[X,Y] \text{ is closed in } (B[X,Y],d_\infty).$$

Take any $BC[X,Y]$-valued sequence $\{f_n\}$ that converges in $(B[X,Y],d_\infty)$ to a mapping $f \in B[X,Y]$. The triangle inequality in (Y,d_Y) ensures that

$$d_Y(f(u),f(v)) \le d_Y(f(u),f_n(u)) + d_Y(f_n(u),f_n(v)) + d_Y(f_n(v),f(v))$$

for each integer n and every $u,v \in X$. Take an arbitrary real number $\varepsilon > 0$. Since $f_n \to f$ in $(B[X,Y],d_\infty)$, it follows that there exists a positive integer n_ε such that $d_\infty(f_n,f) = \sup_{x\in X} d_Y(f_n(x),f(x)) < \frac{\varepsilon}{3}$, and so $d_Y(f_n(x),f(x)) < \frac{\varepsilon}{3}$ for all $x \in X$, whenever $n \ge n_\varepsilon$ (uniform convergence — see Example 3.G). Since each f_n is continuous, it follows that there exists a real number $\delta_\varepsilon > 0$ (which may depend on u and v) such that $d_Y(f_{n_\varepsilon}(u),f_{n_\varepsilon}(v)) < \frac{\varepsilon}{3}$ whenever $d_X(u,v) < \delta_\varepsilon$. Therefore $d_Y(f(u),f(v)) < \varepsilon$ whenever $d_X(u,v) < \delta_\varepsilon$, so that f is continuous. That is, $f \in BC[X,Y]$. Thus, according to Theorem 3.30, $BC[X,Y]$ is a closed subset of the metric space $(B[X,Y],d_\infty)$. Particular case (see Examples 3.D and 3.G): $C[0,1]$ is closed in $(B[0,1],d_\infty)$.

3.6 Dense Sets and Separable Spaces

Let A be a set in a metric space X, and let $\mathcal{U}_A$ be the collection of all open subsets of X included in A:

$$\mathcal{U}_A = \big\{U \in \wp(X):\ U \text{ is open in } X \text{ and } U \subseteq A\big\}.$$

The empty set $\varnothing$ of X always belongs to $\mathcal{U}_A$ so that $\mathcal{U}_A$ is never empty. The union of all sets in $\mathcal{U}_A$ is called the *interior* of A in X, denoted by A° (i.e., $A^\circ = \bigcup \mathcal{U}_A$). According to Theorem 3.15(c), it follows that

$$A^\circ \text{ is open in } X \quad \text{and} \quad A^\circ \subseteq A.$$

If $U \in \mathcal{U}_A$, then it is plain that $U \subseteq \bigcup \mathcal{U}_A = A^\circ$. Thus, with respect to the inclusion ordering of $\wp(X)$,

$$A^\circ \text{ is the largest open subset of } X \text{ that is included in } A,$$

and hence (since A° is open in X)

$$A \text{ is open in } X \quad \text{if and only if} \quad A^\circ = A.$$

From the above displayed results it is readily verified that

$$\varnothing^\circ = \varnothing, \quad X^\circ = X, \quad (A^\circ)^\circ = A^\circ$$

and, if B also is a set in X,

$$A \subseteq B \quad \text{implies} \quad A^\circ \subseteq B^\circ.$$

Moreover, since $A \cap B$ is a subset of both A and B, we get $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. On the other hand, since $(A \cap B)^\circ$ is the largest open subset of X that is included in $A \cap B$, and since $A^\circ \cap B^\circ$ is open (Theorem 3.15(b)) and is included in $A \cap B$ (because $A^\circ \subseteq A$ and $B^\circ \subseteq B$ so that $A^\circ \cap B^\circ \subseteq A \cap B$), it follows that $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$. Therefore, if A and B are subsets of X, then

$$(A \cap B)^\circ \,=\, A^\circ \cap B^\circ.$$

It is shown by induction that the above identity holds for any finite collection of subsets of X. That is, *the interior of the intersection of a finite collection of subsets of X coincides with the intersection of their interiors*. In general (i.e., by allowing infinite collections as well) one has inclusion rather than equality. Indeed, if $\{A_\gamma\}_{\gamma\in\Gamma}$ is an arbitrary indexed family of subsets of X, then

$$\Big(\bigcap_\gamma A_\gamma\Big)^\circ \subseteq \bigcap_\gamma A_\gamma^\circ$$

since $\bigcap_\gamma A_\gamma \subseteq A_\alpha$ and hence $(\bigcap_\gamma A_\gamma)^\circ \subseteq A_\alpha$ for each index $\alpha \in \Gamma$. Similarly,

$$\bigcup_\gamma A_\gamma^\circ \,\subseteq\, \Big(\bigcup_\gamma A_\gamma\Big)^\circ$$

since $\bigcup_\gamma A_\gamma^\circ \subseteq \bigcup_\gamma A_\gamma$ and $\bigcup_\gamma A_\gamma^\circ$ is open in X by Theorem 3.15(c). However, these inclusions are not reversible in general, so that equality does not hold.

Example 3.O. This is the dual of Example 3.M. Consider the setup of Example 3.M and set $C_n = X\backslash A_n$, which is open in $\mathbb{R}$ for each positive integer n, and $C = X\backslash A$, which is not open in $\mathbb{R}$. Since

$$\bigcap_{n=1}^\infty C_n \,=\, \bigcap_{n=1}^\infty (X\backslash A_n) \,=\, X\Big\backslash \bigcup_{n=1}^\infty A_n \,=\, X\backslash A \,=\, C,$$

it follows that *the intersection of an infinite collection of open sets is not necessarily open* (see Theorem 3.15(b)). Moreover, as $C_n^\circ = C_n$ for each n,

$$(X\backslash A)^\circ \,=\, C^\circ \,=\, \Big(\bigcap_{n=1}^\infty C_n\Big)^\circ \,\subset\, \bigcap_{n=1}^\infty C_n^\circ \,=\, C \,=\, X\backslash A,$$

which is a *proper* inclusion. Now set $D = X\backslash B = (-\infty, 1) \cup (2, \infty)$ (so that $D^\circ = D$). Thus $C^\circ \cup D^\circ$ is a *proper* subset of $(C \cup D)^\circ$:

$$\mathbb{R}\backslash\{1\} \,=\, C^\circ \cup D^\circ \,\subset\, (C \cup D)^\circ \,=\, \mathbb{R}.$$

Remark: The duality between "interior" and "closure" is clear:

$$(X\backslash A)^- = X\backslash A^\circ \quad \text{and} \quad (X\backslash A)^\circ = X\backslash A^-$$

for every $A \subseteq X$. Indeed, $U \in \mathcal{U}_A$ if and only if $X\backslash U \in \mathcal{V}_{X\backslash A}$ (i.e., U is open in X and $U \subseteq A$ if and only if $X\backslash U$ is closed in X and $X\backslash A \subseteq X\backslash U$) and, dually, $V \in \mathcal{V}_{X\backslash A}$ if and only if $X\backslash V \in \mathcal{U}_A$. Thus $A^\circ = \bigcup_{U\in\mathcal{U}_A} U = X\backslash\bigcap_{U\in\mathcal{U}_A}(X\backslash U) = X\backslash\bigcap_{V\in\mathcal{V}_{X\backslash A}} V = X\backslash(X\backslash A)^-$ and so $X\backslash A^\circ = (X\backslash A)^-$; which implies (swap A and $X\backslash A$) that $X\backslash(X\backslash A)^\circ = A^-$ and hence $(X\backslash A)^\circ = X\backslash A^-$. This confirms the above identities and also their equivalent forms:

$$A^\circ = X\backslash(X\backslash A)^- \quad \text{and} \quad A^- = X\backslash(X\backslash A)^\circ.$$

Thus it is easy to show that $A^-\backslash(X\backslash A) = A$ and $A\backslash(X\backslash A^\circ) = A^\circ$.

A point $x \in X$ is an *interior point* of A if it belongs to the interior A° of A. It is clear that every interior point of A is a point of A (i.e., $A^\circ \subseteq A$), and it is readily verified that $x \in A$ *is an interior point of* A *if and only if there exists a neighborhood of* x *included in* A (reason: A° is the largest open neighborhood of every interior point of A that is included in A). The interior of the complement of A, $(X\backslash A)^\circ$, is called the *exterior* of A, and a point $x \in X$ is an *exterior point* of A if it belongs to the exterior $(X\backslash A)^\circ$ of A.

A subset A of a metric space X is called *dense* in X (or *dense everywhere*) if its closure A^- coincides with X (i.e., if $A^- = X$). More generally, suppose A and B are subsets of a metric space X such that $A \subseteq B$. A is *dense in* B if $B \subseteq A^-$ or, equivalently, if $A^- = B^-$ (why?). Clearly, if $A \subseteq B$ and $A^- = X$, then $B^- = X$. Note that the only closed set dense in X is X itself.

Proposition 3.31. *Let* A *be a subset of a metric space* X*. The following assertions are pairwise equivalent.*

(a) $A^- = X$ (i.e., A *is dense in* X).

(b) *Every nonempty open subset of* X *meets* A.

(c) $\mathcal{V}_A = \{X\}$.

(d) $(X\backslash A)^\circ = \varnothing$ (i.e., *the complement of* A *has empty interior*).

Proof. Take any nonempty open subset U of X, and take an arbitrary u in $U \subseteq X$. If (a) holds true, then every point of X is adherent to A. In particular, u is adherent to A. Thus Proposition 3.25 ensures that U meets A. Conclusion: (a)⇒(b). Now take an arbitrary proper closed subset V of X so that $\varnothing \neq X\backslash V$ is open in X. If (b) holds true, then $(X\backslash V) \cap A \neq \varnothing$. Thus V does not include A, and so $V \notin \mathcal{V}_A$. Hence (b)⇒(c). Since $A^- \in \mathcal{V}_A$, it follows that (c)⇒(a). The equivalence (a)⇔(d) is obvious from the identity $A^- = X\backslash(X\backslash A)^\circ$. □

The reader has probably observed that the concepts and results so far in this section apply to topological spaces in general. From now on the metric will play its role. Note that *a point in a subset* A *of a metric space* X *is an interior point of* A *if and only if it is the center of a nonempty open ball*

included in A (reason: every nonempty open ball is a neighborhood and every neighborhood includes a nonempty open ball). We shall say that (A, d) is a *dense subspace* of a metric space (X, d) if the subset A of X is dense in (X, d).

Proposition 3.32. *Let (X, d) be a metric space and let A and B be subsets of X such that $\varnothing \neq A \subseteq B \subseteq X$. The following assertions are pairwise equivalent.*

(a) $A^- = B^-$ (i.e., *A is dense in B*).

(b) *Every nonempty open ball centered at any point b of B meets A.*

(c) $\inf_{a \in A} d(b, a) = 0$ *for every* $b \in B$.

(d) *For every point b in B there exists an A-valued sequence $\{a_n\}$ that converges in (X, d) to b.*

Proof. Recall that $A^- = B^-$ if and only if $B \subseteq A^-$. Let b be an arbitrary point in B. Thus assertion (a) can be rewritten as follows.

(a′) Every point b in B is a point of adherence of A.

Now notice that assertions (a′), (b), (c), and (d) are pairwise equivalent by Proposition 3.27. □

Corollary 3.33. *Let F and G be continuous mappings of a metric space X into a metric space Y. If F and G coincide on a dense subset of X, then they coincide on the whole space X.*

Proof. Suppose X is nonempty to avoid trivialities. Let A be a nonempty dense subset of X. Take an arbitrary $x \in X$ and let $\{a_n\}$ be an A-valued sequence that converges in X to x (whose existence is ensured by Proposition 3.32). If $F\colon X \to Y$ and $G\colon X \to Y$ are continuous mappings such that $F|_A = G|_A$, then

$$F(x) \;=\; F(\lim a_n) \;=\; \lim F(a_n) \;=\; \lim G(a_n) \;=\; G(\lim a_n) \;=\; G(x)$$

(Corollary 3.8). Thus $F(x) = G(x)$ for every $x \in X$; that is, $F = G$. □

A metric space (X, d) is *separable* if there exists a countable dense set in X. The density criteria in Proposition 3.3 (with $B = X$) are particularly useful to check separability.

Example 3.P. Take an arbitrary integer $n \geq 1$, an arbitrary real $p \geq 1$, and consider the metric space $(\mathbb{R}^n, d_p)$ of Example 3.A. Since the set of all rational numbers $\mathbb{Q}$ is dense in the real line $\mathbb{R}$ equipped with its usual metric, it follows that $\mathbb{Q}^n$ (the set of all rational n-tuples) is dense in $(\mathbb{R}^n, d_p)$. Indeed, $\mathbb{Q}^- = \mathbb{R}$ implies that $\inf_{\upsilon \in \mathbb{Q}} |\xi - \upsilon| = 0$ for every $\xi \in \mathbb{R}$, which in turn implies that $\inf_{y \in \mathbb{Q}^n} d_p(x, y) = \inf_{y=(\upsilon_1,\ldots,\upsilon_n) \in \mathbb{Q}^n} \big(\sum_{i=1}^n |\xi_i - \upsilon_i|^p\big)^{\frac{1}{p}} = 0$ for every vector $x = (\xi_1, \ldots, \xi_n)$ in $\mathbb{R}^n$. Hence $(\mathbb{Q}^n)^- = \mathbb{R}^n$ according to Proposition 3.32. Moreover, since $\#\mathbb{Q}^n = \#\mathbb{Q} = \aleph_0$ (Problems 1.25(c) and 2.8), it follows that $\mathbb{Q}^n$ is countably infinite. Thus $\mathbb{Q}^n$ is a countable dense subset of $(\mathbb{R}^n, d_p)$, and so

$$(\mathbb{R}^n, d_p) \text{ is a separable metric space.}$$

Now consider the metric space (ℓ_+^p, d_p) for any $p \geq 1$ as in Example 3.B, where ℓ_+^p is the set of all *real-valued* p-summable infinite sequences. Let ℓ_+^0 be the subset of ℓ_+^p made up of all real-valued infinite sequences with a finite number of nonzero entries, and let X be the subset of ℓ_+^0 consisting of all rational-valued infinite sequences with a finite number of nonzero entries. The set ℓ_+^0 is dense in (ℓ_+^p, d_p) — Problem 3.44(b). Since $\mathbb{Q}^- = \mathbb{R}$, it follows that X is dense in (ℓ_+^0, d_p) — the proof is essentially the same as the proof that $\mathbb{Q}^n$ is dense in $(\mathbb{R}^n, d_p)$. Thus $X^- = (\ell_+^0)^- = \ell_+^p$, and so X is dense in (ℓ_+^p, d_p). Next we show that X is countably infinite. In fact, X is a linear space over the rational field $\mathbb{Q}$ and $\dim X = \aleph_0$ (see Example 2.J). Thus $\#X = \max\{\#\mathbb{Q}, \dim X\} = \aleph_0$ by Problem 2.8. Conclusion: X is a countable dense subset of (ℓ_+^p, d_p), and so

$$(\ell_+^p, d_p) \text{ is a separable metric space.}$$

The same argument is readily extended to complex spaces so that $(\mathbb{C}^n, d_p)$ also is separable, as well as (ℓ_+^p, d_p) when ℓ_+^p is made up of all *complex-valued* p-summable infinite sequences. Finally we show that (see Example 3.D)

$$(C[0,1], d_\infty) \text{ is a separable metric space.}$$

Actually, the set $P[0,1]$ of all polynomials on $[0,1]$ is dense in $(C[0,1], d_\infty)$. This is the well-known *Weierstrass Theorem*, which says that *every continuous function in* $C[0,1]$ *is the uniform limit of a sequence of polynomials in* $P[0,1]$ (i.e., for every $x \in C[0,1]$ there exists a $P[0,1]$-valued sequence $\{p_n\}$ such that $d_\infty(p_n, x) \to 0$). Moreover, it is easy to show that the set X of all polynomials on $[0,1]$ with rational coefficients is dense in $(P[0,1], d_\infty)$, and so X is dense in $(C[0,1], d_\infty)$. Since X is a linear space over the rational field $\mathbb{Q}$, and since $\dim X = \aleph_0$ (essentially the same proof as in Example 2.M), we get by Problem 2.8 that X is countable. Thus X is a countable dense subset of $(C[0,1], d_\infty)$.

A collection $\mathcal{B}$ of open subsets of a metric space X is a *base* (or a *topological base*) for X if every open set in X is the union of some subcollection of $\mathcal{B}$. For instance, the collection of all open balls in a metric space (including the empty ball) is a base for X (cf. Corollary 3.16). Note that the above definition forces the empty set $\varnothing$ of X to be a member of any base for X if the subcollection is nonempty.

Proposition 3.34. *Let* $\mathcal{B}$ *be a collection of open subsets of a metric space* X *that contains the empty set. The following assertions are pairwise equivalent.*

(a) $\mathcal{B}$ *is a base for* X.

(b) *For every nonempty open subset* U *of* X *and every point* x *in* U *there exists a set* B *in* $\mathcal{B}$ *such that* $x \in B \subseteq U$.

(c) *For every* x *in* X *and every neighborhood* N *of* x *there exists a set* B *in* $\mathcal{B}$ *such that* $x \in B \subseteq N$.

Proof. Take an arbitrary open subset U of the metric space X and set $\mathcal{B}_U = \{B \in \mathcal{B}\colon B \subseteq U\}$. If $\mathcal{B}$ is a base for X, then $U = \bigcup \mathcal{B}_U$ by the definition of base. Thus, if $x \in U$, then $x \in B$ for some $B \in \mathcal{B}_U$ so that $x \in B \subseteq U$. That is, (a) implies (b). On the other hand, if (b) holds, then any open subset U of X clearly coincides with $\bigcup \mathcal{B}_U$, which shows that (a) holds true. Finally note that (b) and (c) are trivially equivalent: every neighborhood of x includes an open set containing x, and every open set containing x is a neighborhood of x. □

Theorem 3.35. *A metric space is separable if and only if it has a countable base.*

Proof. Suppose $\mathcal{B} = \{B_n\}$ is a countable base for X. Consider a set $\{b_n\}$ with each b_n taken from each nonempty set B_n in $\mathcal{B}$. Proposition 3.34(b) ensures that for every nonempty open subset U of X there exists a set B_n such that $B_n \subseteq U$, and therefore $U \cap \{b_n\} \neq \varnothing$. Thus Proposition 3.31(b) says that the countable set $\{b_n\}$ is dense in X, and so X is separable. On the other hand, suppose X is separable, which means that there is a countable subset A of X that is dense in X. Consider the collection $\mathcal{B} = \{B_{1/n}(a)\colon\ n \in \mathbb{N} \text{ and } a \in A\}$ of nonempty open balls, which is a double indexed family, indexed by $\mathbb{N} \times A$ (i.e., by two countable sets), and thus a countable collection itself. In other words, $\#\mathcal{B} = \#(\mathbb{N} \times A) = \max\{\#\mathbb{N}, \#A\} = \#\mathbb{N}$ — cf. Problem 1.30(b).

Claim. For every $x \in X$ and every neighborhood N of x there exists a ball in $\mathcal{B}$ containing x and included in N.

Proof. Take an arbitrary $x \in X$ and an arbitrary neighborhood N of x. Let $B_\varepsilon(x)$ be an open ball of radius $\varepsilon > 0$, centered at x, and included in N. Take a positive integer n such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and a point $a \in A$ such that $a \in B_{1/n}(x)$ (recall: since $A^- = X$, it follows by Proposition 3.32(b) that there exists $a \in A$ such that $a \in B_\rho(x)$ for every $x \in X$ and every $\rho > 0$). Obviously, $x \in B_{1/n}(a)$. Moreover, if $y \in B_{1/n}(a)$, then $d(x, y) \leq d(x, a) + d(a, y) < \frac{2}{n} < \varepsilon$ so that $y \in B_\varepsilon(x)$, and hence $B_{1/n}(a) \subseteq B_\varepsilon(x)$. Thus $x \in B_{1/n}(a) \subseteq B_\varepsilon(x) \subseteq N$. □

Therefore the countable collection $\mathcal{B} \cup \{\varnothing\}$ of open balls is a base for X by Proposition 3.34(c). □

Corollary 3.36. *Every subspace of a separable metric space is itself separable.*

Proof. Let S be a subspace of a separable metric space X and, according to Theorem 3.35, let $\mathcal{B}$ be a countable base for X. Set $\mathcal{B}_S = \{S \cap B\colon\ B \in \mathcal{B}\}$, which is a countable collection of subsets of S. Since the sets in $\mathcal{B}$ are open subsets of X, it follows that the sets in $\mathcal{B}_S$ are open relative to S (see Problem 3.38(c)). Take an arbitrary nonempty relatively open subset A of S so that $A = S \cap U$ for some open subset U of X (Problem 3.38(c)). Since $U = \bigcup \mathcal{B}'$ for some subcollection $\mathcal{B}'$ of $\mathcal{B}$, it follows that $A = S \cap \bigcup_{B \in \mathcal{B}'} B = \bigcup_{B \in \mathcal{B}'} S \cap B = \bigcup \mathcal{B}'_S$, where $\mathcal{B}'_S = \{S \cap B\colon\ B \in \mathcal{B}'\}$ is a subcollection of $\mathcal{B}_S$. Thus $\mathcal{B}_S$ is a base for S. Therefore the subspace S has a countable base, which means by the previous theorem that S is separable. □

Let A be a subset of a metric space. An *isolated point* of A is a point in A that is not an accumulation point of A. That is, a point x is an isolated point of A if $x \in A \backslash A^\star$.

Proposition 3.37. *Let A be a subset of a metric space X and let x be a point in A. The following assertions are pairwise equivalent.*

(a) *x is an isolated point of A.*

(b) *There exists an open set U in X such that $A \cap U = \{x\}$.*

(c) *There exists a neighborhood N of x such that $A \cap N = \{x\}$.*

(d) *There exists an open ball $B_\rho(x)$ centered at x such that $A \cap B_\rho(x) = \{x\}$.*

Proof. Assertion (a) is equivalent to assertion (b) by Proposition 3.26. Assertions (b), (c), and (d) are trivially pairwise equivalent. □

A subset A of X consisting entirely of isolated points is a *discrete subset* of X. This means that in the subspace A every set is open, and hence the subspace A is homeomorphic to a discrete space (i.e., to a metric space equipped with the discrete metric). According to Theorem 3.35 and Corollary 3.36, *a discrete subset of a separable metric space is countable.* Thus, *if a metric space has an uncountable discrete subset, then it is not separable.*

Example 3.Q. Let S be a set, let (Y, d) be a metric space, and consider the metric space $(B[S, Y], d_\infty)$ of all bounded mappings of S into (Y, d) equipped with the sup-metric d_∞ (Example 3.C). Suppose Y has more than one point, and let y_0 and y_1 be two distinct points in Y. As usual, let 2^S denote the set of all mappings on S with values either y_0 or y_1 (i.e., the set of all mappings of S into $\{y_0, y_1\}$ so that $2^S = \{y_0, y_1\}^S \subseteq B[S, Y]$). If $f, g \in 2^S$ and $f \neq g$ (i.e., if f and g are two distinct mappings on S with values either y_0 or y_1), then

$$d_\infty(f, g) \;=\; \sup_{s \in S} d(f(s), g(s)) \;=\; d(y_0, y_1) \;\neq\; 0.$$

Therefore, any open ball $B_\rho(g) = \{f \in 2^S\colon\ d_\infty(f, g) < \rho\}$ centered at an arbitrary point g of 2^S with radius $\rho = d(y_0, y_1)/2$ is such that $2^S \cap B_\rho(g) = \{g\}$. This means that every point of 2^S is an isolated point of it, and hence

$$2^S \text{ is a discrete set in } (B[S, Y], d_\infty).$$

If S is an infinite set, then 2^S is an uncountable subset of $B[S, Y]$ (recall: if S is infinite, then $\aleph_0 \leq \#S < \#2^S$ by Theorems 1.4 and 1.5). Thus $(B[S, Y], d_\infty)$ *is not separable whenever* $2 \leq \#Y$ *and* $\aleph_0 \leq \#S$. Concrete example:

$$(\ell_+^\infty, d_\infty) \text{ is not a separable metric space.}$$

Indeed, set $S = \mathbb{N}$ and $Y = \mathbb{C}$ (or $Y = \mathbb{R}$) with its usual metric d, so that $(B[S, Y], d_\infty) = (\ell_+^\infty, d_\infty)$: the set of all scalar-valued bounded sequences

equipped with the sup-metric, as introduced in Example 3.B. The set $2^{\mathbb{N}}$, consisting of all sequences with values either 0 or 1, is an uncountable discrete subset of $(\ell_+^\infty, d_\infty)$.

In a discrete subset every point is isolated. The opposite notion is that of a set where every point is not isolated. A subset A of a metric space X is *dense in itself* if A has no isolated point or, equivalently, if every point in A is an accumulation point of A; that is, if $A \subseteq A^\star$. Since $A^- = A \cup A^\star$ for every subset A of X, it follows that a set A is dense in itself if and only if $A^\star = A^-$. A subset A of X that is both closed in X and dense in itself (i.e., such that $A^\star = A$) is a *perfect set*: a closed set without isolated points. For instance, $\mathbb{Q} \cap [0,1]$ is a countable perfect subset of the metric space $\mathbb{Q}$, but it is *not* perfect in the metric space $\mathbb{R}$ (since it is not closed in $\mathbb{R}$). As a matter of fact, every nonempty perfect subset of $\mathbb{R}$ is uncountable because $\mathbb{R}$ is a "complete" metric space, a concept that we shall define next.

3.7 Complete Spaces

Consider the metric space $(\mathbb{R}, d)$, where d denotes the usual metric on the real line $\mathbb{R}$, and let (A, d) be the subspace of $(\mathbb{R}, d)$ with $A = (0, 1]$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be the A-valued sequence such that $\alpha_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. Does $\{\alpha_n\}$ converge in the metric space (A, d)? It is clear that $\{\alpha_n\}$ converges to 0 in $(\mathbb{R}, d)$, and hence we might at first glance think that it also converges in (A, d). But the point 0 simply does not exist in A, so that it is nonsense to say that "$\alpha_n \to 0$ in (A, d)". In fact, $\{\alpha_n\}$ does not converge in the metric space (A, d). However, the sequence $\{\alpha_n\}$ seems to possess a "special property" that makes it apparently convergent in spite of the particular underlying set A, and the metric space (A, d) in turn seems to bear a "peculiar characteristic" that makes such a sequence fail to converge in it. The "special property" of the sequence $\{\alpha_n\}$ is that it is a *Cauchy sequence* in (A, d) and the "peculiar characteristic" of the metric space (A, d) is that it is not *complete*.

Definition 3.38. Let (X, d) be a metric space. An X-valued sequence $\{x_n\}$ (indexed either by $\mathbb{N}$ or $\mathbb{N}_0$) is a *Cauchy sequence* in (X, d) (or satisfies the *Cauchy criterion*) if for each real number $\varepsilon > 0$ there exists a positive integer n_ε such that

$$n, m \geq n_\varepsilon \quad \text{implies} \quad d(x_m, x_n) < \varepsilon.$$

A usual notation for the Cauchy criterion is $\lim_{m,n} d(x_m, x_n) = 0$. Equivalently, an X-valued sequence $\{x_n\}$ is a Cauchy sequence if $\operatorname{diam}(\{x_k\}_{n\leq k}) \to 0$ as $n \to \infty$ (i.e., $\lim_n \operatorname{diam}(\{x_k\}_{n\leq k}) = 0$).

Basic facts about Cauchy sequences are stated in the following proposition. In particular, it shows that *every convergent sequence is bounded*, and that *a Cauchy sequence has a convergent subsequence if and only if every subsequence of it converges* (see Proposition 3.5).

Proposition 3.39. *Let (X, d) be a metric space.*

(a) *Every convergent sequence in (X, d) is a Cauchy sequence.*

(b) *Every Cauchy sequence in (X, d) is bounded.*

(c) *If a Cauchy sequence in (X, d) has a subsequence that converges in (X, d), then it converges itself in (X, d) and its limit coincides with the limit of that convergent subsequence.*

Proof. (a) Take an arbitrary $\varepsilon > 0$. If an X-valued sequence $\{x_n\}$ converges to $x \in X$, then there exists an integer $n_\varepsilon \geq 1$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ whenever $n \geq n_\varepsilon$. Since $d(x_m, x_n) \leq d(x_m, x) + d(x, x_n)$ for every pair of indices m, n (triangle inequality), it follows that $d(x_m, x_n) < \varepsilon$ whenever $m, n \geq n_\varepsilon$.

(b) If $\{x_n\}$ is a Cauchy sequence, then there exists an integer $n_1 \geq 1$ such that $d(x_m, x_n) < 1$ whenever $m, n \geq n_1$. The set $\{d(x_m, x_n) \in \mathbb{R}\colon\ m, n \leq n_1\}$ has a maximum in $\mathbb{R}$, say β, because it is finite. Thus $d(x_m, x_n) \leq d(x_m, x_{n_1}) + d(x_{n_1}, x_n) \leq 2\max\{1, \beta\}$ for every pair of indices m, n.

(c) Suppose $\{x_{n_k}\}$ is a subsequence of an X-valued Cauchy sequence $\{x_n\}$ that converges to a point $x \in X$ (i.e., $x_{n_k} \to x$ as $k \to \infty$). Take an arbitrary $\varepsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence, it follows that there exists a positive integer n_ε such that $d(x_m, x_n) < \frac{\varepsilon}{2}$ whenever $m, n \geq n_\varepsilon$. Since $\{x_{n_k}\}$ converges to x, it follows that there exists a positive integer k_ε such that $d(x_{n_k}, x) < \frac{\varepsilon}{2}$ whenever $k \geq k_\varepsilon$. Thus, if j is any integer with the property that $j \geq k_\varepsilon$ and $n_j \geq n_\varepsilon$ (for instance, if $j = \max\{n_\varepsilon, k_\varepsilon\}$), then $d(x_n, x) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x) < \varepsilon$ for every $n \geq n_\varepsilon$, and therefore $\{x_n\}$ converges to x. □

Although a convergent sequence always is a Cauchy sequence, the converse may fail. For instance, the $(0, 1]$-valued sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space $((0, 1], d)$, where d is the usual metric on $\mathbb{R}$, that does not converge in $((0, 1], d)$. There are, however, metric spaces with the notable property that Cauchy sequences in them are convergent. Metric spaces possessing this property are so important that we give them a name. A metric space X is *complete* if every Cauchy sequence in X is a convergent sequence in X.

Theorem 3.40. *Let A be a subset of a metric space X.*

(a) *If the subspace A is complete, then A is closed in X.*

(b) *If X is complete and if A is closed in X, then the subspace A is complete.*

Proof. (a) Take an arbitrary A-valued sequence $\{a_n\}$ that converges in X. Since every convergent sequence is a Cauchy sequence, it follows that $\{a_n\}$ is a Cauchy sequence in X, and therefore a Cauchy sequence in the subspace A. If the subspace A is complete, then $\{a_n\}$ converges in A. Conclusion: If A is complete as a subspace of X, then every A-valued sequence that converges in X has its limit in A. Thus, according to the Closed Set Theorem (Theorem 3.30), A is closed in X.

(b) Take an arbitrary A-valued Cauchy sequence $\{a_n\}$. If X is complete, then $\{a_n\}$ converges in X to a point $a \in X$. If A is closed in X, then Theorem 3.30 (the Closed Set Theorem again) ensures that $a \in A$, and hence $\{a_n\}$ converges in the subspace A. Conclusion: If X is complete and A is closed in X, then every Cauchy sequence in the subspace A converges in A. That is, A is complete as a subspace of X. □

An important immediate corollary of the above theorem says that "inside" a *complete metric space* the properties of being closed and complete coincide.

Corollary 3.41. *Let X be a complete metric space. A subset A of X is closed in X if and only if the subspace A is complete.*

Example 3.R. (a) A basic property of the real number system is that every bounded sequence of real numbers has a convergent subsequence. This and Proposition 3.39 ensure that the metric space $\mathbb{R}$ (equipped with its usual metric) is complete; and so is the metric space $\mathbb{C}$ of all complex numbers equipped with its usual metric (reason: if $\{\alpha_k\}$ is a Cauchy sequence in $\mathbb{C}$, then $\{\mathrm{Re}\,\alpha_k\}$ and $\{\mathrm{Im}\,\alpha_k\}$ are both Cauchy sequences in $\mathbb{R}$ so that they converge in $\mathbb{R}$, and hence $\{\alpha_k\}$ converges in $\mathbb{C}$). Since the set $\mathbb{Q}$ of all rational numbers is not closed in $\mathbb{R}$ (recall: $\mathbb{Q}^- = \mathbb{R}$), it follows by Corollary 3.41 that the metric space $\mathbb{Q}$ is not complete. More generally (but similarly),

$$\mathbb{R}^n \text{ and } \mathbb{C}^n \text{ are complete metric spaces}$$

when equipped with any of their metrics d_p for $p \geq 1$ or d_∞ (as in Example 3.A), for every positive integer n, while

$$\mathbb{Q}^n \text{ is not a complete metric space.}$$

(b) Now let $\mathbb{F}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ equipped with their usual metrics. As we have just seen, $\mathbb{F}$ is a complete metric space. For each real number $p \geq 1$ let (ℓ^p_+, d_p) be the metric space of all $\mathbb{F}$-valued p-summable sequences equipped with its usual metric d_p as in Example 3.B. Take an arbitrary Cauchy sequence in (ℓ^p_+, d_p), say $\{x_n\}_{n\in\mathbb{N}}$. Recall that this is a sequence of sequences; that is, $x_n = \{\xi_{n(k)}\}_{k\in\mathbb{N}}$ is a sequence in ℓ^p_+ for each integer $n \in \mathbb{N}$. The Cauchy criterion says: for every $\varepsilon > 0$ there exists an integer $n_\varepsilon \geq 1$ such that $d_p(x_m, x_n) < \varepsilon$ whenever $m, n \geq n_\varepsilon$. Thus

$$|\xi_{m(k)} - \xi_{n(k)}| \leq \Big(\sum_{i=1}^{\infty} |\xi_{m(i)} - \xi_{n(i)}|^p\Big)^{\frac{1}{p}} = d_p(x_m, x_n) < \varepsilon$$

for every $k \in \mathbb{N}$ whenever $m, n \geq n_\varepsilon$. Therefore, for each $k \in \mathbb{N}$ the scalar-valued sequence $\{\xi_{n(k)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{F}$, and hence it converges in $\mathbb{F}$ (since $\mathbb{F}$ is complete) to, say, $\xi_{(k)} \in \mathbb{F}$. Consider the scalar-valued sequence $x = \{\xi_{(k)}\}_{k\in\mathbb{N}}$ consisting of those limits $\xi_{(k)} \in \mathbb{F}$ for every $k \in \mathbb{N}$. First we show that $x \in \ell^p_+$. Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (ℓ^p_+, d_p), it follows by

Proposition 3.39 that it is bounded (i.e., $\sup_{m,n} d_p(x_m, x_n) < \infty$), and hence $\sup_m d_p(x_m, 0) < \infty$ where 0 denotes the null sequence in ℓ_+^p. (Indeed, for every $m \in \mathbb{N}$ the triangle inequality ensures that $d_p(x_m, 0) \le \sup_{m,n} d_p(x_m, x_n) + d_p(x_n, 0)$ for an arbitrary $n \in \mathbb{N}$.) Therefore,

$$\Big(\sum_{k=1}^{j} |\xi_{n(k)}|^p\Big)^{\frac{1}{p}} \le \Big(\sum_{k=1}^{\infty} |\xi_{n(k)}|^p\Big)^{\frac{1}{p}} = d_p(x_n, 0) \le \sup_m d_p(x_m, 0)$$

for every $n \in \mathbb{N}$ and each integer $j \ge 1$. Since $\xi_{n(k)} \to \xi_{(k)}$ in $\mathbb{F}$ as $n \to \infty$ for each $k \in \mathbb{N}$, it follows that

$$\Big(\sum_{k=1}^{j} |\xi_{(k)}|^p\Big)^{\frac{1}{p}} = \lim_n \Big(\sum_{k=1}^{j} |\xi_{n(k)}|^p\Big)^{\frac{1}{p}} \le \sup_m d_p(x_m, 0)$$

for every $j \in \mathbb{N}$. Thus

$$\Big(\sum_{k=1}^{\infty} |\xi_{(k)}|^p\Big)^{\frac{1}{p}} = \sup_j \Big(\sum_{k=1}^{j} |\xi_{(k)}|^p\Big)^{\frac{1}{p}} \le \sup_m d_p(x_m, 0),$$

which means that $x = \{\xi_{(k)}\}_{k\in\mathbb{N}} \in \ell_+^p$. Next we show that $x_n \to x$ in (ℓ_+^p, d_p). Again, as $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (ℓ_+^p, d_p), for any $\varepsilon > 0$ there exists an integer $n_\varepsilon \ge 1$ such that $d_p(x_m, x_n) < \varepsilon$ whenever $m, n \ge n_\varepsilon$. Thus

$$\sum_{k=1}^{j} |\xi_{n(k)} - \xi_{m(k)}|^p \le \sum_{k=1}^{\infty} |\xi_{n(k)} - \xi_{m(k)}|^p < \varepsilon^p$$

for every integer $j \ge 1$ whenever $m, n \ge n_\varepsilon$. Since $\lim_m \xi_{m(k)} = \xi_{(k)}$ for each $k \in \mathbb{N}$, it follows that $\sum_{k=1}^{j} |\xi_{n(k)} - \xi_{(k)}|^p \le \varepsilon^p$, and hence

$$d_p(x_n, x) = \Big(\sum_{k=1}^{\infty} |\xi_{n(k)} - \xi_{(k)}|^p\Big)^{\frac{1}{p}} = \sup_j \Big(\sum_{k=1}^{j} |\xi_{n(k)} - \xi_{(k)}|^p\Big)^{\frac{1}{p}} \le \varepsilon,$$

whenever $n \ge n_\varepsilon$; which means that $x(n) \to x$ in (ℓ_+^p, d_p). Therefore

$$(\ell^p, d_p) \text{ is a complete metric space}$$

for every $p \ge 1$. Similarly (see Example 3.B), for each $p \ge 1$,

$$(\ell^p, d_p) \text{ is a complete metric space.}$$

Example 3.S. Let S be a nonempty set, let (Y, d) be a metric space, and consider the metric space $(B[S, Y], d_\infty)$ of all bounded mappings of S into (Y, d) equipped with the sup-metric d_∞ (Example 3.C). We claim that

$$(B[S, Y], d_\infty) \text{ is complete if and only if } (Y, d) \text{ is complete.}$$

(a) Indeed, suppose (Y, d) is a complete metric space. Let $\{f_n\}$ be a Cauchy sequence in $(B[S, Y], d_\infty)$. Thus $\{f_n(s)\}$ is a Cauchy sequence in (Y, d) for every $s \in S \neq \varnothing$ (because $d(f_m(s), f_n(s)) \leq \sup_{s\in S} d(f_m(s), f_n(s)) = d_\infty(f_m, f_n)$ for each pair of integers m, n and every $s \in S$), and hence $\{f_n(s)\}$ converges in (Y, d) for every $s \in S$ (since (Y, d) is complete). Set $f(s) = \lim_n f_n(s)$ for each $s \in S$ (i.e., $f_n(s) \to f(s)$ in (Y, d)), which defines a function f of S into Y. We shall show that $f \in B[S, Y]$ and that $f_n \to f$ in $(B[S, Y], d_\infty)$, thus proving that $(B[S, Y], d_\infty)$ is complete whenever (Y, d) is complete. First note that, for each positive integer n and every pair of points s, t in S,

$$d(f(s), f(t)) \leq d(f(s), f_n(s)) + d(f_n(s), f_n(t)) + d(f_n(t), f(t))$$

by the triangle inequality. Now take an arbitrary real number $\varepsilon > 0$. Since $\{f_n\}$ is a Cauchy sequence in $(B[S, Y], d_\infty)$, it follows that there exists a positive integer n_ε such that $d_\infty(f_m, f_n) = \sup_{s\in S} d(f_m(s), f_n(s)) < \varepsilon$, and hence

$$d(f_m(s), f_n(s)) \leq \varepsilon$$

for all $s \in S$, whenever $m, n \geq n_\varepsilon$. Moreover, since $f_m(s) \to f(s)$ in (Y, d) for every $s \in S$, and since the metric is continuous (i.e., $d(\cdot, y)\colon Y \to \mathbb{R}$ is a continuous function from the metric space Y to the metric space $\mathbb{R}$ for each $y \in Y$), it also follows that $d(f(s), f_n(s)) = d(\lim_m f_m(s), f_n(s)) = \lim_m d(f_m(s), f_n(s))$ for each positive integer n and every $s \in S$ (see Problem 3.14 or 3.34 and Corollary 3.8). Thus

$$d(f(s), f_n(s)) \leq \varepsilon$$

for all $s \in S$ whenever $n \geq n_\varepsilon$. Furthermore, since each f_n lies in $B[S, Y]$, it follows that there exists a real number γ_{n_ε} such that

$$\sup_{s,t\in S} d(f_{n_\varepsilon}(s), f_{n_\varepsilon}(t)) \leq \gamma_{n_\varepsilon}.$$

Therefore, for any $\varepsilon > 0$ there exists a positive integer n_ε such that

$$d(f(s), f(t)) \leq 2\varepsilon + \gamma_{n_\varepsilon}$$

for all $s, t \in S$, so that $f \in B[S, Y]$, and

$$d_\infty(f, f_n) = \sup_{s\in S} d(f(s), f_n(s)) \leq \varepsilon$$

whenever $n \geq n_\varepsilon$, so that $f_n \to f$ in $(B[S, Y], d_\infty)$.

(b) Conversely, suppose $(B[S, Y], d_\infty)$ is a complete metric space. Take an arbitrary Y-valued sequence $\{y_n\}$ and set $f_n(s) = y_n$ for each integer n and all $s \in S \neq \varnothing$. This defines a sequence $\{f_n\}$ of constant mappings of S into Y with each f_n clearly in $B[S, Y]$ (a constant mapping is obviously bounded). Note that $d_\infty(f_m, f_n) = \sup_{s\in S} d(f_m(s), f_n(s)) = d(y_m, y_n)$ for every pair of integers m, n. Thus $\{f_n\}$ is a Cauchy sequence in $(B[S, Y], d_\infty)$ if and only if

$\{y_n\}$ is a Cauchy sequence in (Y, d). Moreover, $\{f_n\}$ converges in $(B[S, Y], d_\infty)$ if and only if $\{y_n\}$ converges in (Y, d). (Reason: If $d(y_n, y) \to 0$ for some $y \in Y$, then $d_\infty(f_n, f) \to 0$ where $f \in B[S, Y]$ is the constant mapping $f(s) = y$ for all $s \in S$ and, on the other hand, if $d_\infty(f_n, f) \to 0$ for some $f \in B[S, Y]$, then $d(y_n, f(s)) = d(f_n(s), f(s))$ for each n and every s so that $d(y_n, f(s)) \to 0$ for all $s \in S$ — and hence f must be a constant mapping.) Now suppose (Y, d) is not complete, which implies that there exists a Cauchy sequence in (Y, d), say $\{y_n\}$, that fails to converge in (Y, d). Thus the sequence $\{f_n\}$ of constant mappings $f_n(s) = y_n$ for each integer n and all $s \in S$ is a Cauchy sequence in $(B[S, Y], d_\infty)$ that fails to converge in $(B[S, Y], d_\infty)$, and so $(B[S, Y], d_\infty)$ is not complete. Conclusion: If $(B[S, Y], d_\infty)$ is complete, then (Y, d) is complete.

(c) Concrete example: Set $S = \mathbb{N}$ or $S = \mathbb{Z}$ and $Y = \mathbb{F}$ (either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ equipped with their usual metric). Then

$$(\ell^\infty_+, d_\infty) \text{ and } (\ell^\infty, d_\infty) \text{ are complete metric spaces.}$$

Example 3.T. Consider the set $B[X, Y]$ of all bounded mappings of a nonempty metric space (X, d_X) into a metric space (Y, d_Y) and equip it with the sup-metric d_∞ as in the previous example. Let $BC[X, Y]$ be the set of all continuous mappings from $B[X, Y]$ (Example 3.N), so that $(BC[X, Y], d_\infty)$ is the subspace of $(B[X, Y], d_\infty)$ made up of all bounded continuous mappings of (X, d_X) into (Y, d_Y). If (Y, d_Y) is complete, then $(B[X, Y], d_\infty)$ is complete according to Example 3.S. Since $BC[X, Y]$ is closed in $(B[X, Y], d_\infty)$ (Example 3.N), it follows by Theorem 3.40 that $(BC[X, Y], d_\infty)$ is complete. On the other hand, the very same construction used in item (b) of the previous example shows that $(BC[X, Y], d_\infty)$ is not complete unless (Y, d_Y) is. Conclusion:

$$(BC[X, Y], d_\infty) \text{ is complete if and only if } (Y, d_Y) \text{ is complete.}$$

In particular (see Examples 3.D, 3.G, and 3.N),

$$(C[0, 1], d_\infty) \text{ is a complete metric space}$$

because $\mathbb{R}$ or $\mathbb{C}$ (equipped with their usual metrics, as always) are complete metric spaces (Example 3.R). However, for any $p \geq 1$ (see Problem 3.58),

$$(C[0, 1], d_p) \text{ is not a complete metric space.}$$

The concept of completeness leads to the next useful result on contractions.

Theorem 3.42. (Contraction Mapping Theorem or Method of Successive Approximations or Banach Fixed Point Theorem). *A strict contraction F of a nonempty complete metric space (X, d) into itself has a unique fixed point $x \in X$, which is the limit in (X, d) of every X-valued sequence of the form $\{F^n(x_0)\}_{n \in \mathbb{N}_0}$ for any $x_0 \in X$.*

Proof. Take any $x_0 \in X$. Consider the X-valued sequence $\{x_n\}_{n\in\mathbb{N}_0}$ such that

$$x_n \;=\; F^n(x_0)$$

for each $n \in \mathbb{N}_0$. Recall that F^n denotes the composition of $F\colon X \to X$ with itself n times (and that F^0 is by convention the identity map on X). It is clear that the sequence $\{x_n\}_{n\in\mathbb{N}_0}$ satisfies the difference equation

$$x_{n+1} \;=\; F(x_n)$$

for every $n \in \mathbb{N}_0$. Conversely, if an X-valued sequence $\{x_n\}_{n\in\mathbb{N}_0}$ is recursively defined from any point $x_0 \in X$ onwards as $x_{n+1} = F(x_n)$ for every $n \in \mathbb{N}_0$, then it is of the form $x_n = F^n(x_0)$ for each $n \in \mathbb{N}_0$ (proof: induction). Now suppose $F\colon (X,d) \to (X,d)$ is a strict contraction and let $\gamma \in (0,1)$ be any Lipschitz constant for F so that

$$d(F(x),F(y)) \;\le\; \gamma\, d(x,y)$$

for every x, y in X. A trivial induction shows that

$$d(F^n(x),F^n(y)) \;\le\; \gamma^n\, d(x,y)$$

for every nonnegative integer n and every $x, y \in X$. Next take an arbitrary pair of nonnegative distinct integers, say $m < n$. Note that

$$x_n \;=\; F^n(x_0) \;=\; F^m(F^{n-m}(x_0)) \;=\; F^m(x_{n-m}),$$

and hence

$$d(x_m,x_n) \;=\; d(F^m(x_0),F^m(x_{n-m})) \;\le\; \gamma^m\, d(x_0,x_{n-m}).$$

By using the triangle inequality we get

$$d(x_0,x_{n-m}) \;\le\; \sum_{i=0}^{n-m-1} d(x_i,x_{i+1}),$$

and therefore

$$d(x_m,x_n) \;\le\; \gamma^m \sum_{i=0}^{n-m-1} d(x_i,x_{i+1}) \;\le\; \gamma^m\Big(\sum_{i=0}^{n-m-1} \gamma^i\Big)\, d(x_0,x_1).$$

Another trivial induction shows that $\sum_{i=0}^{k-1}\alpha^i = \frac{1-\alpha^k}{1-\alpha}$ for every real $\alpha \ne 1$ and each integer $k \ge 1$. Thus $\sum_{i=0}^{n-m-1}\gamma^i = \frac{1-\gamma^{n-m}}{1-\gamma} < \frac{1}{1-\gamma}$ for any $\gamma \in (0,1)$ so that

$$d(x_m,x_n) \;<\; \frac{\gamma^m}{1-\gamma}\, d(x_0,x_1) \qquad \text{and} \qquad \gamma^m \to 0.$$

This ensures that $\{x_n\}$ is a Cauchy sequence in (X,d) (reason: for any $\varepsilon > 0$ there is an integer n_ε such that $\frac{\gamma^m}{1-\gamma}\, d(x_0,x_1) < \varepsilon$, which implies $d(x_m,x_n) < \varepsilon$

whenever $n > m \geq n_\varepsilon$). Hence $\{x_n\}$ converges in the complete metric space (X, d). Set $x = \lim x_n \in X$. Since a contraction is continuous, we get by Corollary 3.8 that $\{F(x_n)\}$ converges in (X, d) and $F(\lim x_n) = \lim F(x_n)$. Thus

$$x \;=\; \lim x_n \;=\; \lim x_{n+1} \;=\; \lim F(x_n) \;=\; F(\lim x_n) \;=\; F(x)$$

so that the limit of $\{x_n\}$ is a fixed point of F. Moreover, if y is any fixed point of F, then $d(x, y) = d(F(x), F(y)) \leq \gamma\, d(x, y)$, which implies that $d(x, y) = 0$ (since $\gamma \in (0, 1)$), and so $x = y$. Conclusion: For every $x_0 \in X$ the sequence $\{F^n(x_0)\}$ converges in (X, d), and its limit is the unique fixed point of F. □

3.8 Continuous Extension and Completion

Recall that continuity preserves convergence (Corollary 3.8). Uniform continuity, as one might expect, goes beyond that. In fact, uniform continuity also preserves Cauchy sequences.

Lemma 3.43. *Let $F\colon X \to Y$ be a uniformly continuous mapping of a metric space X into a metric space Y. If $\{x_n\}$ is a Cauchy sequence in X, then $\{F(x_n)\}$ is a Cauchy sequence in Y.*

Proof. The proof is straightforward by the definitions of Cauchy sequence and uniform convergence. Indeed, let d_X and d_Y denote the metrics on X and Y, respectively, and take an arbitrary X-valued sequence $\{x_n\}$. If $F\colon X \to Y$ is uniformly continuous, then for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$d_X(x_m, x_n) < \delta_\varepsilon \quad \text{implies} \quad d_Y(F(x_m), F(x_n)) < \varepsilon.$$

However, associated with δ_ε there exists a positive integer n_ε such that

$$m, n \geq n_\varepsilon \quad \text{implies} \quad d_X(x_m, x_n) < \delta_\varepsilon$$

whenever $\{x_n\}$ is a Cauchy sequence in X. Hence, for every real number $\varepsilon > 0$ there exists a positive integer n_ε such that

$$m, n \geq n_\varepsilon \quad \text{implies} \quad d_Y(F(x_m), F(x_n)) < \varepsilon,$$

which means that $\{F(x_n)\}$ is a Cauchy sequence in Y. □

Thus, if $G\colon X \to Y$ is a uniform homeomorphism between two metric spaces X and Y, then $\{x_n\}$ is a Cauchy sequence in X if and only if $\{G(x_n)\}$ is a Cauchy sequence in Y, and therefore a uniform homeomorphism takes a complete metric space onto a complete metric space.

Theorem 3.44. *Take two uniformly homeomorphic metric spaces. One of them is complete if and only if the other is.*

Proof. Let X and Y be metric spaces and let $G\colon X \to Y$ be a uniform homeomorphism. Take an arbitrary Cauchy sequence $\{y_n\}$ in Y and consider the sequence $\{x_n\}$ in X such that $x_n = G^{-1}(y_n)$ for each n. Lemma 3.43 ensures that $\{x_n\}$ is a Cauchy sequence in X. If X is complete, then $\{x_n\}$ converges in X to, say, $x \in X$. Since G is continuous, it follows by Corollary 3.8 that the sequence $\{y_n\}$, which is such that $y_n = G(x_n)$ for each n, converges in Y to $y = G(x)$. Thus Y is complete. □

The preceding theorem does not hold if uniform homeomorphism is replaced by plain homeomorphism: if X and Y are homeomorphic metric spaces, then it is not necessarily true that X is complete if and only if Y is complete. In other words, *completeness is not a topological invariant* (continuity preserves convergence but not Cauchy sequences). Therefore, there may exist homeomorphic metric spaces such that just one of them is complete. A *Polish space* is a separable metric space homeomorphic to a complete metric space.

Example 3.U. Let $\mathbb{R}$ be the real line with its usual metric. Set $A = (0, 1]$ and $B = [1, \infty)$, both subsets of $\mathbb{R}$. Consider the function $G\colon A \to B$ such that $G(\alpha) = \frac{1}{\alpha}$ for every $\alpha \in A$. As is readily verified, G is a homeomorphism of A onto B, so that A and B are homeomorphic subspaces of $\mathbb{R}$. Now consider the A-valued sequence $\{\alpha_n\}$ with $\alpha_n = \frac{1}{n}$ for each $n \in \mathbb{N}$, which is a Cauchy sequence in A. However, $G(\alpha_n) = n$ for every $n \in \mathbb{N}$, and so $\{G(\alpha_n)\}$ is certainly not a Cauchy sequence in B (since it is not even bounded in B). Thus $G\colon A \to B$ (which is continuous) is not uniformly continuous by Lemma 3.43. Actually, B is a complete subspace of $\mathbb{R}$ since B is a closed subset of the complete metric space $\mathbb{R}$ (Corollary 3.41) and, as we have just seen, A is not a complete subspace of $\mathbb{R}$: the Cauchy sequence $\{\alpha_n\}$ does not converge in A because its continuous image $\{G(\alpha_n)\}$ does not converge in B (Corollary 3.8).

Lemma 3.43 also leads to an extremely useful result on extensions of uniformly continuous mappings of a *dense* subspace of a metric space into a *complete* metric space.

Theorem 3.45. *Every uniformly continuous mapping $F\colon A \to Y$ of a dense subspace A of a metric space X into a complete metric space Y has a unique continuous extension over X, which in fact is uniformly continuous.*

Proof. Suppose the metric space X is nonempty (to avoid trivialities) and let A be a dense subset of X. Take an arbitrary point x in X. Since $A^- = X$, it follows by Proposition 3.32 that there exists an A-valued sequence $\{a_n\}$ that converges in X to x, and hence $\{a_n\}$ is a Cauchy sequence in the metric space X (Proposition 3.39) so that $\{a_n\}$ is a Cauchy sequence in the subspace A of X. Now suppose $F\colon A \to Y$ is a uniformly continuous mapping of A into a metric space Y. Thus, according to Lemma 3.43, $\{F(a_n)\}$ is a Cauchy sequence in Y. If Y is a complete metric space, then the Y-valued sequence $\{F(a_n)\}$ converges in it. Let $y \in Y$ be the (unique) limit of $\{F(a_n)\}$ in Y:

$$y = \lim F(a_n).$$

We shall show now that y, which obviously depends on $x \in X$, does not depend on the A-valued sequence $\{a_n\}$ that converges in X to x. Indeed, let $\{a'_n\}$ be an A-valued sequence converging in X to x, and set

$$y' = \lim F(a'_n).$$

Since both sequences $\{a_n\}$ and $\{a'_n\}$ converge in X to the same limit x, it follows that $d_X(a_n, a'_n) \to 0$ (Problem 3.14(b)), where d_X denotes the metric on X. Thus for every real number $\delta > 0$ there exists an index n_δ such that

$$n \geq n_\delta \quad \text{implies} \quad d_X(a_n, a'_n) < \delta.$$

Moreover, since the mapping $F\colon A \to Y$ is uniformly continuous, for every real number $\varepsilon > 0$ there exists a real number $\delta_\varepsilon > 0$ such that

$$d_X(a, a') < \delta_\varepsilon \quad \text{implies} \quad d_Y(F(a), F(a')) < \varepsilon$$

for all a and a' in A, where d_Y denotes the metric on Y. Conclusion: Given any $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$, associated with which there is an n_{δ_ε}, such that

$$n \geq n_{\delta_\varepsilon} \quad \text{implies} \quad d_Y(F(a_n), F(a'_n)) < \varepsilon.$$

Thus (Problem 3.14(c)) $0 \leq d_Y(y, y') \leq \varepsilon$ for all $\varepsilon > 0$, and so $d_Y(y, y') = 0$. That is, $y = y'$. Therefore, for each $x \in X$ set

$$\widehat{F}(x) = \lim F(a_n)$$

in Y, where $\{a_n\}$ is any A-valued sequence that converges in X to x. This defines a mapping $\widehat{F}\colon X \to Y$ of X into Y.

Claim 1. $\widehat{F}$ is an extension of F over X.

Proof. Take an arbitrary a in A and consider the A-valued constant sequence $\{a_n\}$ such that $a_n = a$ for every index n. As the Y-valued sequence $\{F(a_n)\}$ is constant, it trivially converges in Y to $F(a)$. Thus $\widehat{F}(a) = F(a)$ for every a in A. That is, $\widehat{F}|_A = F$. This means that $F\colon A \to Y$ is a restriction of $\widehat{F}\colon X \to Y$ to $A \subseteq X$ or, equivalently, $\widehat{F}$ is an extension of F over X. □

Claim 2. $\widehat{F}$ is uniformly continuous.

Proof. Take a pair of arbitrary points x and x' in X. Let $\{a_n\}$ and $\{a'_n\}$ be any pair of A-valued sequences converging in X to x and x', respectively (recall: the existence of these sequences is ensured by Proposition 3.32 because A is dense in X). Note that $d_X(a_n, a'_n) \leq d_X(a_n, x) + d_X(x, x') + d_X(x', a'_n)$ for every index n by the triangle inequality in X. Thus, as $a_n \to x$ and $a'_n \to x'$ in X, for any $\delta > 0$ there exists an index n_δ such that (Definition 3.4)

$$d_X(x, x') < \delta \quad \text{implies} \quad d_X(a_n, a'_n) < 3\delta \text{ for every } n \geq n_\delta.$$

Since $F\colon A \to Y$ is uniformly continuous, it follows by Definition 3.6 that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$, which depends only on ε, such that

$$d_X(a_n, a'_n) < 3\delta_\varepsilon \quad \text{implies} \quad d_Y(F(a_n), F(a'_n)) < \varepsilon.$$

Thus, associated with each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ (depending only on ε), which in turn ensures the existence of an index n_{δ_ε}, such that

$$d_X(x, x') < \delta_\varepsilon \quad \text{implies} \quad d_Y(F(a_n), F(a'_n)) < \varepsilon \ \text{ for every } \ n \geq n_{\delta_\varepsilon}.$$

Moreover, since $F(a_n) \to \widehat{F}(x)$ and $F(a'_n) \to \widehat{F}(x')$ in Y by the very definition of $\widehat{F}\colon X \to Y$, it follows by Problem 3.14(c) that

$$d_Y(F(a_n), F(a'_n)) < \varepsilon \ \text{ for every } \ n \geq n_{\delta_\varepsilon} \quad \text{implies} \quad d_Y(\widehat{F}(x), \widehat{F}(x')) \leq \varepsilon.$$

Therefore, given an arbitrary $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$d_X(x, x') < \delta_\varepsilon \quad \text{implies} \quad d_Y(\widehat{F}(x), \widehat{F}(x')) \leq \varepsilon$$

for all $x, x' \in X$. Thus $\widehat{F}\colon X \to Y$ is uniformly continuous (Definition 3.6). □

Finally, since $\widehat{F}\colon X \to Y$ is continuous, it follows by Corollary 3.33 that if $\widehat{G}\colon X \to Y$ is a continuous extension of $F\colon A \to Y$ over X, then $\widehat{G} = \widehat{F}$ (because A is dense in X and $\widehat{G}|_A = \widehat{F}|_A = F$). Therefore, $\widehat{F}$ is the unique continuous extension of F over X. □

Corollary 3.46. *Let X and Y be complete metric spaces, and let A and B be dense subspaces of X and Y, respectively. If $G\colon A \to B$ is a uniform homeomorphism of A onto B, then there exists a unique uniform homeomorphism $\widehat{G}\colon X \to Y$ of X onto Y that extends G over X* (i.e., $\widehat{G}|_A = G$).

Proof. Since A is dense in X, Y is complete, and $G\colon A \to B \subseteq Y$ is uniformly continuous, it follows by the previous theorem that G has a unique uniformly continuous extension $\widehat{G}\colon X \to Y$. Also, the inverse $G^{-1}\colon B \to A$ of $G\colon A \to B$ has a unique uniformly continuous extension $\widehat{G^{-1}}\colon Y \to X$. Now observe that $(\widehat{G^{-1}}\widehat{G})|_A = G^{-1}G = I_A$, where $I_A\colon A \to A$ is the identity on A (reason: $\widehat{G}|_A = G\colon A \to B$ and $\widehat{G^{-1}}|_B = G^{-1}\colon B \to A$). The identity I_A is uniformly continuous (because its domain and range are subspaces of the same metric space X), and hence it has a unique continuous extension on X (by the previous theorem) which clearly is $I_X\colon X \to X$, the identity on X (recall: I_X in fact is uniformly continuous because its domain and range are equipped with the same metric). Thus $\widehat{G^{-1}}\widehat{G} = I_X$, since $\widehat{G^{-1}}\widehat{G}$ is continuous (composition of continuous mappings) and is an extension of the uniformly continuous mapping $G^{-1}G = I_A$ over X. Similarly, $\widehat{G}\widehat{G^{-1}} = I_Y$ where $I_Y\colon Y \to Y$ is the identity on Y. Therefore $\widehat{G^{-1}} = \widehat{G}^{-1}$. Summing up: $\widehat{G}\colon X \to Y$ is an invertible uniformly continuous mapping with a uniformly continuous inverse (i.e., a uniform homeomorphism) which is the unique uniformly continuous extension of $G\colon A \to B$ over X. □

Recall that every surjective isometry is a uniform homeomorphism. Suppose the uniform homeomorphism G of the above corollary is a surjective isometry. Take an arbitrary pair of points x and x' in X so that $\widehat{G}(x) = \lim G(a_n)$ and $\widehat{G}(x') = \lim G(a'_n)$ in Y, where $\{a_n\}$ and $\{a'_n\}$ are A-valued sequences converging in X to x and x', respectively (cf. proof of Theorem 3.45). Since G is an isometry, it follows by Problem 3.14(b) that

$$d_Y(\widehat{G}(x), \widehat{G}(x')) = \lim d_Y(G(a_n), G(a'_n)) = \lim d_X(a_n, a'_n) = d_X(x, x').$$

Thus $\widehat{G}$ is an isometry as well, and so a surjective isometry (since $\widehat{G}$ is a homeomorphism). This proves the following further corollary of Theorem 3.45.

Corollary 3.47. *Let A and B be dense subspaces of complete metric spaces X and Y, respectively. If $J\colon A \to B$ is a surjective isometry of A onto B, then there exists a unique surjective isometry $\widehat{J}\colon X \to Y$ of X onto Y that extends J over X* (i.e., $\widehat{J}|_A = J$).

If a metric space X is a subspace of a complete metric space Z, then its closure X^- in Z is a complete metric space by Theorem 3.40. In this case X can be thought of as being "completed" by joining to it all its accumulation points from Z (recall: $X^- = X \cup X^\star$), and X^- can be viewed as a "completion" of X. However, if a metric space X is not specified as being a subspace of a complete metric space Z, then the above approach of simply taking the closure of X in Z obviously collapses; but the idea of "completion" behind such an approach survives. To begin with, recall that two metric spaces, say X and $\widetilde{X}$, are isometrically equivalent if there exists a surjective isometry of one of them onto the other (notation: $X \cong \widetilde{X}$). Isometrically equivalent metric spaces are regarded (as far as purely metric-space structure is concerned) as being essentially the same metric space. If $\widetilde{X}$ is a subspace of a complete metric space, then its closure $\widetilde{X}^-$ in that complete metric space is itself a complete metric space. With this in mind, consider the following definition.

Definition 3.48. If the image of an isometry on a metric space X is a dense subspace of a metric space $\widehat{X}$, then X is said to be *densely embedded* in $\widehat{X}$. If a metric space X is densely embedded in a complete metric space $\widehat{X}$, then $\widehat{X}$ is a *completion* of X.

In other words, if $J\colon X \to \widehat{X}$ is an isometry and $J(X)^- = \widehat{X}$, then X is densely embedded in $\widehat{X}$. Moreover, if $\widehat{X}$ is complete, then $\widehat{X}$ is a completion of X. Even if a metric space fails to be complete it can always be densely embedded in a complete metric space. Lemma 3.43 plays a central role in the proof of this result, which is restated below.

Theorem 3.49. *Every metric space has a completion.*

Proof. Let (X, d_X) be an arbitrary metric space and let $CS(X)$ denote the collection of all Cauchy sequences in (X, d_X). If $\boldsymbol{x} = \{x_n\}$ and $\boldsymbol{y} = \{y_n\}$ are

sequences in $CS(X)$, then the real-valued sequence $\{d_X(x_n, y_n)\}$ converges in $\mathbb{R}$ (see Problem 3.53(a)). Thus, for each pair $(\boldsymbol{x}, \boldsymbol{y})$ in $CS(X)\times CS(X)$, set

$$d(\boldsymbol{x}, \boldsymbol{y}) \;=\; \lim d_X(x_n, y_n).$$

This defines a function $d\colon CS(X)\times CS(X) \to \mathbb{R}$ which is a pseudometric on $CS(X)$. Indeed, nonnegativeness and symmetry are trivially verified, and the triangle inequality in $(CS(X), d)$ follows at once by the triangle inequality in (X, d_X). Consider a relation $\sim$ on $CS(X)$ defined as follows. If $\boldsymbol{x} = \{x_n\}$ and $\boldsymbol{x}' = \{x'_n\}$ are Cauchy sequences in (X, d_X), then

$$\boldsymbol{x}' \sim \boldsymbol{x} \quad \text{if} \quad d(\boldsymbol{x}', \boldsymbol{x}) = 0.$$

Proposition 3.3 asserts that $\sim$ is an equivalence relation on $CS(X)$. Let $\widehat{X}$ be the collection of all equivalence classes $[\boldsymbol{x}] \subseteq CS(X)$ with respect to $\sim$ for every sequence $\boldsymbol{x} = \{x_n\}$ in $CS(X)$. In other words, set $\widehat{X} = CS(X)/\!\sim$, the quotient space of $CS(X)$ modulo $\sim$. For each pair $([\boldsymbol{x}], [\boldsymbol{y}])$ in $\widehat{X}\times\widehat{X}$, set

$$d_{\widehat{X}}([\boldsymbol{x}], [\boldsymbol{y}]) \;=\; d(\boldsymbol{x}, \boldsymbol{y})$$

for an arbitrary pair $(\boldsymbol{x}, \boldsymbol{y})$ in $[\boldsymbol{x}]\times[\boldsymbol{y}]$ (i.e., $d_{\widehat{X}}([\boldsymbol{x}], [\boldsymbol{y}]) = \lim d_X(x_n, y_n)$ where $\{x_n\}$ and $\{y_n\}$ are any Cauchy sequences from the equivalence classes $[\boldsymbol{x}]$ and $[\boldsymbol{y}]$, respectively). Proposition 3.3 also asserts that this actually defines a function $d_{\widehat{X}}\colon \widehat{X}\times\widehat{X} \to \mathbb{R}$, and that such a function $d_{\widehat{X}}$ is a metric on $\widehat{X}$. Thus

$$(\widehat{X}, d_{\widehat{X}}) \text{ is a metric space.}$$

Now consider the mapping $K\colon X \to \widehat{X}$ defined as follows. For each $x \in X$ take the constant sequence $\boldsymbol{x} = \{x_n\} \in CS(X)$ such that $x_n = x$ for all indices n, and set $K(x) = [\boldsymbol{x}] \in \widehat{X}$. That is, for each x in X, $K(x)$ is the equivalence class in $\widehat{X}$ containing the constant sequence with entries equal to x. Note that

$$K\colon (X, d_X) \to (\widehat{X}, d_{\widehat{X}}) \text{ is an isometry.}$$

Indeed, if $x, y \in X$, let $\boldsymbol{x} = \{x_n\}$ and $\boldsymbol{y} = \{y_n\}$ be constant sequences with $x_n = x$ and $y_n = y$ for all n. Then $d_{\widehat{X}}(K(x), K(y)) = d_X(x_n, y_n) = d_X(x, y)$.

Claim 1. $K(X)^- = \widehat{X}$.

Proof. Take any $[\boldsymbol{x}] \in \widehat{X}$ and any $\{x_n\} \in [\boldsymbol{x}]$ so that $\{x_n\}$ is a Cauchy sequence in (X, d_X). Thus for each $\varepsilon > 0$ there is an index n_ε such that $d_X(x_n, x_{n_\varepsilon}) < \varepsilon$ for every $n \geq n_\varepsilon$. Set $[\boldsymbol{x}_\varepsilon] = K(x_{n_\varepsilon}) \in K(X)$: the equivalence class in $\widehat{X}$ containing the constant sequence with entries equal to x_{n_ε}. Therefore, for each $[\boldsymbol{x}] \in \widehat{X}$ and each $\varepsilon > 0$ there exists an $[\boldsymbol{x}_\varepsilon] \in K(X)$ such that $d_{\widehat{X}}([\boldsymbol{x}_\varepsilon], [\boldsymbol{x}]) = \lim d_X(x_n, x_{n_\varepsilon}) < \varepsilon$. Hence $K(X)$ is dense in $(\widehat{X}, d_{\widehat{X}})$ (Proposition 3.32). □

Claim 2. The metric space $(\widehat{X}, d_{\widehat{X}})$ is complete.

Proof. Take an arbitrary Cauchy sequence $\{[\boldsymbol{x}]_k\}_{k\geq 1}$ in $(\widehat{X}, d_{\widehat{X}})$. Since $K(X)$ is dense in $(\widehat{X}, d_{\widehat{X}})$, for each $k \geq 1$ there exists $[\boldsymbol{y}]_k \in K(X)$ such that

$$d_{\widehat{X}}([\boldsymbol{x}]_k, [\boldsymbol{y}]_k) \, < \, \tfrac{1}{k}$$

(cf. Proposition 3.32). Then, since

$$d_{\widehat{X}}([\boldsymbol{y}]_j, [\boldsymbol{y}]_k) \, \leq \, d_{\widehat{X}}([\boldsymbol{y}]_j, [\boldsymbol{x}]_j) + d_{\widehat{X}}([\boldsymbol{x}]_j, [\boldsymbol{x}]_k) + d_{\widehat{X}}([\boldsymbol{x}]_k, [\boldsymbol{y}]_k)$$

for every $j, k \geq 1$, and since $\{[\boldsymbol{x}]_k\}_{k\geq 1}$ is a Cauchy sequence in $(\widehat{X}, d_{\widehat{X}})$, it follows that the $K(X)$-valued sequence

$$\{[\boldsymbol{y}]_k\}_{k\geq 1} \text{ is a Cauchy sequence in } (\widehat{X}, d_{\widehat{X}}).$$

Now take any $k \geq 1$ and notice that, as $[\boldsymbol{y}]_k$ lies in $K(X)$, there exists $y_k \in X$ such that the *constant* sequence $\boldsymbol{y}_k = \{y_k\}_{n\geq 1}$ belongs to the equivalence class $[\boldsymbol{y}]_k = K(y_k)$, and so $y_k = K^{-1}([\boldsymbol{y}]_k)$. Indeed, $K\colon X \to K(X) \subseteq \widehat{X}$ is a surjective isometry of X onto the subspace $K(X)$ of $\widehat{X}$ so that $K^{-1}\colon K(X) \to X$ is again a surjective isometry, thus uniformly continuous. Therefore, since $\{[\boldsymbol{y}]_k\}_{k\geq 1}$ is a Cauchy sequence in $(K(X), d_{\widehat{X}})$, it follows by Lemma 3.43 that

$$\{y_k\}_{k\geq 1} \text{ is a Cauchy sequence in } (X, d_X).$$

Then $\boldsymbol{y} = \{y_k\}_{k\geq 1} \in CS(X)$, and so the equivalence class $[\boldsymbol{y}]$ is in $\widehat{X}$. Moreover,

$$\{[\boldsymbol{x}]_k\}_{k\geq 1} \text{ converges in } (\widehat{X}, d_{\widehat{X}}) \text{ to } [\boldsymbol{y}] \in \widehat{X}.$$

Indeed, for every $k \geq 1$,

$$0 \, \leq \, d_{\widehat{X}}([\boldsymbol{x}]_k, [\boldsymbol{y}]) \, \leq \, d_{\widehat{X}}([\boldsymbol{x}]_k, [\boldsymbol{y}]_k) + d_{\widehat{X}}([\boldsymbol{y}]_k, [\boldsymbol{y}]) \, \leq \, \tfrac{1}{k} + d_{\widehat{X}}([\boldsymbol{y}]_k, [\boldsymbol{y}]).$$

Take $\boldsymbol{y} = \{y_n\}_{n\geq 1} \in [\boldsymbol{y}]$ and, for each $k \geq 1$, take the *constant* sequence $\boldsymbol{y}_k = \{y_k\}_{n\geq 1} \in [\boldsymbol{y}]_k$ such that $y_k = K^{-1}([\boldsymbol{y}])$. By the definition of the metric $d_{\widehat{X}}$ on $\widehat{X}$, $d_{\widehat{X}}([\boldsymbol{y}]_k, [\boldsymbol{y}]) = \lim_n d_X(y_n, y_k)$ for every $k \geq 1$, and so $\lim_k d_{\widehat{X}}([\boldsymbol{y}]_k, [\boldsymbol{y}]) = 0$ because $\{y_k\}_{k\geq 1}$ is a Cauchy sequence in (X, d_X). Therefore,

$$d_{\widehat{X}}([\boldsymbol{x}]_k, [\boldsymbol{y}]) \to 0 \quad \text{as} \quad k \to \infty.$$

Conclusion: Every Cauchy sequence in $(\widehat{X}, d_{\widehat{X}})$ converges in $(\widehat{X}, d_{\widehat{X}})$, so that $(\widehat{X}, d_{\widehat{X}})$ is a complete metric space. □

Summing up: $X \cong K(X)$, $K(X)^- = \widehat{X}$, and $\widehat{X}$ is complete. That is, X is densely embedded in a complete metric space $\widehat{X}$, which means that $\widehat{X}$ is a completion of X. □

Corollary 3.47 leads to the proof that a completion of a metric space is essentially unique; that is, the completion of a metric space is unique up to a surjective isometry.

Theorem 3.50. *Any two completions of a metric space are isometrically equivalent.*

Proof. Let X be a metric space and, according to Theorem 3.49, let $\widehat{X}$ and $\widehat{X}'$ be two completions of X. This means that there exist surjective isometries

$$J\colon \widetilde{X} \to X \quad \text{and} \quad J'\colon \widetilde{X}' \to X,$$

where $\widetilde{X}$ is a dense subspace of $\widehat{X}$ and $\widetilde{X}'$ is a dense subspace of $\widehat{X}'$. Recall that a surjective isometry is invertible, and also that its inverse is again a surjective isometry. Thus set

$$\widetilde{J} = J^{-1}J'\colon \widetilde{X}' \to \widetilde{X},$$

which, as a composition of surjective isometries, is a surjective isometry itself. Since $\widetilde{X}$ and $\widetilde{X}'$ are dense subspaces of the complete metric spaces $\widehat{X}$ and $\widehat{X}'$, it follows by Corollary 3.47 that there exists a unique surjective isometry

$$\widehat{J}\colon \widehat{X}' \to \widehat{X}$$

that extends $\widetilde{J}$ over $\widehat{X}'$. Thus $\widehat{X}'$ and $\widehat{X}$ are isometrically equivalent. □

According to Definition 3.48, a complete metric space $\widehat{X}$ is a completion of a metric space X if there exists a dense subspace of $\widehat{X}$, say $\widetilde{X}$, which is isometrically equivalent to X; that is, if there exists a surjective isometry

$$J\colon \widetilde{X} \to X$$

of $\widetilde{X}$ onto X for some dense subspace $\widetilde{X}$ of $\widehat{X}$. Now let Y be another metric space, consider a mapping

$$F\colon X \to Y$$

of X into Y, and let

$$\widehat{F}\colon \widehat{X} \to Y$$

be a mapping of a completion $\widehat{X}$ of X into Y such that $\widehat{F}(x) = F(J(x))$ for every $x \in \widetilde{X}$. That is, $\widehat{F}$ is an extension of the composition FJ over $\widehat{X}$ or, equivalently, $FJ\colon \widetilde{X} \to Y$ is the restriction of $\widehat{F}$ to $\widetilde{X}$. It is customary to refer to $\widehat{F}$ as an *extension of F over the completion $\widehat{X}$ of X* (which in fact is a slight abuse of terminology). The situation so far is illustrated by the following commutative diagram (recall: $\widehat{F}|_{\widetilde{X}} = FJ$).

$$\begin{array}{ccccc} \widetilde{X} \cong X & \xleftarrow{\ J\ } & \widetilde{X} & \subseteq & \widehat{X} = \widetilde{X}^- \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle \widehat{F}|_{\widetilde{X}}} & \swarrow{\scriptstyle \widehat{F}} & \\ & & Y & & \end{array}$$

The next theorem says that, if F is uniformly continuous and Y is complete, then there exists an essentially unique continuous extension $\widehat{F}$ of F over a completion $\widehat{X}$ of X.

Theorem 3.51. *Let $\widehat{X}$ be a completion of a metric space X and let Y be a complete metric space. Every uniformly continuous mapping $F\colon X \to Y$ has a uniformly continuous extension $\widehat{F}\colon \widehat{X} \to Y$ over the completion $\widehat{X}$ of X. Moreover, $\widehat{F}$ is unique up to a surjective isometry.*

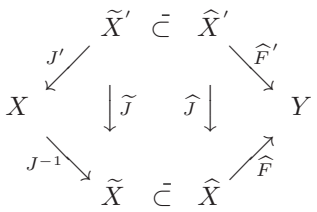
Proof. Existence. Let $\widehat{X}$ be a completion of a metric space X. Thus there exists a dense subspace $\widetilde{X}$ of the metric space $\widehat{X}$, and a surjective isometry $J\colon \widetilde{X} \to X$ of $\widetilde{X}$ onto X. Suppose $F\colon X \to Y$ is a uniformly continuous mapping of X into a metric space Y. Consider the composition $FJ\colon \widetilde{X} \to Y$, which is uniformly continuous as well (reason: J is uniformly continuous). Since $\widetilde{X}$ is dense in $\widehat{X}$, Y is complete, and $FJ\colon \widetilde{X} \to Y$ is uniformly continuous, it follows by Theorem 3.45 that there exists a unique continuous extension $\widehat{F}\colon \widehat{X} \to Y$ of $FJ\colon \widetilde{X} \to Y$ over $\widehat{X}$, which in fact is uniformly continuous. Thus $\widehat{F}\colon \widehat{X} \to Y$ is a uniformly continuous extension of $F\colon X \to Y$ over the completion $\widehat{X}$ of X.

Uniqueness. Suppose $\widehat{F}'\colon \widehat{X}' \to Y$ is another continuous extension of $F\colon X \to Y$ over some completion $\widehat{X}'$ of X so that $\widehat{F}'|_{\widetilde{X}'} = FJ'$, where $\widetilde{X}'$ is a dense subspace of $\widehat{X}'$ and $J'\colon \widetilde{X}' \to X$ is a surjective isometry of $\widetilde{X}'$ onto X. Set $\widetilde{J} = J^{-1}J'\colon \widetilde{X}' \to \widetilde{X}$ as in the proof of Theorem 3.50, and let $\widehat{J}\colon \widehat{X}' \to \widehat{X}$ be the surjective isometry of $\widehat{X}'$ onto $\widehat{X}$ such that $\widehat{J}|_{\widetilde{X}'} = \widetilde{J}$. Thus, since $\widehat{F}|_{\widetilde{X}} = FJ$,

$$\widehat{F}\widehat{J}|_{\widetilde{X}'} = FJ\widetilde{J} = FJJ^{-1}J' = \widehat{F}'|_{\widetilde{X}'}.$$

Therefore, the continuous mappings $\widehat{F}\widehat{J}\colon \widehat{X}' \to Y$ (composition of two continuous mappings) and $\widehat{F}'\colon \widehat{X}' \to Y$ coincide on a dense subset $\widetilde{X}'$ of $\widehat{X}'$, and hence $\widehat{F}' = \widehat{F}\widehat{J}$ by Corollary 3.33. Conclusion: If $\widehat{F}'\colon \widehat{X}' \to Y$ is another continuous extension of F over some completion $\widehat{X}'$ of X, then there exists a surjective isometry $\widehat{J}\colon \widehat{X}' \to \widehat{X}$ such that $\widehat{F}' = \widehat{F}\widehat{J}$. In other words, a continuous extension of F over a completion of X is unique up to a surjective isometry. □

The commutative diagram, where $\bar{\subset}$ denotes dense inclusion,



illustrates the uniqueness proofs of Theorems 3.50 and 3.51.

3.9 The Baire Category Theorem

We close our discussion on complete metric spaces with an important classification of subsets of a metric space into two categories. The basic notion behind such a classification is the following one. A subset A of a metric space X is *nowhere dense* (or *rare*) in X if $(A^-)^\circ = \varnothing$ (i.e., if the interior of its closure is empty). Clearly, a closed subset of X is nowhere dense in X if and only if it has an empty interior. Note that $(A\backslash A^\circ)^\circ = \varnothing$ for every subset A of a metric space X. Indeed, A° is the largest open subset of X that is included in A, so that the only open subset of X that is included in $A\backslash A^\circ$ is the empty set $\varnothing$ of X. Therefore, *if V is a closed subset of X, then $V\backslash V^\circ$ is nowhere dense in X* (reason: $V\backslash V^\circ = V^-\backslash V^\circ = \partial V$, which is closed in X — see Problem 3.41). Dually, *if U is an open subset of X, then $U^-\backslash U$ is nowhere dense in X* (recall that $U^-\backslash U = (X\backslash V)^-\backslash(X\backslash V) = (X\backslash V^\circ)\backslash(X\backslash V) = V\backslash V^\circ$). Carefully note that $\partial\mathbb{Q} = \mathbb{Q}^-\backslash\mathbb{Q}^\circ = (\mathbb{Q}^-)^\circ = \mathbb{R}$ in the metric space $\mathbb{R}$ (usual metric).

Proposition 3.52. *A singleton $\{x\}$ on a point x in a metric space X is nowhere dense in X if and only if x is not an isolated point of X.*

Proof. Recall that every singleton in a metric space X is a closed subset of X (Problem 3.37), and hence $\{x\} = \{x\}^-$ for every x in X. According to Proposition 3.37, a point x in X is an isolated point of X if and only if the singleton $\{x\}$ is an open set in X; that is, if and only if $\{x\}^\circ = \{x\}$. Thus a point x in X is not an isolated point of X if and only if $\{x\}^\circ \subset \{x\}$ (i.e., $\{x\}^\circ \neq \{x\}$) or, equivalently, $\{x\}^\circ = \varnothing$ (since the empty set is the only proper subset of any singleton). But $\{x\}^\circ = \varnothing$ if and only if $(\{x\}^-)^\circ = \varnothing$ (because $\{x\} = \{x\}^-$ for every singleton $\{x\}$), which means that $\{x\}$ is nowhere dense in X. □

The next proposition gives alternative characterizations of nowhere dense sets that will be required in the sequel.

Proposition 3.53. *Let X be a metric space and let A be a subset of X. The following assertions are pairwise equivalent.*

(a) $(A^-)^\circ = \varnothing$ (i.e., *A is nowhere dense in X*).

(b) *For every nonempty open subset U of X there exists a nonempty open subset U' of X such that $U' \subset U$ and $U' \cap A = \varnothing$.*

(c) *For every nonempty open subset U of X and every real number $\rho > 0$ there exists an open ball B_ε with radius $\varepsilon \in (0, \rho)$ such that $B_\varepsilon^- \subset U$ and $B_\varepsilon^- \cap A = \varnothing$.*

Proof. Suppose A is nonempty (otherwise the proposition is trivial).

Proof of (a)⇔(b). Take an arbitrary nonempty open subset U of X. If $(A^-)^\circ = \varnothing$, then $U\backslash A^- \neq \varnothing$ (i.e., A^- includes no nonempty open set), $U\backslash A^-$ is open in X (since $U\backslash A^- = (X\backslash A^-) \cap U$), $U\backslash A^- \subset U$, and $(U\backslash A^-) \cap A = \varnothing$. Thus

(a) implies (b). Conversely, suppose $(A^-)^\circ \neq \varnothing$ so that there exists an open subset U_0 of X such that $\varnothing \neq U_0 \subseteq A^-$. Then every point of U_0 is a point of adherence of A, and hence every nonempty open subset U_0' of U_0 meets A (cf. Proposition 3.25). Therefore, the denial of (a) implies the denial of (b). Equivalently, (b) implies (a).

Proof of (b)$\Leftrightarrow$(c). If (b) holds true, then (c) holds true for every open ball included in U'. Precisely, suppose (b) holds true. If u is any point of the open set U', then there exists a radius $\rho > 0$ such that $B_\rho(u) \subset U'$. Take an open ball $B_\varepsilon(u)$ with center at u and radius $\varepsilon \in (0, \rho)$. Since $B_\varepsilon(u)^- \subseteq B_\rho(u)$, it follows that $B_\varepsilon(u)^- \subset U$ and $B_\varepsilon(u)^- \cap A = \varnothing$. Thus (b) implies (c). Conversely, suppose (c) holds true and set $U' = B_\varepsilon$, so that (c) implies (b). □

By using Proposition 3.53 it is easy to show that $A \cup B$ is nowhere dense in X whenever the sets A and B are both nowhere dense in X. Thus a trivial induction ensures that any *finite* union of nowhere dense subsets of a metric space X is again nowhere dense in X. However, a countable union of nowhere dense subsets of X does not need to be nowhere dense in X. A subset A of a metric space X is of *first category* (or *meager*) in X if it is a countable union of nowhere dense subsets of X. That is, A is of first category in X if $A = \bigcup_{n\in\mathbb{N}} A_n$, where each A_n is nowhere dense in X. The complement of a set of first category in X is a *residual* (or *comeager*) in X. A subset B of X is of *second category* (or *nonmeager*) in X if it is not of first category in X.

Example 3.V. Let X be a metric space. Recall that a subset of X is dense in itself if and only if it has no isolated point. Thus, according to Proposition 3.52, if X is nonempty and dense in itself, then every singleton in X is nowhere dense in X. Moreover, a nonempty countable subset of X is a countable union of singletons in X. Therefore, if A is a nonempty countable subset of X, and if X is dense in itself, then A is a countable union of nowhere dense subsets of X. Summing up: *If a metric space X is dense in itself* (i.e., *has no isolated point*), *then every countable subset of it is of first category in X*. For instance, $\mathbb{Q}$ is a (dense) subset of first category in $\mathbb{R}$. Equivalently, *if a metric space X has no isolated point, then every subset of second category in X is uncountable.*

The following basic properties of sets of first category are (almost) immediate consequences of the definition. Note that assertions (a) and (b), but not assertion (c), in the following proposition still hold if we replace "sets of first category" by "nowhere dense sets".

Proposition 3.54. *Let X be a metric space.*

(a) *A subset of a set of first category in X is of first category in X.*

(b) *The intersection of an arbitrary collection of subsets of X is a set of first category in X if at least one of the subsets is of first category in X.*

(c) *The union of a countable collection of sets of first category in X is a set of first category in X.*

Proof. If $B = \bigcup_n B_n$ and if $A \subseteq B \subseteq X$, then $A = \bigcup_n A_n$, with each $A_n = B_n \cap A \subseteq B_n$, and so $(A_n^-)^\circ \subseteq (B_n^-)^\circ$. Hence (a) holds true by the definitions of nowhere dense set and set of first category. Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary collection of subsets of X. Since $\bigcap_{\gamma\in\Gamma} A_\gamma \subseteq A_\alpha$ for every $\alpha \in \Gamma$, it follows by item (a) that $\bigcap_{\gamma\in\Gamma} A$ is of first category in X whenever at least one of the sets A_α in $\{A_\gamma\}_{\gamma\in\Gamma}$ is of first category in X. This proves assertion (b). If $\{A_n\}$ is a countable collection of subsets of X, and if each A_n is a countable union of nowhere dense subsets of X, then $\bigcup_n A_n$ is itself a countable union of nowhere dense subsets of X (recall: a countable union of a countable collection is again a countable collection — Corollary 1.11). Thus $\bigcup_n A_n$ is a set of first category in X, which concludes the proof of assertion (c). □

Example 3.V may suggest that sets of second category are particularly important. The next theorem, which plays a fundamental role in the theory of metric spaces, shows that they really are very important.

Theorem 3.55. (Baire Category Theorem). *Every nonempty open subset of a complete metric space X is of second category in X.*

Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ be an arbitrary countable collection of nowhere dense subsets of a metric space X and set $A = \bigcup_{n\in\mathbb{N}} A_n \subseteq X$. Let U be an arbitrary nonempty open subset of X.

Claim. For each integer $k \geq 1$ there exists a collection $\{B_n\}_{n=1}^{k+1}$ of open balls B_n with radius $\varepsilon_n \in (0, \frac{1}{n})$ such that: $B_n^- \subset U$ and $B_n^- \cap A_n = \varnothing$ for each $n = 1, \ldots, k+1$, and $B_{n+1} \subset B_n$ for every $n = 1, \ldots, k$.

Proof. Since each A_n is nowhere dense in X, it follows by Proposition 3.53 that there exist open balls B_1 and B_2 with center at x_1 and x_2 (both included in U) and positive radius $\varepsilon_1 < 1$ and $\varepsilon_2 < \frac{1}{2}$, respectively, such that $B_i^- \subset U$ and $B_i \cap A_i = \varnothing$, for $i = 1, 2$, and $B_2 \subset B_1$. Thus the claimed result holds for $k = 1$. Suppose it holds for some $k \geq 1$, and take $0 < \varepsilon_{k+1} < \min\{\varepsilon_k, \frac{1}{k+1}\}$. Proposition 3.53 ensures again the existence of an open ball B_{k+1} with center at $x_{k+1} \in U$ and radius ε_{k+1} such that $B_{k+1}^- \subset U$ and $B_{k+1} \cap A_{k+1} = \varnothing$. It is plain that $B_{k+1} \subset B_k$, and so the claimed result holds for $k+1$ whenever it holds for some $k \geq 1$, which concludes the proof by induction. □

Consider the collection $\{B_n\}_{n\in\mathbb{N}} = \bigcup_{k\in\mathbb{N}} \{B_n\}_{n=1}^{k+1}$. Since each open ball $B_n = B_{\varepsilon_n}(x_n) \subset U$ is such that $0 < \varepsilon_n < \frac{1}{n}$, it follows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ of centers $x_n \in U$ is a Cauchy sequence in X (reason: for each $\varepsilon > 0$ take a positive integer $n_\varepsilon > \frac{1}{\varepsilon}$ so that, if $n > m \geq n_\varepsilon$, then $B_{\varepsilon_n}(x_n) \subset B_{\varepsilon_m}(x_m)$ with $\varepsilon_m \leq \frac{1}{m} \leq \frac{1}{n_\varepsilon} < \varepsilon$, and hence $d(x_m, x_n) < \varepsilon$). Now suppose the metric space X is complete. This ensures that the Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ converges in X to, say, $x \in X$. Take an arbitrary integer $i \geq 1$. Since x is the limit of the sequence $\{x_n\}_{n\geq i}$ (a subsequence of $\{x_n\}_{n\in\mathbb{N}}$), and since $\{x_n\}_{n\geq i} \subset B_i$

($B_{n+1} \subset B_n$ for every $n \in \mathbb{N}$), it follows that $x \in B_i^-$ (i.e., x is an adherent point of B_i). Therefore, $x \notin A$ because $B_i^- \cap A_i = \varnothing$ for every $i \in \mathbb{N}$ and $A = \bigcup_{i\in\mathbb{N}} A_i$. However, $x \in U$ because $B_i^- \subset U$ for all $i \in \mathbb{N}$. Hence $x \in U\backslash A$, and so $U \neq A$. Summing up: If U is a nonempty open subset of a complete metric space X, and if A is a set of first category in X (i.e., if A is a countable union of nowhere dense sets in X), then $U \neq A$. Conclusion: Every nonempty open subset of a complete metric space X is not a set of first category in X. □

In particular, as a metric space is always open in itself, we get at once the following corollary of Theorem 3.55 on *nonempty* complete metric spaces.

Corollary 3.56. *A complete metric space is of second category in itself.*

Of course, Corollary 3.56 refers to *nonempty* complete metric spaces. As a matter of fact, the metric spaces in the next three results are clearly nonempty too, although, for simplicity, we shall drop "nonempty" from their statements.

Corollary 3.57. *The complement of a set of first category in a complete metric space is a dense subset of second category in that space.*

Proof. Let X be a *nonempty* complete metric space. The union of two sets of first category in X is a set of first category in X (Proposition 3.54). Since the union of a subset A of X and its complement $X\backslash A$ is the whole space X, it follows by Corollary 3.56 that A and $X\backslash A$ cannot both be of first category in X. Thus $X\backslash A$ is of second category in X whenever A is of first category in X. Moreover, if $(X\backslash A)^- \neq X$, then A° is a nonempty open subset of X (Proposition 3.31), and so A° is a set of second category in X (Theorem 3.55), which implies that A is a set of second category in X (reason: if A is of first category in X, then A° is of first category in X because $A^\circ \subseteq A$ — Proposition 3.54). Thus, if A is of first category in X, then $X\backslash A$ is dense in X. □

In other words, *if $X \neq \varnothing$ is a complete metric space, then every residual in X is both dense in X and of second category in X.*

Theorem 3.58. *If a complete metric space is a countable union of closed sets, then at least one of them has nonempty interior.*

Proof. According to Corollary 3.56, a *nonempty* complete metric space is not a countable union of nowhere dense subsets of it. Thus, if a nonempty complete metric space X is a countable union of subsets of X, then at least one of them is not nowhere dense in X (i.e., the closure of at least one of them has nonempty interior). □

Theorem 3.58 is a particularly useful version of the Baire Category Theorem. A further version of it, which is the dual statement of Theorem 3.58, reads as follows. (This in fact is the classical Theorem of Baire.)

Theorem 3.59. *Every countable intersection of open and dense subsets of a complete metric space X is dense in X.*

Proof. Let $\{U_n\}$ be a countable collection of open and dense subsets of a *nonempty* complete metric space X. Set $V_n = X\backslash U_n$ so that $\{V_n\}$ is a countable collection of closed subsets of X with empty interior (recall: $U_n^- = X$ means $(X\backslash U_n)^\circ = \varnothing$ by Proposition 3.31). If $(\bigcap_n U_n)^- \neq X$, then $X\backslash(\bigcap_n U_n)^- \neq \varnothing$, and hence $X\backslash\bigcap_n U_n \neq \varnothing$. However, $X\backslash\bigcap_n U_n = \bigcup_n (X\backslash U_n)$ — De Morgan laws — so that $\bigcup_n V_n \neq \varnothing$, which implies $(\bigcup_n V_n)^- \neq \varnothing$. Thus, according to Theorem 3.58, the nonempty subspace $(\bigcup_n V_n)^-$ of X is not complete (reason: each V_n is a closed subset of $(\bigcup_n V_n)^-$ — see Problem 3.38 — with $V_n^\circ = \varnothing$). On the other hand, Corollary 3.41 ensures that $(\bigcup_n V_n)^-$ is a complete subspace of the complete metric space X (since $(\bigcup_n V_n)^-$ is a closed subset of X), which leads to a contradiction. Conclusion: $(\bigcap_n U_n)^- = X$. □

The Baire Category Theorem is a nonconstructive existence theorem. For instance, for an arbitrary countable collection $\{A_n\}$ of nowhere dense sets in a nonempty complete metric space X, Corollary 3.57 asserts the existence of a dense set of points in X with the property that none of them lies in $\{A_n\}$ for every n, but it does not tell us how to find those points. However, the unusual (and remarkable) fact about the Baire Category Theorem is that, while its hypothesis (completeness) has been defined in a metric space and is not a topological invariant (completeness is preserved by uniform homeomorphism but not by plain homeomorphism — see Example 3.U), its conclusion is of purely topological nature and is a topological invariant. For instance, the conclusion in Theorem 3.55 (being of second category) is a topological invariant in a general topological space. Indeed, if $G\colon X \to Y$ is a homeomorphism between topological spaces X and Y and A is an arbitrary subset of X, then it is easy to show that $G(A)^- = G(A^-)$ and $G(A)^\circ = G(A^\circ)$. Thus the property of being nowhere dense is a topological invariant (i.e., $(A^-)^\circ = \varnothing$ if and only if $(G(A)^-)^\circ = \varnothing$), and so is the property of being of first or second category (since $G(A) = G(\bigcup_n A_n) = \bigcup_n G(A_n)$ whenever $A = \bigcup_n A_n$). Such a purely topological conclusion suggests the following definition. A topological space is a *Baire space* if the conclusion of the classical Theorem of Baire holds on it. Precisely, a Baire space is a topological space X on which every countable intersection of open and dense subsets of X is dense in X. Thus Theorem 3.59 simply says that *every complete metric space is a Baire space*.

Example 3.W. We shall now unfold three further consequences of the Baire Category Theorem, each resulting from one of the above three versions of it.

(a) If A is a set of first category in a complete metric space X, then Corollary 3.57 says that $(X\backslash A)^- = X$; equivalently, $A^\circ = \varnothing$. Conclusion: *A set of first category in a complete metric space has empty interior*. Corollary: *A closed set of first category in a complete metric space is nowhere dense in that space*: if $A = A^-$ is a set of first category in a complete metric space, then $(A^-)^\circ = \varnothing$.

(b) Recall that a set without isolated points (i.e., dense in itself) in a complete metric space may be countable (example: $\mathbb{Q}$ in $\mathbb{R}$). Suppose A is a nonempty perfect subset of a complete metric space X, which means that A is a closed set without isolated points. If A is a countable set, then it is the countable union of all singletons in it. Since every point in A is not an isolated point of A, it follows by Proposition 3.52 that every singleton in A is nowhere dense in the subspace A, so that every singleton in A has empty interior in A (recall: a singleton in a metric space A is closed in A — Problem 3.37). Then A is the countable union of closed sets in A, all of them with empty interior in A, and therefore the subspace A is not complete according to Theorem 3.58. However, since A is a closed subset of a complete metric space X, it follows by Corollary 3.41 that the subspace A is complete. Thus the assumption that A is countable leads to a contradiction. Conclusion: *A nonempty perfect set in a complete metric space is uncountable*.

(c) A subset of a metric space X is a G_δ (read: *G-delta*) if it is a countable intersection of open subsets of X, and an F_σ (read: *F-sigma*) if it is a countable union of closed subsets of X. First observe that, if the complement $X\backslash A$ of a subset A of X is a countable union of subsets of X, then A includes a G_δ. In fact, if $X\backslash A = \bigcup_n C_n$, then $\bigcap_n (X\backslash C_n^-) \subseteq \bigcap_n (X\backslash C_n) = X\backslash \bigcup_n C_n = X\backslash(X\backslash A) = A$, and so A includes a G_δ; viz., $\bigcap_n (X\backslash C_n^-)$. Moreover, if each C_n is nowhere dense in X (i.e., $(C_n^-)^\circ = \varnothing$), then $(X\backslash C_n^-)^- = X$ (see Proposition 3.31) so that $X\backslash C_n^-$ is open and dense in X for every index n. Thus, according to Theorem 3.59, $\bigcap_n (X\backslash C_n^-)$ is a dense G_δ in X whenever X is a complete metric space. Summing up: If $X\backslash A$ is of first category (i.e., a countable union of nowhere dense subsets of X) in a complete metric space X, then A includes a dense G_δ. Conversely, if a subset A of a metric space X includes a G_δ, say $\bigcap_n U_n \subseteq A$ where each U_n is open in X, then $X\backslash A \subseteq X\backslash \bigcap_n U_n = \bigcup_n X\backslash U_n$. If the G_δ is dense in X (i.e., $(\bigcap_n U_n)^- = X$), then $(X\backslash \bigcap_n U_n)^\circ = \varnothing$ (see Proposition 3.31 again) so that $[(X\backslash U_n)^-]^\circ = (X\backslash U_n)^\circ \subseteq (\bigcup_m X\backslash U_m)^\circ = \varnothing$, and so each $X\backslash U_n$ is nowhere dense in X. Thus $\bigcup_n X\backslash U_n$ is a set of first category in X, which implies that $X\backslash A$ is of first category in X as well (because every subset of a set of first category is itself of first category — Proposition 3.54). Summing up: If a set A in a metric space X includes a dense G_δ, then $X\backslash A$ is of first category in X. Conclusion: *A subset of a complete metric space is a residual* (i.e., its complement is of first category) *if and only if it includes a dense G_δ*. (This generalizes Corollary 3.57.) Dually, *a subset of a complete metric space is of first category if and only if it is included in an F_σ with empty interior*. (This generalizes the results of item (a).)

3.10 Compact Sets

Recall that a collection $\mathcal{A}$ of nonempty subsets of a set X is a *covering* of a subset A of X (or $\mathcal{A}$ *covers* $A \subseteq X$) if $A \subseteq \bigcup \mathcal{A}$. If $\mathcal{A}$ is a covering of A, then

any subcollection of $\mathcal{A}$ that also covers A is a *subcovering* of A. A covering of A comprising only open subsets of X is called an *open covering*.

Definition 3.60. A metric space X is *compact* if every open covering of X includes a finite subcovering. A subset A of a metric space X is *compact* if it is compact as a subspace of X.

The notion of compactness plays an extremely important role in general topology. Note that any topology $\mathcal{T}$ on a metric space X clearly is an open covering of X which trivially has a finite subcovering; namely, the collection $\{X\}$ consisting of X alone. However, the definition of a compact space demands that *every* open covering of it has a finite subcovering. The idea behind the definition of a compact space is that even open coverings made up of "very small" open sets have a finite subcovering. Also note that the definition of a compact subspace A of a metric space X is given in terms of the relative topology on A: an open covering of the subspace A consists of relatively open subsets of A. The next elementary result says that this can be equally defined in terms of the topology on X.

Proposition 3.61. *A subset A of a metric space X is compact if and only if every covering of A made up of open subsets of X has a finite subcovering.*

Proof. If $\mathcal{U}$ is a covering of A (i.e., $A \subseteq \bigcup \mathcal{U}$) consisting of open subsets of X, then $\{U \cap A\colon\ U \in \mathcal{U}\}$ is an open covering of the subspace A (see Problem 3.38). Conversely, every open covering $\mathcal{U}_A$ of the subspace A consisting of relatively open subsets of A is of the form $\{U \cap A\colon\ U \in \mathcal{U}\}$ for some covering $\mathcal{U}$ of A consisting of open subsets of X (reason: $U_A \in \mathcal{U}_A$ if and only if $U_A = A \cap U$ for some open subset U of X — see Problem 3.38 again). □

The properties of being a closed subset and of being a compact subset of a metric space are certainly different from each other (trivial example: every metric space is closed in itself). However, "inside" a *compact metric space* these properties coincide.

Theorem 3.62. *Let A be a subset of a metric space X.*

(a) *If A is compact, then A is closed in X.*

(b) *If X is compact and if A is closed in X, then A is compact.*

Proof. (a) Let A be a compact subset of a metric space X. If either $A = \varnothing$ or $A = X$, then A is trivially closed in X. Thus suppose $\varnothing \neq X \backslash A \neq X$ and take an arbitrary point x in $X \backslash A$. Since x is distinct from every point in A, it follows that for each $a \in A$ there exists an open neighborhood A_a of a and an open neighborhood X_a of x such that $A_a \cap X_a = \varnothing$ (reason: every metric space is a Hausdorff space — see Problem 3.37). But $A \subseteq \bigcup_{a \in A} A_a$ so that $\{A_a\}_{a \in A}$ is a covering of A consisting of nonempty open subsets of X. If A is compact, then there is a finite subset of A, say $\{a_i\}_{i=1}^n$, such that $A \subseteq \bigcup_{i=1}^n A_{a_i}$ (Proposition

3.61). Set $U_x = \bigcap_{i=1}^n X_{a_i}$, which is an open neighborhood of x (recall: each X_{a_i} is an open neighborhood of x). Since $A_{a_i} \cap U_x = \varnothing$ for each i, it follows that $(\bigcup_{i=1}^n A_{a_i}) \cap U_x = \varnothing$, and hence $A \cap U_x = \varnothing$. Therefore $U_x \subseteq X \backslash A$. Thus $X \backslash A$ is open in X (it includes an open neighborhood of each one of its points).

(b) Let A be a closed subset of a compact metric space X. Take an arbitrary covering of A, say $\mathcal{U}_A$, consisting of open subsets of X. Thus $\mathcal{U}_A \cup \{X \backslash A\}$ is an open covering of X. As X is compact, this covering includes a finite subcovering, say $\mathcal{U}$, so that $\mathcal{U} \backslash \{X \backslash A\} \subseteq \mathcal{U}_A$ is a finite subcovering of $\mathcal{U}_A$. Therefore, every covering of A consisting of open subsets of X has a finite subcovering, and hence (Proposition 3.61) A is compact. □

Corollary 3.63. *Let X be a compact metric space. A subset A of X is closed in X if and only if it is compact.*

A subset of a metric space X is *relatively compact* (or *conditionally compact*) if it has a compact closure. It is clear by Corollary 3.63 that *every subset of a compact metric space is relatively compact*. Another important property of a compact set is that *the continuous image of a compact set is compact*.

Theorem 3.64. *Let $F: X \to Y$ be a continuous mapping of a metric space X into a metric space Y.*

(a) *If A is a compact subset of X, then $F(A)$ is compact in Y.*

(b) *If X is compact, then F is a closed mapping.*

(c) *If X is compact and F is injective, then F is a homeomorphism of X onto $F(X)$.*

Proof. (a) Let $\mathcal{U}$ be a covering of $F(A)$ (i.e., $F(A) \subseteq \bigcup_{U \in \mathcal{U}} U$) consisting of open subsets U of Y. If F is continuous, then $F^{-1}(U)$ is an open subset of X for every $U \in \mathcal{U}$ according to Theorem 3.12. Set $\mathcal{F}^{-1}(\mathcal{U}) = \{F^{-1}(U)\colon U \in \mathcal{U}\}$; a collection of open subsets of X. Clearly (see Problem 1.2), $A \subseteq F^{-1}(F(A)) \subseteq F^{-1}\big(\bigcup_{U \in \mathcal{U}} U\big) = \bigcup_{U \in \mathcal{U}} F^{-1}(U)$ so that $\mathcal{F}^{-1}(\mathcal{U})$ is a covering of A made up of open subsets of X. If A is compact, then (Proposition 3.61) there exists a finite subcollection of $\mathcal{F}^{-1}(\mathcal{U})$ covering A; that is, there exists $\{U_i\}_{i=1}^n \subseteq \mathcal{U}$ such that $A \subseteq \bigcup_{i=1}^n F^{-1}(U_i) \subseteq X$. Thus $F(A) \subseteq F\big(\bigcup_{i=1}^n F^{-1}(U_i)\big) \subseteq \bigcup_{i=1}^n U_i \subseteq Y$ (have another look at Problem 1.2), and so $F(A)$ is compact by Proposition 3.61.

(b) If X is compact and if A is a closed subset of X, then A is compact by Theorem 3.62(b). Hence $F(A)$ is a compact subset of Y by item (a), so that $F(A)$ is closed in Y according to Theorem 3.62(a).

(c) If X is compact and F is injective, then F is a continuous invertible closed mapping of X onto $F(X)$ by item (b), and therefore a homeomorphism of X onto $F(X)$ (Theorem 3.24). □

Since compactness is preserved under continuous mappings, it is preserved by homeomorphisms, and so *compactness is a topological invariant*. Moreover, a one-to-one continuous correspondence between compact metric spaces is a homeomorphism. These are straightforward corollaries of Theorem 3.64.

Corollary 3.65. *If X and Y are homeomorphic metric spaces, then one is compact if and only if the other is.*

Corollary 3.66. *If X and Y are compact metric spaces, then every injective continuous mapping of X onto Y is a homeomorphism.*

Probably the reader has already noticed two important features: the metric has not played its role yet in this section, and the concepts of completeness and compactness share some common properties (e.g., compare Theorems 3.40 and 3.62). Indeed, the compactness proofs so far apply to topological spaces that are not necessarily metrizable. Actually, they all apply to Hausdorff spaces (metrizable or not), and Theorems 3.62(b) and 3.64(a) do hold for general topological spaces (not necessarily Hausdorff). As for the connection between completeness and compactness, first note that the notion of completeness introduced in Section 3.7 needs a metric. Moreover, as we have just seen, compactness is a topological invariant while completeness is not preserved by plain homeomorphism (just by uniform homeomorphism — Theorem 3.44 and Example 3.U). However, as we shall see in the next section, in a metric space compactness implies completeness.

Some of the most important results of mathematical analysis deal with continuous mappings on compact metric spaces. Theorem 3.64 is a special instance of such results that leads to many relevant corollaries (e.g., Corollary 3.85 in the next section is an extremely useful corollary of Theorem 3.64). Another important result along this line reads as follows.

Theorem 3.67. *Every continuous mapping of a compact metric space into an arbitrary metric space is uniformly continuous.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces and take an arbitrary real number $\varepsilon > 0$. If $F\colon X \to Y$ is a continuous mapping, then for each $x \in X$ there exists a real number $\delta_\varepsilon(x) > 0$ such that

$$d_X(x', x) < 2\delta_\varepsilon(x) \quad \text{implies} \quad d_Y(F(x'), F(x)) < \varepsilon.$$

Let $B_{\delta_\varepsilon(x)}(x)$ be the open ball with center at the point x and radius $\delta_\varepsilon(x)$. Consider the collection $\{B_{\delta_\varepsilon(x)}(x)\}_{x\in X}$, which surely covers X (i.e., $X = \bigcup_{x\in X} B_{\delta_\varepsilon(x)}(x)$). If X is compact (Definition 3.60), then this covering of X includes a finite subcovering, say $\bigcup_{i=1}^n B_{\delta_\varepsilon(x_i)}(x_i)$ with $x_i \in X$ for $i = 1, \ldots, n$. Take any $x' \in X$. Since $\bigcup_{i=1}^n B_{\delta_\varepsilon(x_i)}(x_i)$ covers X,

$$d_X(x_j, x') < \delta_\varepsilon(x_j)$$

for some $j = 1, \dots, n$ (i.e., $X = \bigcup_{i=1}^{n} B_{\delta_\varepsilon(x_i)}(x_i)$ implies that every point of X belongs to some ball $B_{\delta_\varepsilon(x_j)}(x_j)$). Thus

$$d_Y(F(x_j), F(x')) < \varepsilon.$$

Set $\delta_\varepsilon = \min\{\delta_\varepsilon(x_i)\}_{i=1}^{n}$, which is a positive number. If

$$d_X(x', x) < \delta_\varepsilon,$$

then $d_X(x, x_j) \le d_X(x, x') + d_X(x', x_j) < \delta_\varepsilon + \delta_\varepsilon(x_j) \le 2\delta_\varepsilon(x_j)$ by the triangle inequality, and hence $d_Y(F(x), F(x_j)) < \varepsilon$. Therefore, since $d_Y(F(x), F(x')) \le d_Y(F(x), F(x_j)) + d_Y(F(x_j), F(x'))$, it follows that

$$d_Y(F(x'), F(x)) < 2\varepsilon.$$

Conclusion: Given an arbitrary $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$d_X(x', x) < \delta_\varepsilon \quad \text{implies} \quad d_Y(F(x'), F(x)) < 2\varepsilon$$

for all $x, x' \in X$. That is, $F: X \to Y$ is uniformly continuous. □

We shall now investigate alternative characterizations of compact sets that, unlike the fundamental concept posed in Definition 3.60, will be restricted to metrizable spaces.

Definition 3.68. Let A be a subset of a metric space (X, d). A subset A_ε of A is an *ε-net* for A if for every point x of A there exists a point y in A_ε such that $d(x, y) < \varepsilon$. A subset A of X is *totally bounded* in (X, d) if for every real number $\varepsilon > 0$ there exists a finite ε-net for A.

Proposition 3.69. *Let A be a subset of a metric space X. The following assertions are equivalent.*

(a) *A is totally bounded.*

(b) *For every real number $\varepsilon > 0$ there exists a finite partition of A into sets of diameter less than ε.*

Proof. Take any $\varepsilon > 0$ and set $\rho = \frac{\varepsilon}{2}$. If there is a finite ρ-net A_ρ for A, then the finite collection of open balls $\{B_\rho(y)\}_{y \in A_\rho}$ covers A. That is, $A \subseteq \bigcup_{y \in A_\rho} B_\rho(y)$ since every $x \in A$ belongs to $B_\rho(y)$ for some $y \in A_\rho$ whenever A_ρ is a ρ-net for A. Set $A_y = B_\rho(y) \cap A$ for each $y \in A_\rho$ so that $A = \bigcup_{y \in A_\rho} A_y$. A disjointification (Problem 1.18) of the finite collection $\{A_y\}_{y \in A_\rho}$ is a finite partition of A into sets of diameter not greater than $\max_{y \in A_\rho} \operatorname{diam}(B_\rho(y) \cap A) \le 2\rho = \varepsilon$. Thus (a) implies (b) according to Definition 3.68. On the other hand, if $\{A_i\}_{i=1}^{n}$ is a finite partition of A into (nonempty) sets of diameter less than ε, then by taking one point a_i of each set A_i we get a finite set $\{a_i\}_{i=1}^{n}$ which is an ε-net for A. Therefore (b) implies (a). □

Note that *every finite subset of a metric space X is totally bounded*: it is a finite ε-net for itself for every positive ε. In particular, the empty set of X is totally bounded: for every positive ε there is no point in $\varnothing$ within a distance greater than ε for every point of X. It is also readily verified that *every subset of a totally bounded set is totally bounded* (indeed, $A_\varepsilon \cap B$ is an ε-net for any set $B \subseteq A$ whenever A_ε is an ε-net for A). Moreover, *the closure of a totally bounded set is again totally bounded* (reason: A_ε is a 2ε-net for A^- whenever A_ε is an ε-net for A).

Proposition 3.70. *Let A be a subset of a metric space X. If A has a finite ε-net for some $\varepsilon > 0$, then A is bounded in X.*

Proof. Suppose a nonempty subset A of a metric space (X, d) has a finite ε-net for some $\varepsilon > 0$. Take $x, y \in A$ arbitrary so that there exist $a, b \in A_\varepsilon$ for which $d(x, a) < \varepsilon$ and $d(x, b) < \varepsilon$. So $d(x, y) \le d(x, a) + d(a, b) + d(b, y) < d(a, b) + 2\varepsilon$ by the triangle inequality. Hence $\operatorname{diam}(A) \le \operatorname{diam}(A_\varepsilon) + 2\varepsilon$. Therefore, since $\operatorname{diam}(A_\varepsilon) < \infty$ (A_ε is a finite set), it follows that $\operatorname{diam}(A) < \infty$. □

Corollary 3.71. *Every totally bounded set is bounded.*

Example 3.X. The converse fails. That is, a bounded set is not necessarily totally bounded. For instance, the closed unit ball $B_1[\,0\,]$ centered at the null sequence 0 in the metric space (ℓ^2_+, d_2) of Example 3.B is obviously bounded in (ℓ^2_+, d_2); actually, $\operatorname{diam}(B_1[\,0\,]) = 2$. We show that there is no finite ε-net for $B_1[\,0\,]$ with $\varepsilon \le \frac{1}{2}\sqrt{2}$, and hence $B_1[\,0\,]$ is not totally bounded in (ℓ^2_+, d_2). Indeed, consider the countable subset $E = \{e_j\}_{j\in\mathbb{N}}$ of $B_1[\,0\,]$ made up of all scalar-valued sequences $e_j = \{\delta_{jk}\}_{k=1}^\infty$; that is, each sequence e_j has just one nonzero entry (equal to 1) at the jth position. If A_ε is an ε-net for $B_1[\,0\,]$, then A_ε contains a point within a distance less than ε for each e_j in E, and hence A_ε must have a point in each open ball $B_\varepsilon(e_j)$. Since $d_2(e_i, e_j) = \sqrt{2}$ whenever $i \ne j$, it follows that $B_\varepsilon(e_i) \cap B_\varepsilon(e_j) = \varnothing$ (i.e., $B_\varepsilon(e_j) \backslash B_\varepsilon(e_i) = B_\varepsilon(e_j)$) whenever $i \ne j$ and $\varepsilon \le \frac{1}{2}\sqrt{2}$. Thus, if $\varepsilon \le \frac{1}{2}\sqrt{2}$, then for each $e_j \in E$ there exists $b_j \in A_\varepsilon \cap B_\varepsilon(e_j) \backslash B_\varepsilon(e_i)$ for every $i \ne j$. This establishes an injective function from E to A_ε, and therefore $\#E \le \#A_\varepsilon$. Thus A_ε is at least countably infinite (i.e., $\aleph_0 \le \#A_\varepsilon$). Conclusion: $B_1[\,0\,]$ is a closed and bounded set in the complete metric space (ℓ^2_+, d_2) that is not totally bounded in (ℓ^2_+, d_2).

Proposition 3.72. *A totally bounded metric space is separable.*

Proof. Suppose X is a totally bounded metric space. For each positive integer n let X_n be a finite $\frac{1}{n}$-net for X. Set $A = \bigcup_{n\in\mathbb{N}} X_n$, which is a countable subset of X (Corollary 1.11) and dense in X. Thus X is separable. To verify that A is dense in X proceed as follows. Take an arbitrary x in X. For each positive integer n there exists $x_n \in X_n$ such that $d(x, x_n) < \frac{1}{n}$ (since X_n is a $\frac{1}{n}$-net for X), and the A-valued sequence $\{x_n\}_{n\in\mathbb{N}}$ converges in X to x. Thus $A^- = X$ by Proposition 3.32. □

Lemma 3.73. *A set A in a metric space X is totally bounded if and only if every A-valued sequence has a Cauchy subsequence.*

Proof. If A is a finite set, then the result holds trivially (in this case every A-valued sequence has a constant subsequence). Thus suppose A is an infinite set and let d denote the metric on X.

(a) We shall say that an A-valued sequence $\{x_k\}_{k\in\mathbb{N}_0}$ has Property $P_n(\varepsilon)$, for some integer $n \in \mathbb{N}$, if there exists an $\varepsilon > 0$ such that $d(x_j, x_k) \geq \varepsilon$ for every pair $\{j, k\}$ of distinct integers $j, k = 0, 1, \ldots, n$.

Claim. If A is not totally bounded, then there exists an $\varepsilon > 0$ and an A-valued sequence that has Property $P_n(\varepsilon)$ for every $n \in \mathbb{N}$.

Proof. Suppose the infinite set A is not totally bounded and let ε be any positive real number for which there is no finite ε-net for A. In particular (and trivially), no singleton in A is an ε-net for A, and hence there exists a pair of points in A, say x_0 and x_1, for which $d(x_0, x_1) \geq \varepsilon$. Thus every A-valued sequence whose first two entries coincide with x_0 and x_1 has Property $P_1(\varepsilon)$. Suppose there exists an A-valued sequence $\{x_k\}_{k\in\mathbb{N}_0}$ that has Property $P_n(\varepsilon)$ for some integer $n \in \mathbb{N}_0$, so that $d(x_j, x_k) \geq \varepsilon$ for every $j, k = 0, 1, \ldots, n$ such that $j \neq k$. Since the set $\{x_k\}_{k=0}^n$ is not an ε-net for A (recall that there is no finite ε-net for A), it follows that there exists a point in A, say x'_{n+1}, for which $d(x_k, x'_{n+1}) \geq \varepsilon$ for every $k = 0, 1, \ldots, n$. Replace x_{n+1} with x'_{n+1} so that the resulting sequence has Property $P_{n+1}(\varepsilon)$. This concludes the proof by induction. □

If an A-valued sequence $\{x_k\}_{k\in\mathbb{N}_0}$ has Property $P_n(\varepsilon)$ for every $n \in \mathbb{N}$, then $d(x_j, x_k) \geq \varepsilon$ for every nonnegative distinct integers j and k, and hence it has no Cauchy subsequence. Conclusion: If A is not totally bounded, then there exists an A-valued sequence that has no Cauchy subsequence. Equivalently, if every A-valued sequence has a Cauchy subsequence, then A is totally bounded.

(b) Conversely, suppose A is totally bounded and let $\{x_k\}_{k\in\mathbb{N}}$ be an arbitrary A-valued sequence. According to Proposition 3.69 there exists a finite partition $\mathcal{A}$ of A into sets of diameter less than 1. Since $\mathcal{A}$ is a finite partition of A, it follows that at least one of its sets, say $A_1 \subseteq A$, has the property that the (infinite) A-valued sequence $\boldsymbol{x} = \{x_k\}_{k\in\mathbb{N}}$ has an (infinite) subsequence, say $\boldsymbol{x}_1 = \{x_{(1)_k}\}_{k\in\mathbb{N}}$, whose entries lie in A_1. Note that A_1 is totally bounded (because A is). Thus there exists a finite partition $\mathcal{A}_1$ of A_1, consisting of subsets of A_1 with diameter less than $\frac{1}{2}$, such that at least one of its sets, say $A_2 \subseteq A_1 \subseteq A$, has the property that the A_1-valued sequence $\boldsymbol{x}_1 = \{x_{(1)_k}\}_{k\in\mathbb{N}}$ has a subsequence, say $\boldsymbol{x}_2 = \{x_{(2)_k}\}_{k\in\mathbb{N}}$, whose entries lie in A_2. This leads to the inductive construction of a decreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of A with $\operatorname{diam}(A_n) < \frac{1}{n}$, each including a subsequence $\boldsymbol{x}_n = \{x_{(n)_k}\}_{k\in\mathbb{N}}$ of the A-valued sequence $\{x_k\}_{k\in\mathbb{N}}$, for every $n \in \mathbb{N}$. Moreover, the sequence of sequences $\{\boldsymbol{x}_n\}_{n\in\mathbb{N}}$ (i.e., the $A^{\mathbb{N}}$-valued sequence whose entries are the A_n-valued sequences $\boldsymbol{x}_n$ for each $n \in \mathbb{N}$) has the property that $\boldsymbol{x}_{n+1}$ is a

subsequence of $\boldsymbol{x}_n$ for each $n \in \mathbb{N}$. The kernel of the proof relies on the *diagonal procedure* (Section 1.9). Consider the sequence $\{x_{(n)_n}\}_{n\in\mathbb{N}}$, where each $x_{(n)_n}$ is the nth entry of $\boldsymbol{x}_n$, which has the following properties: (1) it is an A-valued sequence (each $x_{(n)_n}$ lies in $A_n \subseteq A$), (2) it is a subsequence of $\{x_k\}_{k\in\mathbb{N}}$ (since $\boldsymbol{x}_{n+1}$ is a subsequence of $\boldsymbol{x}_n$ for each $m \in \mathbb{N}$ and $\boldsymbol{x}_1$ is a subsequence of $\boldsymbol{x} = \{x_k\}_{k\in\mathbb{N}}$), and (3) $d(x_{(m)_m}, x_{(n)_n}) < \frac{1}{m}$ if $m < n$ (because $x_{(m)_m} \in A_m$, $x_{(n)_n} \in A_n$, $A_n \subseteq A_m$, and $\operatorname{diam}(A_m) < \frac{1}{m}$ for every $m \in \mathbb{N}$). Therefore, the "diagonal" sequence $\{x_{(n)_n}\}_{n\in\mathbb{N}}$ is a Cauchy subsequence of $\{x_k\}_{k\in\mathbb{N}}$. □

Total boundedness is not a topological invariant, but it is preserved by uniform homeomorphism. In fact, the same example that shows that completeness is not preserved by plain homeomorphism (Example 3.U) also shows that total boundedness is not preserved by plain homeomorphism (the sets $(0, 1]$ and $[1, \infty)$ are homeomorphic, but $(0, 1]$ is totally bounded — see Example 3.Y below — while $[1, \infty)$ is not even bounded).

Theorem 3.74. *Let $F\colon X \to Y$ be a uniformly continuous mapping of a metric space X into a metric space Y. If A is a totally bounded subset of X, then $F(A)$ is totally bounded in Y.*

Proof. Let A be a nonempty subset of X and consider its image $F(A)$ in Y under a mapping $F\colon X \to Y$ (if A is empty the result is trivially verified). Take an arbitrary $F(A)$-valued sequence $\{y_n\}$ and consider any A-valued sequence $\{x_n\}$ such that $y_n = F(x_n)$ for every n. If A is totally bounded, then Lemma 3.73 ensures that $\{x_n\}$ has a Cauchy subsequence, say $\{x_{n_k}\}$. If $F\colon X \to Y$ is uniformly continuous, then $\{F(x_{n_k})\}$ is a Cauchy sequence in Y (Lemma 3.43) which is a subsequence of $\{y_n\}$. Thus every $F(A)$-valued sequence has a Cauchy subsequence; that is, $F(A)$ is totally bounded by Lemma 3.73. □

In particular, if $F\colon X \to Y$ is a surjective uniformly continuous mapping of a totally bounded metric space X onto a metric space Y, then Y must be totally bounded.

Corollary 3.75. *If X and Y are uniformly homeomorphic metric spaces, then one is totally bounded if and only if the other is.*

Total boundedness is sometimes referred to as *precompactness*. Lemma 3.73 links completeness to compactness in a metric space. It actually leads to the proof that *a metric space is compact if and only if it is complete and totally bounded*. We shall prove this important assertion in the next section.

3.11 Sequential Compactness

The notions of compactness and total boundedness can be thought of as topological counterparts of the set-theoretic notion of "finiteness" in the sense that they may suggest "approximate finiteness" (see Propositions 3.61 and 3.69).

Definition 3.76. A metric space X is *sequentially compact* if every X-valued sequence has a subsequence that converges in X. A subset A of a metric space X is *sequentially compact* if it is sequentially compact as a subspace of X.

Proposition 3.77. *Let A be a subset of a metric space X. The following assertions are equivalent.*

(a) *A is sequentially compact.*

(b) *Every infinite subset of A has at least one accumulation point in A.*

Proof. If A is empty, then the result holds trivially (it is sequentially compact because there is no A-valued sequence, and it satisfies condition (b) because it includes no infinite subset). Thus let A be a nonempty set. Recall that the limits of the convergent subsequences of a given sequence are precisely the accumulation points of its range.

Proof of (a)⇒(b). If B is an infinite subset of A, then B includes a countably infinite set, and so there exists a B-valued sequence $\{b_n\}_{n\in\mathbb{N}}$ of distinct points. If A is sequentially compact, then this A-valued sequence has a subsequence that converges in X to a point $a \in A$ (Definition 3.76). If $b_n \neq a$ for all $n \in \mathbb{N}$, then $\{b_n\}_{n\in\mathbb{N}}$ is a $B\backslash\{a\}$-valued sequence of distinct points that converges in X to a, and therefore $a \in A$ is an accumulation point of B (Proposition 3.28). If $b_m = a$ for some $m \in \mathbb{N}$, then remove this (unique) point from $\{b_n\}_{n\in\mathbb{N}}$ and get a $B\backslash\{a\}$-valued sequence of distinct points that converges in X to a, so that $a \in A$ is again an accumulation point of B. Conclusion: (a) implies (b).

Proof of (b)⇒(a). Let A be a subset of X for which assertion (b) holds true. In particular, every countably infinite subset of A has an accumulation point. Then every (infinite) A-valued sequence has a convergent subsequence, and hence A is sequentially compact. That is, (b) implies (a). □

A subset A of a metric space X (which may be X itself) is said to have the *Bolzano–Weierstrass Property* if it satisfies condition (b) of Proposition 3.77. Then *a metric space is sequentially compact if and only if it has the Bolzano–Weierstrass Property.* Note: *Every finite subset of a metric space is sequentially compact*, since it includes no infinite subset, and so it has the Bolzano–Weierstrass Property. In particular, the empty set is sequentially compact.

Theorem 3.78. *A metric space is sequentially compact if and only if it is totally bounded and complete.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary X-valued sequence. If the metric space X is both totally bounded and complete, then $\{x_n\}_{n\in\mathbb{N}}$ has a Cauchy subsequence (because X is totally bounded — Lemma 3.73), which converges in X (because X is complete). Thus X is sequentially compact (Definition 3.76). On the other hand, sequential compactness clearly implies total boundedness (see Definition 3.76, Proposition 3.39(a), and Lemma 3.73). Moreover, if X is se-

quentially compact and if $\{x_n\}_{n\in\mathbb{N}}$ is an X-valued Cauchy sequence, then this sequence has a convergent subsequence (Definition 3.76), and hence $\{x_n\}_{n\in\mathbb{N}}$ converges in X (Proposition 3.39(c)). Therefore, in a sequentially compact metric space X, every Cauchy sequence converges; that is, X is complete. □

The next lemma establishes another necessary and sufficient condition for sequential compactness. We shall apply this lemma to prove the equivalence between compactness and sequential compactness in a metric space.

Lemma 3.79. (Cantor). *A metric space X is sequentially compact if and only if every decreasing sequence $\{V_n\}_{n\in\mathbb{N}}$ of nonempty closed subsets of X has a nonempty intersection* (i.e., *is such that* $\bigcap_{n\in\mathbb{N}} V_n \neq \varnothing$).

Proof. (a) Let $\{V_n\}_{n\in\mathbb{N}}$ be a decreasing sequence (i.e., $V_{n+1} \subseteq V_n$ for every $n \in \mathbb{N}$) of nonempty closed subsets of a metric space X. For each $n \in \mathbb{N}$ let v_n be a point of V_n and consider the X-valued sequence $\{v_n\}_{n\in\mathbb{N}}$. If X is sequentially compact, then $\{v_n\}_{n\in\mathbb{N}}$ has a subsequence that converges in X to, say, $v \in X$ (Definition 3.76). Now take an arbitrary $m \in \mathbb{N}$ and note that this convergent subsequence is eventually in V_m (because $\{V_n\}_{n\in\mathbb{N}}$ is decreasing), and hence it has a V_m-valued subsequence that converges in X to v (Proposition 3.5). Since V_m is closed in X, it follows by the Closed Set Theorem that $v \in V_m$ (Theorem 3.30). Therefore, as m is arbitrary, $v \in \bigcap_{m\in\mathbb{N}} V_m$.

(b) Conversely, let $\{x_k\}_{k\in\mathbb{N}}$ be an arbitrary X-valued sequence and set $X_n = \{x_k \in X\colon\ k \geq n\}$ for each $n \in \mathbb{N}$. Note that $\{X_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty subsets of X, and so is the sequence of closed subsets of X, $\{X_n^-\}_{n\in\mathbb{N}}$, consisting of the closures of each X_n. If $\bigcap_{n\in\mathbb{N}} X_n^- \neq \varnothing$, then there exists $x \in X_n^-$ for all $n \in \mathbb{N}$. Take an arbitrary real number $\varepsilon > 0$ and consider the open ball $B_\varepsilon(x)$. Since $B_\varepsilon(x) \cap X_n \neq \varnothing$ for every $n \in \mathbb{N}$ (reason: set $A = X_n$ and $B = X_n^-$ in Proposition 3.32(b)), it follows that for every $n \in \mathbb{N}$ there exist integers $k \geq n$ for which $x_k \in B_\varepsilon(x)$. Thus every nonempty open ball centered at x meets the range of the sequence $\{x_k\}_{k\in\mathbb{N}}$ infinitely often, and so x is the limit of some convergent subsequence of $\{x_k\}_{k\in\mathbb{N}}$ (Proposition 3.28). Conclusion: Every X-valued sequence has a convergent subsequence, which means that X is sequentially compact. □

Theorem 3.80. *A metric space is compact if and only if it is sequentially compact.*

Proof. (a) Suppose X is a compact metric space (Definition 3.60). Let $\{V_n\}_{n=1}^\infty$ be an arbitrary decreasing sequence of nonempty closed subsets of X. Set $U_n = X\backslash V_n$ for each $n \in \mathbb{N}$ so that $\{U_n\}_{n=1}^\infty$ is an increasing sequence of proper open subsets of X. If $\{U_n\}_{n=1}^\infty$ covers X (i.e., if $\bigcup_{n=1}^\infty U_n = X$), then $U_m = \bigcup_{n=1}^m U_n = X$ for some $m \in \mathbb{N}$ (since $\{U_n\}_{n=1}^\infty$ is increasing and X is compact), which contradicts the fact that $U_n \neq X$ for every $n \in \mathbb{N}$. Outcome: $\bigcup_{n=1}^\infty U_n \neq X$ and hence (De Morgan laws) $\bigcap_{n=1}^\infty V_n \neq \varnothing$. Therefore X is sequentially compact by Lemma 3.79.

(b) On the other hand, suppose X is a sequentially compact metric space. Since X is separable (Proposition 3.72 and Theorem 3.78), it follows by Theorem 3.35 that X has a countable base $\mathcal{B}$ of open subsets of X. Let $\mathcal{U}$ be an arbitrary open covering of X.

Claim. There exists a countable subcollection $\mathcal{U}'$ of $\mathcal{U}$ that covers X.

Proof. For each $U \in \mathcal{U}$ set $\mathcal{B}_U = \{B \in \mathcal{B}\colon\ B \subseteq U\}$. Since $\mathcal{B}$ is a base for X, and U is an open subset of X, we get by the definition of base that $U = \bigcup \mathcal{B}_U$. The collection $\mathcal{B}' = \bigcup_{U\in\mathcal{U}} \mathcal{B}_U$ of open subsets of X has the following properties.

$$\#\mathcal{B}' \leq \#\mathcal{B} \quad \text{and} \quad \bigcup \mathcal{U} \subseteq \bigcup \mathcal{B}'.$$

Indeed, since $\mathcal{B}_U \subseteq \mathcal{B}$ for every $U \in \mathcal{U}$, it follows that $\bigcup_{U\in\mathcal{U}} \mathcal{B}_U \subseteq \mathcal{B}$. Thus $\mathcal{B}' \subseteq \mathcal{B}$ so that $\#\mathcal{B}' \leq \#\mathcal{B}$. Moreover, if U is an arbitrary set in $\mathcal{U}$, then $U = \bigcup \mathcal{B}_U = \bigcup_{B\in\mathcal{B}_U} B \subseteq \bigcup_{B\in\mathcal{B}'} B = \bigcup \mathcal{B}'$, and hence $\bigcup \mathcal{U} \subseteq \bigcup \mathcal{B}'$. Another property of the collection $\mathcal{B}'$ is that every set in $\mathcal{B}'$ is included in some set in $\mathcal{U}$ (reason: if $B' \in \mathcal{B}' = \bigcup_{U\in\mathcal{U}} \mathcal{B}_U$, then $B' \in \mathcal{B}_{U'} \subseteq \bigcup \mathcal{B}_{U'} = U'$ for some $U' \in \mathcal{U}$). For each set B' in $\mathcal{B}'$ take one set U' in $\mathcal{U}$ that includes B', and consider the subcollection $\mathcal{U}'$ of $\mathcal{U}$ consisting of all those sets U'. The very construction of $\mathcal{U}'$ establishes a surjective map of $\mathcal{B}'$ onto $\mathcal{U}'$ that embeds $\bigcup \mathcal{B}'$ in $\bigcup \mathcal{U}'$. Thus

$$\#\,\mathcal{U}' \leq \#\mathcal{B}' \quad \text{and} \quad \bigcup \mathcal{B}' \subseteq \bigcup \mathcal{U}'.$$

Therefore, by transitivity,

$$\#\,\mathcal{U}' \leq \#\mathcal{B} \quad \text{and} \quad \bigcup \mathcal{U} \subseteq \bigcup \mathcal{U}'.$$

Conclusion: $\mathcal{U}'$ is a countable subcollection of $\mathcal{U}$ (because $\mathcal{B}$ is a countable base for X) which covers X (because $\mathcal{U}$ covers X). □

If $\mathcal{U}'$ is finite, then it is itself a finite subcovering of $\mathcal{U}$ so that X is compact. If $\mathcal{U}'$ is countably infinite, then it can be indexed by $\mathbb{N}$ so that $\mathcal{U}' = \{U_n\}_{n=1}^{\infty}$, where each U_n is in $\mathcal{U}$. Set $V_n = X \backslash \bigcup_{i=1}^{n} U_i$ for each $n \in \mathbb{N}$ so that $\{V_n\}_{n=1}^{\infty}$ is a decreasing sequence of closed subsets of X. Since $\bigcup \mathcal{U}' = \bigcup_{n=1}^{\infty} U_n = X$ (because $\mathcal{U}'$ covers X), it follows that $\bigcap_{n=1}^{\infty} V_n = \varnothing$. Thus, according to Lemma 3.79, at least one of the sets in $\{V_n\}_{n=1}^{\infty}$, say V_m, must be empty (because X is sequentially compact). Then $\bigcap_{n=1}^{m} V_n = \varnothing$, and so $\bigcup_{n=1}^{m} U_n = X$. Conclusion: $\mathcal{U}$ includes a finite subcovering, viz., $\{U_n\}_{n=1}^{m}$, so that X is compact. □

The theorems have been proved. Let us now harvest the corollaries.

Corollary 3.81. *If X is a metric space, then the following assertions are pairwise equivalent.*

(a) *X is compact.*

(b) *X is sequentially compact.*

(c) *X is complete and totally bounded.*

Proof. Theorems 3.78 and 3.80. □

As we have already observed, completeness and total boundedness are preserved by uniform homeomorphisms but not by plain homeomorphisms, whereas compactness is preserved by plain homeomorphisms. Thus completeness and total boundedness are not topological invariants. However, when taken together they mean compactness, which is a topological invariant.

Corollary 3.82. *A compact subset of any metric space is closed and bounded.*

Proof. Theorem 3.62(a) and Corollaries 3.71 and 3.81. □

Recall that the converse fails. Indeed, we exhibited in Example 3.X a closed and bounded subset of a (complete) metric space that is not totally bounded, and hence not compact.

Theorem 3.83. (Heine–Borel). *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. The condition is clearly necessary by Corollary 3.8. We shall prove that it is also sufficient. Consider the real line $\mathbb{R}$ equipped with its usual metric. Let V_ρ be any nondegenerate closed and bounded interval, say $V_\rho = [\alpha,\ \alpha+\rho]$ for some real number α and some $\rho > 0$. Take an arbitrary $\varepsilon > 0$ and let n_ε be a positive integer large enough so that $\rho < (n_\varepsilon + 1)\frac{\varepsilon}{2}$. For each integer $k = 0, 1, \ldots, n_\varepsilon$ consider the interval $A_k = [\alpha + k\frac{\varepsilon}{2},\ \alpha + (k+1)\frac{\varepsilon}{2})$ of diameter $\frac{\varepsilon}{2}$. Since $A_j \cap A_k = \varnothing$ whenever $j \neq k$, and $V_\rho \subset [\alpha,\ \alpha + (n_\varepsilon + 1)\frac{\varepsilon}{2}) = \bigcup_{k=0}^{n_\varepsilon} A_k$, it follows that $\{A_k \cap V_\rho\}_{k=0}^{n_\varepsilon}$ is a finite partition of V_ρ into sets of diameter less than ε. Thus every closed and bounded interval of the real line is totally bounded (Proposition 3.69). Now equip $\mathbb{R}^n$ with any of the metrics d_∞ or d_p for some $p \geq 1$ as in Example 3.A (recall: these are uniformly equivalent metrics on $\mathbb{R}^n$ — Problem 3.33). Take an arbitrary bounded subset B of $\mathbb{R}^n$ and consider the closed interval V_ρ of diameter $\rho = \operatorname{diam}(B)$ such that $B \subseteq V_\rho^n$. Since V_ρ is totally bounded in $\mathbb{R}$, it follows by Problem 3.64(a) that the Cartesian product V_ρ^n is totally bounded in $\mathbb{R}^n$. Hence, as a subset of a totally bounded set, B is totally bounded. Conclusion: *Every bounded subset of $\mathbb{R}^n$ is totally bounded.* Moreover, since $\mathbb{R}^n$ is a complete metric space (when equipped with any of these metrics — Example 3.R(a)), it follows by Theorem 3.40(b) that every closed subset of $\mathbb{R}^n$ is a complete subspace of $\mathbb{R}^n$. Therefore every closed and bounded subset of $\mathbb{R}^n$ is compact (Corollary 3.81). □

The Heine–Borel Theorem is readily extended to $\mathbb{C}^n$ (again equipped with any of the uniformly equivalent metrics d_∞ or d_p for some $p \geq 1$ as in Example 3.A): *A subset of $\mathbb{C}^n$ is compact if and only if it is closed and bounded in $\mathbb{C}^n$.*

Corollary 3.84. *Let X be a complete metric space and let A be a subset of X.*

(a) *A is compact if and only if it is closed and totally bounded in X.*

(b) *A is relatively compact if and only if it is totally bounded in X.*

Proof. By Theorem 3.40(b) and Corollaries 3.81 and 3.82 we get the result in (a), which in turn leads to the result in (b) by recalling that A^- is totally bounded if and only if A is totally bounded. □

Corollary 3.85. *A continuous image of a compact set is closed and bounded.*

Proof. Theorem 3.64(a) and Corollary 3.82. □

Theorem 3.86. (Weierstrass). *If $\varphi: X \to \mathbb{R}$ is a continuous real-valued function on a metric space X, then φ assumes both a maximum and a minimum value on each nonempty compact subset of X.*

Proof. If φ is a continuous real-valued function defined on a (nonempty) compact metric space, then its (nonempty) range $\mathcal{R}(\varphi)$ is both closed and bounded in the real line $\mathbb{R}$ equipped with its usual metric (Corollary 3.85). Thus the bounded subset $\mathcal{R}(\varphi)$ of $\mathbb{R}$ has an infimum and a supremum in $\mathbb{R}$, which actually lie in $\mathcal{R}(\varphi)$ because $\mathcal{R}(\varphi)$ is closed in $\mathbb{R}$ (recall: a closed subset contains all its adherent points). □

Example 3.Y. Consider the set $C[X, Y]$ of all continuous mappings of a metric space X into a metric space Y, and let $B[X, Y]$ be the set of all bounded mappings of X into Y. According to Corollary 3.85,

$$C[X, Y] \subseteq B[X, Y] \text{ whenever } X \text{ is compact.}$$

Thus the sup-metric d_∞ on $B[X, Y]$ (Example 3.C) is inherited by $C[X, Y]$ if X is compact, and hence $(C[X, Y], d_\infty)$ is a subspace of $(B[X, Y], d_\infty)$. In other words, if X is compact, then the sup-metric d_∞ is well defined on $C[X, Y]$ so that, in this case, $(C[X, Y], d_\infty)$ is a metric space. In particular, $C[0, 1] \subset B[0, 1]$ because $[0, 1]$ is compact in $\mathbb{R}$ by the Heine–Borel Theorem (Theorem 3.83), and so $(C[0, 1], d_\infty)$ is a subspace of $(B[0, 1], d_\infty)$, as we had anticipated in Examples 3.D and 3.G, and used in Examples 3.N and 3.T. Moreover, since the absolute value function $|\cdot|: [0, 1] \to \mathbb{R}$ is continuous, since $(x - y) \in C[0, 1]$ for every $x, y \in C[0, 1]$, since the interval $[0, 1]$ is compact, and since the composition of continuous functions is continuous, it follows by the Weierstrass Theorem (Theorem 3.86) that

$$d_\infty(x, y) = \max_{t \in [0,1]} |x(t) - y(t)| \quad \text{for every} \quad x, y \in C[0, 1].$$

Now let $BC[X, Y]$ be the set of all bounded continuous mappings of X into Y and equip it with the sup-metric d_∞ as in Example 3.T. If X is compact, then $C[X, Y] = BC[X, Y]$ and $(C[X, Y], d_\infty)$ is a metric space that coincides with the metric space $(BC[X, Y], d_\infty)$. Since $(BC[X, Y], d_\infty)$ is complete if and only if Y is complete (Example 3.T), it follows that

$$(C[X, Y], d_\infty) \text{ is complete if } X \text{ is compact and } Y \text{ is complete.}$$

Example 3.Z. Suppose (X, d_X) is a compact metric space, let (Y, d_Y) be any metric space, and consider the metric space $(C[X,Y], d_\infty)$ of Example 3.Y. Let Φ be a subset of $C[X,Y]$. We shall investigate a necessary and sufficient condition that Φ be totally bounded. To begin with, let us pose the following definitions.

(i) A subset Φ of $C[X,Y]$ is *pointwise totally bounded* if for each x in X the set $\Phi(x) = \{f(x) \in Y\colon\ f \in \Phi\}$ is totally bounded in Y. Similarly, Φ is *pointwise bounded* if the set $\Phi(x)$ is bounded in Y for each x in X (i.e., if $\sup_{f,g\in\Phi} d_Y(f(x), g(x)) < \infty$ for each $x \in X$).

(ii) A subset Φ of $C[X,Y]$ is *equicontinuous at a point* $x_0 \in X$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$ for every $f \in \Phi$ (note: δ depends on ε and may depend on x_0 but it does not depend on f — hence the term "equicontinuity"). Φ is *equicontinuous* (or *equicontinuous on* X) if it is equicontinuous at every point of X.

Remark: If a subset Φ is equicontinuous on X, and if for each $\varepsilon > 0$ there exists a $\delta > 0$ (which depends only on ε) such that $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon$ for all $x, x' \in X$ and every $f \in \Phi$, then Φ is said to be *uniformly equicontinuous* on X. Uniform equicontinuity coincides with equicontinuity on a compact space (Theorem 3.67).

(a) *A totally bounded Φ is pointwise totally bounded and equicontinuous.*

Indeed, take $\varepsilon > 0$, $x_0 \in X$, and $f \in \Phi$ arbitrary. If Φ is totally bounded, then there is a finite ε-net Φ_ε for Φ so that the set $\Phi_\varepsilon(x_0) = \{g(x_0) \in Y\colon\ g \in \Phi_\varepsilon\}$ is a finite ε-net for $\Phi(x_0)$. Hence, if Φ is totally bounded, then $\Phi(x_0)$ is totally bounded for every x_0 in X, which means that Φ is pointwise totally bounded. Moreover, since Φ_ε is an ε-net for Φ, there exists $g \in \Phi_\varepsilon$ such that $d_\infty(f, g) < \varepsilon$, and so, for every $x \in X$,

$$d_Y(f(x), g(x)) \le \sup_{x\in X} d_Y(f(x), g(x)) = d_\infty(f, g) < \varepsilon.$$

Therefore,

$$\begin{aligned} d_Y(f(x), f(x_0)) &\le d_Y(f(x), g(x)) + d_Y(g(x), g(x_0)) + d_Y(g(x_0), f(x_0)) \\ &\le 2\varepsilon + d_Y(g(x), g(x_0)) \end{aligned}$$

for every $x \in X$ and every $g \in \Phi_\varepsilon$. Since each $g \in \Phi_\varepsilon$ is continuous, it follows that for each $g \in \Phi_\varepsilon$ there exists a $\delta_g = \delta_g(\varepsilon, x_0) > 0$ such that $d_X(x, x_0) < \delta_g$ implies $d_Y(g(x), g(x_0)) < \varepsilon$. Since Φ_ε is a finite ε-net for Φ, set $\delta = \delta(\varepsilon, x_0) = \min\{\delta_g\}_{g\in\Phi_\varepsilon}$ so that $d_Y(g(x), g(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. Thus there exists a $\delta > 0$ (that does not depend on f) such that

$$d_X(x, x_0) < \delta \quad \text{implies} \quad d_Y(f(x), f(x_0)) < 3\varepsilon.$$

Hence, if Φ is totally bounded, then Φ is equicontinuous at an arbitrary x_0 in X, which means that Φ is equicontinuous on X. Summing up: If Φ is totally bounded, then Φ is pointwise totally bounded and equicontinuous on X.

(b) *A pointwise totally bounded and equicontinuous Φ is totally bounded.*

This is the converse of (a). In fact, recall that X is separable (because it is compact — Proposition 3.72 and Corollary 3.81) and take a countable dense subset A of X. Consider the (infinite) A-valued sequence $\{a_i\}_{i\geq 1}$ consisting of an enumeration of all points of A (followed by an arbitrary repetition of points of A if A is finite). Let $\boldsymbol{f} = \{f_n\}_{n\geq 1}$ be an arbitrary Φ-valued sequence, and suppose Φ is pointwise totally bounded so that $\Phi(x)$ is totally bounded in Y for every $x \in X$. Thus, according to Lemma 3.73, for each $x \in X$ the $\Phi(x)$-valued sequence $\{f_n(x)\}_{n\geq 1}$ has a Cauchy subsequence. In particular, $\{f_n(a_1)\}_{n\geq 1}$ has a Cauchy subsequence, say $\{f_n^{(1)}(a_1)\}_{n\geq 1}$. Set $\boldsymbol{f}_1 = \{f_n^{(1)}\}_{n\geq 1}$, which is a Φ-valued subsequence of $\boldsymbol{f}$ such that $\{f_n^{(1)}(a_1)\}_{n\geq 1}$ is a Cauchy sequence in Y. Consider for each $x \in X$ the $\Phi(x)$-valued sequence $\{f_n^{(1)}(x)\}_{n\geq 1}$. Since $\Phi(x)$ is totally bounded for every $x \in X$, it follows by Lemma 3.73 that $\{f_n^{(1)}(x)\}_{n\geq 1}$ has a Cauchy subsequence for every $x \in X$. In particular, $\{f_n^{(1)}(a_2)\}_{n\geq 1}$ has a Cauchy sequence, say $\{f_n^{(2)}(a_2)\}_{n\geq 1}$. Set $\boldsymbol{f}_2 = \{f_n^{(2)}\}_{n\geq 1}$, which is a Φ-valued subsequence of $\boldsymbol{f}_1$ such that $\{f_n^{(2)}(a_2)\}_{n\geq 1}$ is a Cauchy sequence in Y and both $\{f_n^{(1)}(a_1)\}_{n\geq 1}$ and $\{f_n^{(1)}(a_2)\}_{n\geq 1}$ are Cauchy sequences in Y as well (reason: $\boldsymbol{f}_2$ is a subsequence of $\boldsymbol{f}_1$, and hence $\{f_n^{(2)}(a_1)\}_{n\geq 1}$ is a subsequence of the Cauchy sequence $\{f_n^{(1)}(a_1)\}_{n\geq 1}$). This leads to the inductive construction of a sequence $\{\boldsymbol{f}_k\}_{k\geq 1}$ of Φ-valued subsequences of $\boldsymbol{f}$ with the following properties.

Property (1). Each $\boldsymbol{f}_{k+1} = \{f_n^{(k+1)}\}_{n\geq 1}$ is a subsequence of $\boldsymbol{f}_k = \{f_n^{(k)}\}_{n\geq 1}$.

Property (2). $\{f_n^{(k)}(a_i)\}_{n\geq 1}$ is a Cauchy sequence in Y whenever $1 \leq i \leq k$.

As it happened in part (b) of the proof of Lemma 3.73, the *diagonal procedure* plays a central role in this proof too. Take any integer $i \geq 1$. By Property (1) the Y-valued sequence $\{f_n^{(n)}(a_i)\}_{n\geq i}$ is a subsequence of $\{f_n^{(i)}(a_i)\}_{n\geq 1}$, which is a Cauchy sequence in Y by Property (2). Thus $\{f_n^{(n)}(a_i)\}_{n\geq i}$ is a Cauchy sequence in Y, and so is $\{f_n^{(n)}(a_i)\}_{n\geq 1}$. Therefore, the "diagonal" sequence $\{f_n^{(n)}\}_{n\geq 1}$, which is a subsequence of the Φ-valued sequence $\boldsymbol{f} = \{f_n\}_{n\geq 1}$ (cf. Property (1)), is such that

$$\{f_n^{(n)}(a)\}_{n\geq 1} \text{ is a Cauchy sequence in } Y \text{ for every } a \in A.$$

Now take $\varepsilon > 0$ and $x \in X$ arbitrary. Suppose Φ is equicontinuous on X. Thus there exists a $\delta_\varepsilon = \delta_\varepsilon(x) > 0$ such that

$$d_X(x', x) < \delta_\varepsilon \quad \text{implies} \quad d_Y\big(f_n^{(n)}(x'), f_n^{(n)}(x)\big) < \varepsilon$$

for all n. Since A is dense in X, it follows that there exists $a \in A$ such that $d_X(a, x) < \delta_\varepsilon$, and hence

$$d_Y\big(f_n^{(n)}(a), f_n^{(n)}(x)\big) < \varepsilon$$

for all n. But $\{f_n^{(n)}(a)\}_{n\geq 1}$ is a Cauchy sequence, which means that there exists a positive integer $n_\varepsilon = n_\varepsilon(a)$ such that

$$d_Y\big(f_m^{(m)}(a), f_n^{(n)}(a)\big) \,<\, \varepsilon$$

whenever $m, n \geq n_\varepsilon$. Then, by the triangle inequality,

$$\begin{aligned} d_Y\big(f_m^{(m)}(x), f_n^{(n)}(x)\big) \,&\leq\, d_Y\big(f_m^{(m)}(x), f_m^{(m)}(a)\big) \,+\, d_Y\big(f_m^{(m)}(a), f_n^{(n)}(a)\big) \\ &+\, d_Y\big(f_n^{(n)}(a), f_n^{(n)}(x)\big) \,<\, 3\varepsilon \end{aligned}$$

whenever $m, n \geq n_\varepsilon$. Therefore, as n_ε does not depend on x,

$$d_\infty\big(f_m^{(m)}, f_n^{(n)}\big) \,=\, \sup_{x\in X} d_Y\big(f_m^{(m)}(x), f_n^{(n)}(x)\big) \,<\, 3\varepsilon$$

whenever $m, n \geq n_\varepsilon$, and so the subsequence $\{f_n^{(n)}\}_{n\geq 1}$ of $\boldsymbol{f} = \{f_n\}_{n\geq 1}$ is a Cauchy sequence in Φ. Thus Φ is totally bounded by Lemma 3.73. Summing up: If X is compact, and if Φ is pointwise totally bounded and equicontinuous, then Φ is totally bounded.

(c) Arzelà–Ascoli Theorem. *If X is compact, then a subset of the metric space $(C[X,Y], d_\infty)$ is totally bounded if and only if it is pointwise totally bounded and equicontinuous.*

This is a consequence of (a) and (b). The next corollary follows at once (cf. Example 3.Y and Corollary 3.84).

(d) *If X is a compact metric space and Y is a complete metric space, then a subset of the metric space $(C[X,Y], d_\infty)$ is compact if and only if it is pointwise totally bounded, equicontinuous, and closed in $(C[X,Y], d_\infty)$.*

Recall that total boundedness coincides with plain boundedness in the real line (see proof of Theorem 3.83). Thus we get the following particular case of (d): *A subset Φ of the metric space $(C[0,1], d_\infty)$ is compact if and only if it is pointwise bounded, equicontinuous and closed in $(C[0,1], d_\infty)$.* Note: In this case pointwise boundedness means $\sup_{x\in\Phi}|x(t)| < \infty$ for each $t \in [0,1]$.

Suggested Reading

Bachman and Narici [1] [2]
Brown and Pearcy [3]
Dieudonné [1]
Dugundji [1]
Gelbaum and Olmsted [1]
Goffman and Pedrick [1]
Kaplansky [2]
Kantorovich and Akilov [1]
Kelley [1]
Kolmogorov and Fomin [1]
McDonald and Weiss [1]
Munkres [1]
Naylor and Sell [1]
Royden [1]
Schwartz [1]
Simmons [1]
Smart [1]
Sutherland [1]

Problems

Problem 3.1. Let (X, d) be a metric space. Then, for every x, y, z in X,

(a) $$|d(x, y) - d(y, z)| \leq d(x, z).$$

Hint: Use symmetry and the triangle inequality only.

Incidentally, the above inequality shows that the metric axioms (i) to (iv) in Definition 3.1 are not independent. For instance, the property $d(x, y) \geq 0$ in axiom (i) follows from symmetry and the triangle inequality. That is, the statement "$d(x, y) \geq 0$ for every $x, y \in X$" in fact is a theorem derived by axioms (iii) and (iv). Moreover, for every u, v, x, y in X,

(b) $$|d(x, u) - d(v, y)| \leq d(x, v) + d(u, y).$$

Hint: $d(x, u) \leq d(x, v) + d(v, y) + d(y, u)$ and, similarly, $d(v, y) \leq d(v, x) + d(x, u) + d(u, y)$ — use symmetry.

Problem 3.2. Suppose $d_1 \colon X \times X \to \mathbb{R}$ and $d_2 \colon X \times X \to \mathbb{R}$ are metrics on a set X. Define the functions $d \colon X \times X \to \mathbb{R}$ and $d' \colon X \times X \to \mathbb{R}$ by

$$d(x, y) = d_1(x, y) + d_2(x, y) \quad \text{and} \quad d'(x, y) = \max\big\{d_1(x, y), d_2(x, y)\big\}$$

for every $x, y \in X$. Show that both d and d' are metrics on X.

Problem 3.3. Let p and q be real numbers. If $p > 1$ and if $q = \frac{p}{p-1} > 1$ is the unique solution to the equation $\frac{1}{p} + \frac{1}{q} = 1$ (or, equivalently, the unique solution to the equation $p + q = pq$), then p and q are said to be *Hölder conjugates* of each other. Prove the following inequalities.

(a) If $p > 1$ and $q > 1$ are Hölder conjugates, and if $x = (\xi_1, \ldots, \xi_n)$ and $y = (\upsilon_1, \ldots, \upsilon_n)$ are arbitrary n-tuples in $\mathbb{C}^n$, then

$$\sum_{i=1}^{n} |\xi_i \upsilon_i| \leq \Big(\sum_{i=1}^{n} |\xi_i|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} |\upsilon_i|^q\Big)^{\frac{1}{q}}.$$

Hint: Show that $\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ for every pair of positive real numbers α and β whenever p and q are Hölder conjugates. Set $\|x\|_p = \big(\sum_{i=1}^{n} |\xi_i|^p\big)^{\frac{1}{p}}$.

Moreover,

$$\sum_{i=1}^{n} |\xi_i \upsilon_i| \leq \max_{1 \leq i \leq n} |\xi_i| \sum_{i=1}^{n} |\upsilon_i|.$$

These are the *Hölder inequalities for finite sums*.

(b) Let $x = \{\xi_k\}$ and $y = \{\upsilon_k\}$ be arbitrary complex-valued sequences (i.e., sequences in $\mathbb{C}^{\mathbb{N}}$). If $p > 1$ and $q > 1$ are Hölder conjugates, then

$$\sum_{k=1}^{\infty} |\xi_k \upsilon_k| \le \Big(\sum_{k=1}^{\infty} |\xi_k|^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^{\infty} |\upsilon_k|^q\Big)^{\frac{1}{q}}$$

whenever $\sum_{k=1}^{\infty}|\xi_k|^p < \infty$ and $\sum_{k=1}^{\infty}|\upsilon_k|^q < \infty$; and

$$\sum_{k=1}^{\infty} |\xi_k \upsilon_k| \le \sup_{k\in\mathbb{N}} |\xi_k| \sum_{k=1}^{\infty} |\upsilon_k|$$

whenever $\sup_{k\in\mathbb{N}}|\xi_k| < \infty$ and $\sum_{k=1}^{\infty}|\upsilon_k| < \infty$. These are called *Hölder inequalities for infinite sums*.

(c) Finally, let Ω be a nonempty subset of $\mathbb{C}$, and let x and y be arbitrary complex-valued functions on Ω (i.e., functions in $\mathbb{C}^{\Omega}$). If $p > 1$ and $q > 1$ are Hölder conjugates, then

$$\int_{\Omega} |x\, y|\, d\omega \le \Big(\int_{\Omega} |x|^p\, d\omega\Big)^{\frac{1}{p}} \Big(\int_{\Omega} |y|^q\, d\omega\Big)^{\frac{1}{q}}$$

for every pair of integrable functions x, y in $\mathbb{C}^{\Omega}$ such that $\int_{\Omega} |x|^p\, d\omega < \infty$ and $\int_{\Omega} |y|^q\, d\omega < \infty$. Moreover, if $x, y \in \mathbb{C}^{\Omega}$ are integrable functions such that $\sup_{\omega\in\Omega}|x(\omega)| < \infty$ and $\int_{\Omega} |y|\, d\omega < \infty$, then

$$\int_{\Omega} |x\, y|\, d\omega \le \sup_{\omega\in\Omega} |x(\omega)| \int_{\Omega} |y|\, d\omega.$$

These are the *Hölder inequalities for integrals*.

Problem 3.4. Let p be any real number such that $p \ge 1$. Use the preceding problem to verify the following results.

(a) *Minkowski inequalities for finite sums*. Take an arbitrary pair of n-tuples in $\mathbb{C}^n$, say $x = (\xi_1, \ldots, \xi_n)$ and $y = (\upsilon_1, \ldots, \upsilon_n)$ in $\mathbb{C}^n$. Then

$$\Big(\sum_{i=1}^{n} |\xi_i + \upsilon_i|^p\Big)^{\frac{1}{p}} \le \Big(\sum_{i=1}^{n} |\xi_i|^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{n} |\upsilon_i|^p\Big)^{\frac{1}{p}}$$

and

$$\max_{1\le i\le n} |\xi_i + \upsilon_i| \le \max_{1\le i\le n} |\xi_i| + \max_{1\le i\le n} |\upsilon_i|.$$

Hint: Verify that $|\xi + \upsilon| \le |\xi| + |\upsilon|$ for every pair $\{\xi, \upsilon\}$ of complex numbers, and show that $(\alpha + \beta)^p = (\alpha + \beta)^{p-1}\alpha + (\alpha + \beta)^{p-1}\beta$ for every pair $\{\alpha, \beta\}$ of nonnegative real numbers.

(b) *Minkowski inequalities for infinite sums*. If $x = \{\xi_k\}$ and $y = \{\upsilon_k\}$ are sequences in $\mathbb{C}^{\mathbb{N}}$ (i.e., if $x = \{\xi_k\}$ and $y = \{\upsilon_k\}$ are complex-valued sequences) such that $\sum_{k=1}^{\infty}|\xi_k|^p < \infty$ and $\sum_{k=1}^{\infty}|\upsilon_k|^p < \infty$, then

$$\Big(\sum_{k=1}^{\infty}|\xi_k+\upsilon_k|^p\Big)^{\frac{1}{p}} \le \Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^{\infty}|\upsilon_k|^p\Big)^{\frac{1}{p}}.$$

Moreover,

$$\sup_{k\in\mathbb{N}}|\xi_k+\upsilon_k| \le \sup_{k\in\mathbb{N}}|\xi_k| + \sup_{k\in\mathbb{N}}|\upsilon_k|$$

whenever $\sup_{k\in\mathbb{N}}|\xi_k| < \infty$ and $\sup_{k\in\mathbb{N}}|\upsilon_k| < \infty$.

(c) *Minkowski inequalities for integrals*. If x and y are integrable functions in $\mathbb{C}^{\Omega}$ (i.e., integrable complex-valued functions on Ω) with $\int_\Omega |x|^p\,d\omega < \infty$ and $\int_\Omega |y|^p\,d\omega < \infty$, where Ω is a nonempty subset of $\mathbb{C}$, then

$$\Big(\int_\Omega |x+y|^p\,d\omega\Big)^{\frac{1}{p}} \le \Big(\int_\Omega |x|^p\,d\omega\Big)^{\frac{1}{p}} + \Big(\int_\Omega |y|^p\,d\omega\Big)^{\frac{1}{p}}.$$

If $\sup_{\omega\in\Omega}|x(\omega)| < \infty$ and $\sup_{\omega\in\Omega}|y(\omega)| < \infty$, then

$$\sup_{\omega\in\Omega}|x(\omega)+y(\omega)| \le \sup_{\omega\in\Omega}|x(\omega)| + \sup_{\omega\in\Omega}|y(\omega)|.$$

Problem 3.5. (a) Let $\{\alpha_k\}$ be a sequence of *nonnegative* real numbers such that $\sum_{k=1}^{\infty}\alpha_k < \infty$. Take an arbitrary real number $r > 1$. Verify that

$$\sum_{k=1}^{\infty}\alpha_k^r \le \Big(\sum_{k=1}^{\infty}\alpha_k\Big)^r.$$

Hint: $\alpha_k \le \sum_{k=1}^{\infty}\alpha_k$ and so $\sum_{k=1}^{\infty}\big(\alpha_k/\sum_{k=1}^{\infty}\alpha_k\big)^r \le \sum_{k=1}^{\infty}\big(\alpha_k/\sum_{k=1}^{\infty}\alpha_k\big) = 1$.

(b) *Jensen inequality*. If p and q are real numbers such that $0 < p < q$, then

$$\Big(\sum_{k=1}^{\infty}|\xi_k|^q\Big)^{\frac{1}{q}} \le \Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{\frac{1}{p}}$$

for every $x = \{\xi_k\}$ in $\mathbb{C}^{\mathbb{N}}$ such that $\sum_{k=1}^{\infty}|\xi_k|^p < \infty$ (i.e., for every p-summable complex-valued sequence). Prove.

Hint: Apply the inequality in (a) for $\sum_{k=1}^{\infty}|\xi_k|^q = \sum_{k=1}^{\infty}|\xi_k|^{pr}$ with $r = \frac{q}{p} > 1$.

(c) Let ℓ_+^p and ℓ_+^∞ be the sets of all p-summable and bounded sequences from $\mathbb{F}^{\mathbb{N}}$, respectively, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ (i.e., the sets of all p-summable and bounded scalar-valued sequences, respectively; see Example 3.B). Show that

$$1 \le p < q \quad \text{implies} \quad \ell_+^p \subset \ell_+^q \subset \ell_+^\infty,$$

and verify that these are in fact proper inclusions.

Problem 3.6. If A and B are nonempty and bounded subsets of a metric space (X, d), then

$$\sup_{x\in A,\, y\in B} d(x,y) \;\le\; \operatorname{diam}(A) + \operatorname{diam}(B) + d(A,B).$$

Hint: $d(x,y) \le d(x,a) + d(a,b) + d(b,y)$ for every $x, a \in A$ and every $y, b \in B$.

Now conclude that $A \cup B$ is bounded. Show that the union of a *finite* collection of bounded subsets of X is a bounded subset of X.

Problem 3.7. Let (X, d) be a metric space and define $d_i\colon X\times X \to \mathbb{R}$ for $i = 1, 2$ as follows.

$$d_1(x,y) = \frac{d(x,y)}{1 + d(x,y)} \quad \text{and} \quad d_2(x,y) = \min\big\{1, d(x,y)\big\}$$

for every $x, y \in X$. Show that d_1 and d_2 are metrics on X. Moreover, verify that every set in the metric spaces (X, d_1) and (X, d_2) is bounded.

Problem 3.8. Let p, q, and r be positive numbers such that

$$\frac{1}{p} + \frac{1}{q} \;=\; \frac{1}{r}.$$

(a) Prove the following extension of the Hölder inequality for integrals.

$$\Big(\int_\Omega |x\,y|^r\, d\omega\Big)^{\frac{1}{r}} \;\le\; \Big(\int_\Omega |x|^p\, d\omega\Big)^{\frac{1}{p}} \Big(\int_\Omega |y|^q\, d\omega\Big)^{\frac{1}{q}}$$

for all integrable functions x, y in $\mathbb{C}^\Omega$ such that $\int_\Omega |x|^p\, d\omega < \infty$ and $\int_\Omega |y|^q\, d\omega < \infty$. (Similar inequalities hold for finite and infinite sums.)

Hint: Verify that $\frac{p}{r}$ and $\frac{q}{r}$ are Hölder conjugates.

(b) Let S be a bounded interval of the real line and let d_p be the usual metric on $R^p(S)$ (see Example 3.E). Show that

$$1 \le r < p \quad \text{implies} \quad R^p(S) \subset R^r(S).$$

Hint: $d_r(x,y) \le \operatorname{diam}(S)^{\frac{p-r}{pr}} d_p(x,y)$ for every x, y in $R^p(S)$ if $1 \le r < p$.

Problem 3.9. Let $\{(X_i, d_i)\}_{i=1}^n$ be a finite collection of metric spaces and set $Z = \prod_{i=1}^n X_i$, the Cartesian product of their underlying sets. Consider the functions $d_p\colon Z\times Z \to \mathbb{R}$ (for each real number $p \ge 1$) and $d_\infty\colon Z\times Z \to \mathbb{R}$ defined as follows.

$$d_p(x,y) \;=\; \Big(\sum_{i=1}^n d_i(x_i,y_i)^p\Big)^{\frac{1}{p}} \quad \text{and} \quad d_\infty(x,y) \;=\; \max\big\{d_i(x_i,y_i)\big\}_{i=1}^n$$

for every $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $Z = \prod_{i=1}^n X_i$. Show that these are metrics on Z.

Remark: Let $d: Z \times Z \to \mathbb{R}$ be any of the above metrics. The Cartesian product $\prod_{i=1}^{n} X_i$ equipped with d is referred to as a *product space* of the metric spaces $\{(X_i, d_i)\}_{i=1}^{n}$, and the metric d as a *product metric*. Sometimes, when the metric d has been previously specified, it is convenient to denote the product space $\left(\prod_{i=1}^{n} X_i\,, d\right)$ by $\prod_{i=1}^{n}(X_i, d_i)$. This notation is particularly suitable for a product space of metric spaces with the same underlying set but different metrics. For instance, $(X, d_1)\times(X, d_2)$ and $(X, d_2)\times(X, d_1)$ are different metric spaces with the same underlying set $X \times X$.

Problem 3.10. Consider the real line $\mathbb{R}$ with its usual metric. Let $\{\alpha_n\}$ be a real-valued sequence (indexed by $\mathbb{N}$ or $\mathbb{N}_0$) and recall the definitions of bounded, increasing, decreasing, and monotone sequences:

(i) $\{\alpha_n\}$ is bounded if $\sup_n |\alpha_n| < \infty$.
(ii) $\{\alpha_n\}$ is increasing if $\alpha_n \le \alpha_{n+1}$ for every index n.
(iii) $\{\alpha_n\}$ is decreasing if $\alpha_{n+1} \le \alpha_n$ for every index n.
(iv) $\{\alpha_n\}$ is monotone if it is either increasing or decreasing.

Prove the following statements.

(a) If $\{\alpha_n\}$ converges, then it is bounded.

(b) If $\{\alpha_n\}$ is monotone and bounded, then it converges.

Therefore, for a real-valued monotone sequence, boundedness becomes equivalent to convergence. Now suppose $\{\alpha_n\}$ is a nonnegative sequence (i.e., $0 \le \alpha_n$ for every index n) and let $\{\beta_n\}$ be another real-valued sequence.

(c) If $0 \le \alpha_n \le \beta_n$ for every n and $\beta_n \to 0$, then $\alpha_n \to 0$.

Next let $\{\alpha_n\}$ and $\{\beta_n\}$ be arbitrary scalar-valued sequences (either in $\mathbb{R}$ or $\mathbb{C}$).

(d) $\sup_n |\alpha_n| \inf_n |\beta_n| \le \sup_n(|\alpha_n|\,|\beta_n|)$.

Hint: $|\alpha_n| \inf_n |\beta_n| \le |\alpha_n|\,|\beta_n|$.

(e) If $\alpha_n \to \alpha$ and $\beta_n \to \beta$, then $\alpha_n + \beta_n \to \alpha + \beta$.

(f) If $\alpha_n \to \alpha$ and $\beta_n \to \beta$, then $\alpha_n\,\beta_n \to \alpha\,\beta$.

Hint: $\alpha_n\beta_n = (\alpha_n - \alpha)\beta_n + \alpha(\beta_n - \beta) + \alpha\,\beta$.

Show that (e) and (f) can be extended to *finite* sums and *finite* products of convergent sequences. A subsequence $\{n_k\}$ of the sequence of all nonnegative integers is of *bounded increments* if $\sup_{k\ge 0}(n_{k+1} - n_k) < \infty$. Use the extension of (e) to finite sums to prove the following proposition.

(g) If $\{\alpha_n\}$ is a sequence of nonnegative numbers, and if there exists a subsequence of bounded increments $\{n_k\}$ such that $\alpha_{n_k+j} \to 0$ as $k \to \infty$ for every $j \ge 0$, then $\alpha_n \to 0$ as $n \to \infty$.

Hint: Set $J = \sup_{k\geq 0}(n_{k+1} - n_k)$ and write any nonnegative integer as $n = n_k + j_k \leq n_k + J$ for some $0 \leq j_k \leq J$. The extension of (e) ensures that $\sum_{j=1}^{J} \alpha_{n_k+j} \to 0$ as $n_k \to \infty$. Therefore, $\sup_{0\leq j_k\leq J} \alpha_{n_k+j_k} \to 0$ as $n_k \to \infty$. Now show that for every $\varepsilon > 0$ there is an integer k_ε for which $\alpha_n = \alpha_{n_k+j_k} < \sup_{0\leq j_k\leq J} \alpha_{n_k+j_k} < \varepsilon$ whenever $n_k \geq n_{k_\varepsilon}$. Thus conclude that $\alpha_n < \varepsilon$ whenever $n \geq n_{k_\varepsilon} + J$.

Problem 3.11. Consider again the real line $\mathbb{R}$ with its usual metric and let $\{\alpha_n\}$ be a sequence of *nonnegative* real numbers. For each integer $n \geq 1$ set

$$\sigma_n = \sum_{i=1}^{n} \alpha_i.$$

$\{\sigma_n\}$ is called the *sequence of partial sums* of $\{\alpha_n\}$, which clearly is nonnegative and increasing. Let us introduce the following usual notation and terminology.

(i) If $\{\sigma_n\}$ converges, then we say that $\sum_{k=1}^{\infty} \alpha_k$ converges.
(ii) If $\{\sigma_n\}$ is bounded, then we write $\sum_{k=1}^{\infty} \alpha_k < \infty$.
(iii) For each index n write $\sum_{k=n+1}^{\infty} \alpha_k = \sup_m \sigma_m - \sigma_n$.

The purpose of this problem is to show that the assertions

$$\sum_{k=1}^{\infty} \alpha_k \text{ converges,}$$

$$\sum_{k=1}^{\infty} \alpha_k < \infty, \quad \text{and}$$

$$\sum_{k=n+1}^{\infty} \alpha_k \to 0 \quad \text{as} \quad n \to \infty$$

are pairwise equivalent. Prove the following propositions.

(a) If $\sup_m \sigma_m < \infty$, then $\sigma_n \to \sup_m \sigma_m$ as $n \to \infty$.

(b) If $\{\sigma_n\}$ converges, then $\lim_n \sigma_n = \sup_m \sigma_m$.

Obviously, the above convergences are all in $\mathbb{R}$ with its usual metric.

Problem 3.12. Let $\{\alpha_n\}$ be a sequence of *nonnegative* real numbers and equip the real line with its usual metric. Prove the following assertions.

(a) If $\sum_{n=1}^{\infty} \alpha_n < \infty$, then $\lim_n \alpha_n = 0$.

(b) If $\{\alpha_n\}$ is decreasing and $\sum_{n=1}^{\infty} \alpha_n < \infty$, then $\lim_n n\alpha_n = 0$.

Hint: Show that if $\lim_n \alpha_{2n} = \lim_n \alpha_{2n-1} = \alpha$, then $\lim_n \alpha_n = \alpha$.

Now exhibit a pair of nonnegative sequences $\{\beta_n\}$ and $\{\gamma_n\}$ with the following properties.

(c) $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sup_n n\beta_n = \infty$.

(d) $\{\gamma_n\}$ is decreasing, $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$, and $\sup_n n\gamma_n = \infty$.

Problem 3.13. Let $\{\beta_n\}$ be a real-valued bounded sequence. The real line is a boundedly complete lattice in the natural ordering $\leq$ of $\mathbb{R}$ (Example 1.C). Thus both $\sup_{n\leq k}\beta_k = \sup\{\beta_k\colon\ n \leq k\}$ and $\inf_{n\leq k}\beta_k = \inf\{\beta_k\colon\ n \leq k\}$ exist in $\mathbb{R}$ for each index n. Set

$$\alpha_n = \inf_{n\leq k} \beta_k \quad \text{and} \quad \gamma_n = \sup_{n\leq k} \beta_k$$

for each n, consider the real-valued sequences $\{\alpha_n\}$ and $\{\gamma_n\}$, and equip the real line $\mathbb{R}$ with its usual metric.

(a) Show that both sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ converge in $\mathbb{R}$.

Hint: They are bounded monotone sequences.

Since these sequences converge in $\mathbb{R}$, set

$$\alpha = \lim_n \alpha_n = \lim_n \inf_{n\leq k} \beta_k \quad \text{and} \quad \gamma = \lim_n \gamma_n = \lim_n \sup_{n\leq k} \beta_k.$$

The limits $\alpha \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ of the bounded monotone sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ are usually denoted by

$$\alpha = \liminf_n \beta_n \quad \text{and} \quad \gamma = \limsup_n \beta_n,$$

and are called *limit inferior* (or *lower limit*) and *limit superior* (or *upper limit*) of the sequence $\{\beta_n\}$, respectively (see Problem 1.19). First show that

(b) $\liminf_n \beta_n \leq \limsup_n \beta_n$.

Now prove the following propositions which say that *a real-valued sequence converges if and only if its lower and upper limits coincide*.

(c) If $\{\beta_n\}$ converges in $\mathbb{R}$, then $\liminf_n \beta_n = \limsup_n \beta_n = \lim_n \beta_n$.

Hint: Show that $|\alpha_n - \beta| \leq |\beta_n - \beta|$ for each n and every $\beta \in \mathbb{R}$. Similarly, show that $|\beta_n - \gamma| \leq |\gamma_n - \gamma|$ for each n and every $\gamma \in \mathbb{R}$.

(d) If $\liminf_n \beta_n = \limsup_n \beta_n = \beta$, then $\{\beta_n\}$ converges in $\mathbb{R}$ to β.

Hint: Show that $\alpha_n \leq \beta_n \leq \gamma_n$ for each n, then apply Problem 3.10(c).

Remark: If the real-valued sequence $\{\beta_n\}$ is not bounded above, then $\{\beta_k\}_{n\leq k}$ is not bounded above for every index n. In this case we write $\sup_{n\leq k}\beta_k = \infty$ for every n and also $\limsup_n \beta_n = \infty$. Similarly, if $\{\beta_n\}$ is not bounded below, then $\{\beta_k\}_{n\leq k}$ is not bounded below for every index n. In this case we write $\inf_{n\leq k}\beta_k = -\infty$ for every n and $\liminf_n \beta_n = -\infty$. Observe that the notation $\limsup_n \beta_n = \infty$ and $\liminf_n \beta_n = -\infty$ are mere formalisms; they simply say that $\{\beta_n\}$ is not bounded above or below, respectively. It should also be noticed that the concepts and results in this problem naturally generalize from sequences to nets of real numbers.

Warning: Let $\{\beta_n\}$ be an arbitrary real-valued sequence (bounded or not). Consider the sets $B = \{\beta_n\}$ and $B_n = \{\beta_k\}_{n\leq k}$ for each n, which are the ranges of the sequences $\{\beta_n\}$ and $\{\beta_k\}_{n\leq k}$. Since $\{B_n\}$ is a decreasing sequence of subsets of $B \subseteq \mathbb{R}$ in the inclusion ordering, it follows by Problem 1.19 that

$$\lim_n B_n \;=\; \liminf_n B_n \;=\; \limsup_n B_n \;=\; \bigcap_n B_n,$$

which always exists as a set in the power set $\wp(\mathbb{R})$ (e.g., if $\beta_n = n$ for each $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} B_n = \varnothing$). Carefully note that the present problem deals with the natural ordering $\leq$ of $\mathbb{R}$, and not with the inclusion ordering $\subseteq$ of $\wp(\mathbb{R})$.

Problem 3.14. Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences (both indexed by $\mathbb{N}$ or $\mathbb{N}_0$) in a metric space (X, d). Set $x = \lim x_n$ and $y = \lim y_n$ in X. Prove the following propositions.

(a) $d(x_n, u) \to d(x, u)$ and $d(v, y_n) \to d(v, y)$ for each $u, v \in X$.

(b) $d(x_n, y_n) \to d(x, y)$.

Hint: Problems 3.1(b) and 3.10(c).

(c) If there exists $\alpha > 0$ and an integer n_0 such that $d(x_n, y_n) < \alpha$ for every $n \geq n_0$, then $d(x, y) \leq \alpha$.

Hint: Use the triangle inequality to show that $d(x, y) < 2\varepsilon + \alpha$ for every $\varepsilon > 0$, so that $d(x, y) \leq \inf_{\varepsilon>0}(2\varepsilon + \alpha)$.

Problem 3.15. Two sequences $\{x_n\}$, $\{y_n\}$ in a metric space (X, d) are *equiconvergent* if $\lim d(x_n, y_n) = 0$. Prove the following propositions.

(a) $\{x_n\}$ converges to x in (X, d) if and only if it is equiconvergent with the constant sequence $\{x'_n\}$ where $x'_n = x$ for all n.

(b) Two sequences that converge to the same limit are equiconvergent.

(c) If one of two equiconvergent sequences is convergent, then so is the other and both have the same limit.

Problem 3.16. Obviously, uniform continuity implies continuity.

(a) Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ for every $x \in \mathbb{R}$ is continuous but not uniformly continuous.

(b) Prove that every Lipschitzian mapping is uniformly continuous.

(c) Show that the function $g\colon [0, \infty) \to [0, \infty)$ defined by $g(x) = x^{\frac{1}{2}}$ for every $x \in [0, \infty)$ is uniformly continuous but not Lipschitzian.

Problem 3.17. Consider the setup and the mappings of Example 3.H.

(a) Show that $F\colon (X, d_2) \to (Y, d_2)$ of Example 3.H(a) is nowhere continuous.

Hint: Take $x_0 \in X$ and $\delta > 0$ arbitrary. Let α be any positive real number such that $\frac{\delta^3}{18} \le \alpha^2 < \frac{\delta^3}{6}$ and consider the function $e_\delta : \mathbb{R} \to \mathbb{R}$ defined by

$$e_\delta(t) = \begin{cases} \alpha, & 0 \le t \le \frac{3}{\delta}, \\ -\alpha, & \frac{3}{\delta} < t \le \frac{6}{\delta}, \\ 0, & \text{otherwise.} \end{cases}$$

Verify that e_δ lies in X so that $x_0 + e_\delta \in X$. Set $x_\delta = x_0 + e_\delta$ in X, $y_\delta = F(x_\delta)$ and $y_0 = F(x_0)$ in Y. Compute $y_\delta(t) - y_0(t)$ for $t \in \mathbb{R}$ and conclude that $d_2(x_\delta, x_0) < \delta$ and $d_2(F(x_\delta), F(x_0)) \ge 1$.

(b) Show that $F : (R^2(S), d_2) \to (R^2(S), d_2)$ of Example 3.H(b) is Lipschitzian (and hence uniformly continuous — *hint*: Hölder inequality).

Problem 3.18. Let $C'[0,1]$ be the subset of $C[0,1]$ consisting of all differentiable functions from $C[0,1]$ whose derivatives lie in $C[0,1]$. Consider the subspace $(C'[0,1], d_\infty)$ of the metric space $(C[0,1], d_\infty)$ — see Example 3.D. Let $D : C'[0,1] \to C[0,1]$ be the mapping that assigns to each $x \in C'[0,1]$ its derivative in $C[0,1]$.

(a) Show that $D : (C'[0,1], d_\infty) \to (C[0,1], d_\infty)$ is nowhere continuous.

Now consider the function $d : C'[0,1] \times C'[0,1] \to \mathbb{R}$ defined by

$$d(x,y) = d_\infty(x,y) + d_\infty(D(x), D(y))$$

for every $x, y \in C'[0,1]$, which is a metric on $C'[0,1]$ (cf. Problem 3.2).

(b) Show that $D : (C'[0,1], d) \to (C[0,1], d_\infty)$ is a contraction (and therefore Lipschitzian, and hence uniformly continuous).

Problem 3.19. Consider the real line $\mathbb{R}$ equipped with its usual metric. Set $B[0,\infty) = B[[0,\infty), \mathbb{R}]$ and $R^1[0,\infty) = R^1([0,\infty))$ as in Examples 3.C and 3.E. Let d_∞ be the sup-metric on $B[0,\infty)$ and let d_1 be the usual metric on $R^1[0,\infty)$. Consider the set of all real-valued functions x on $[0,\infty)$ that are 1-integrable (i.e., x is Riemann integrable on $[0,\infty)$ and $\int_0^\infty |x(s)|\, ds < \infty$) and bounded (i.e., $\sup_{s \ge 0} |x(s)| < \infty$). Allowing a slight abuse of notation, write

$$X[0,\infty) = B[0,\infty) \cap R^1[0,\infty)$$

and set, for any real number $\alpha > 0$,

$$X[0,\alpha] = \big\{ x \in X[0,\infty) : \ x(s) = 0 \text{ for all } s > \alpha \big\}.$$

For each $x \in X[0,\infty)$ consider the function $y : [0,\infty) \to \mathbb{R}$ defined by

$$y(t) = \int_0^t x(s)\, ds$$

for every $t \geq 0$. Let $BC[0,\infty) \subseteq B[0,\infty)$ denote the set of all real-valued bounded and continuous functions on $[0,\infty)$.

(a) Show that $y \in BC[0,\infty)$.

Now consider the mapping $F\colon X[0,\infty) \to BC[0,\infty)$ that assigns to each function x in $X[0,\infty)$ the function $y = F(x)$ in $BC[0,\infty)$ defined by the above integral. For simplicity we shall use the same notation F for the restriction $F|_{X[0,\alpha]}$ of F to $X[0,\alpha]$. By using the appropriate definitions, show that

(b) $F\colon (X[0,\infty), d_1) \to (BC[0,\infty), d_\infty)$ is a contraction,

(c) $F\colon (X[0,\infty), d_\infty) \to (BC[0,\infty), d_\infty)$ is nowhere continuous,

(d) $F\colon (X[0,\alpha], d_\infty) \to (BC[0,\infty), d_\infty)$ is Lipschitzian.

Problem 3.20. Let I be the identity map of $C[0,1]$ onto itself and consider the metrics d_∞ and d_p (for any $p \geq 1$) on $C[0,1]$ as in Example 3.D. Show that

(a) $I\colon (C[0,1], d_\infty) \to (C[0,1], d_p)$ is a contraction,

(b) $I\colon (C[0,1], d_p) \to (C[0,1], d_\infty)$ is nowhere continuous.

Hint: Take the $C[0,1]$-valued sequence $\{x_n\}$ of Example 3.F. Apply Theorem 3.7 to show that $I\colon (C[0,1], d_p) \to (C[0,1], d_\infty)$ is not continuous at $0 \in C[0.1]$.

Problem 3.21. Recall that $\ell_+^p \subset \ell_+^\infty$ for every $p \geq 1$ (Problem 3.5) and consider the subspace (ℓ_+^p, d_∞) of the metric space $(\ell_+^\infty, d_\infty)$. Let I be the identity map of ℓ_+^p onto itself. Show that, for each $p \geq 1$,

(a) $I\colon (\ell_+^p, d_p) \to (\ell_+^p, d_\infty)$ is a contraction,

(b) $I\colon (\ell_+^p, d_\infty) \to (\ell_+^p, d_p)$ is nowhere continuous.

Hint: Use Theorem 3.7 to show that $I\colon (\ell_+^p, d_\infty) \to (\ell_+^p, d_p)$ is not continuous at $0 \in \ell_+^p$. Take $x_n = n^{-\frac{1}{p}}(1, 1, \ldots, 1, 0, 0, \ldots)$.

Problem 3.22. Let $\mathbb{F}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ and let $\mathbb{F}^{\mathbb{N}}$ be the collection of all scalar-valued sequences indexed by $\mathbb{N}$. Consider the metric space (ℓ_+^p, d_p) of Example 3.B for an arbitrary $p \geq 1$ and, for every $a = \{\alpha_k\} \in \mathbb{F}^{\mathbb{N}}$, consider the following subset of ℓ_+^p.

$$X_a^p \;=\; \big\{x = \{\xi_k\} \in \ell_+^p\colon\ \textstyle\sum_{k=1}^\infty |\alpha_k \xi_k|^p < \infty\big\}.$$

Let $D_a\colon (X_a^p, d_p) \to (\ell_+^p, d_p)$ be the *diagonal mapping* that assigns to each sequence $x = \{\xi_k\}$ in X_a^p the sequence $\{\alpha_k \xi_k\}$ in ℓ_+^p; that is,

$$D_a(x) \;=\; \{\alpha_k \xi_k\} \quad \text{ for every } \quad x = \{\xi_k\} \in X_a^p.$$

(a) Show that $X_a^p = \ell_+^p$ and D_a is Lipschitzian if $a \in \ell_+^\infty$ (i.e., if $\sup_k |\alpha_k| < \infty$).

(b) Show that D_a is not continuous if $\alpha_k = k$ for each $k \in \mathbb{N}$.

Hint: Use Theorem 3.7. Take $x_n = n^{-1}(0, \ldots, 0, 1, 0, \ldots)$.

Problem 3.23. Let α be an arbitrary nonnegative real number and set

$$\beta = \beta(\alpha) = \begin{cases} 1 - \alpha, & \alpha \le 1, \\ \dfrac{\alpha - 1}{\alpha}, & \alpha \ge 1. \end{cases}$$

Consider the real-valued sequence $\{\xi_n\}$ recursively defined as follows.

(i) $$\xi_0 = 0 \quad \text{and} \quad \xi_{n+1} = \tfrac{1}{2}(\beta + \xi_n^2)$$

for every $n \ge 0$. Verify that

(a) $\{\xi_n\}$ is an increasing sequence,

(b) $0 \le \xi_n \le \beta$ for all $n \ge 0$,

(c) $\{\xi_n\}$ converges in $\mathbb{R}$, and

(d) $\lim \xi_n = 1 - (1 - \beta)^{\frac{1}{2}}$.

Hint: According to Problem 3.16(a), the function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ for every $x \in \mathbb{R}$ is continuous. Use Theorem 3.7.

Thus prove the *square root algorithm*: For every nonnegative real number α,

$$\alpha^{\frac{1}{2}} = \begin{cases} 1 - \lim \xi_n, & \alpha \le 1, \\ (1 - \lim \xi_n)^{-1}, & \alpha \ge 1, \end{cases}$$

where $\{\xi_n\}$ is recursively defined as in (i).

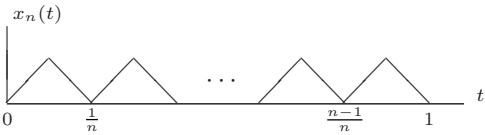
Problem 3.24. Take an arbitrary $C[0,1]$-valued sequence $\{x_n\}$ that converges in $(C[0,1], d_\infty)$ to $x \in C[0,1]$. Take an arbitrary $[0,1]$-valued sequence $\{t_n\}$ that converges in $\mathbb{R}$ to $t \in [0,1]$. Show that

$$x_n(t_n) \to x(t) \quad \text{in} \quad \mathbb{R}.$$

Hint: $\alpha_k + \beta_k \to \alpha + \beta$ whenever $\alpha_k \to \alpha$ and $\beta_k \to \beta$ in $\mathbb{R}$ (Problem 3.10(e)). Now use Problem 3.10(c) and Theorem 3.7.

Problem 3.25. Consider the standard notion of curve length in the plane and let $D[0,1]$ denote the subset of $C[0,1]$ consisting of all real-valued functions on $[0,1]$ whose graph has a finite length. Let $\varphi\colon D[0,1] \to \mathbb{R}$ be the mapping that assigns to each function $x \in D[0,1]$ the length of its graph (e.g., if $x \in C[0,1]$ is given by $x(t) = (t - t^2)^{\frac{1}{2}}$ for every $t \in [0,1]$, then $x \in D[0,1]$ and $\varphi(x) = \frac{\pi}{2}$). Now consider the $D[0,1]$-valued sequence $\{x_n\}$ defined as follows. For each

$n \geq 1$ the graph of x_n forms n *equilateral* triangles of the same height when intercepted with the horizontal axis.



Use this sequence to show that the mapping φ is not continuous when $D[0,1]$ is equipped with the sup-metric d_∞ and $\mathbb{R}$ is equipped with its usual metric. (*Hint*: Theorem 3.7.)

Problem 3.26. Let $C[0,\infty)$ denote the set of all real-valued continuous functions on $[0,\infty)$ and set

$$XC[0,\infty) \,=\, X[0,\infty) \cap C[0,\infty) \,=\, B[0,\infty) \cap R^1[0,\infty) \cap C[0,\infty),$$

where the set $X[0,\infty)$ was defined in Problem 3.19 as the intersection of $B[0,\infty)$ and $R^1[0,\infty)$. Now consider the mapping $\varphi\colon XC[0,\infty) \to \mathbb{R}$ given by

$$\varphi(x) \,=\, \int_0^\infty |x(t)|\, dt$$

for every $x \in XC[0,\infty)$. Let $\{x_n\}$ be the $XC[0,\infty)$-valued sequence such that, for each $n \geq 1$,

$$x_n(t) \,=\, \frac{1}{n^2} \begin{cases} t, & t \in [0,n], \\ 2n - t, & t \in [n,2n], \\ 0, & \text{otherwise.} \end{cases}$$

Use this sequence to show that $\varphi\colon XC[0,\infty) \to \mathbb{R}$ is not continuous when $XC[0,\infty)$ is equipped with the sup-metric d_∞ from $B[0,\infty)$ and $\mathbb{R}$ is equipped with its usual metric. (*Hint*: Theorem 3.7 — compare with Example 3.I.)

Problem 3.27. Let (X,d) be a metric space. A real-valued function $\varphi\colon X \to \mathbb{R}$ is *upper semicontinuous at a point* x_0 of X if for each real number β such that $\varphi(x_0) < \beta$ there exists a positive number δ such that

$$d(x,x_0) < \delta \quad \text{implies} \quad \varphi(x) < \beta.$$

It is *upper semicontinuous on* X if it is upper semicontinuous at every point of X. Similarly, a real-valued function $\psi\colon X \to \mathbb{R}$ is *lower semicontinuous at a point* x_0 of X if for each real number α such that $\alpha < \psi(x_0)$ there exists a positive number δ such that

$$d(x,x_0) < \delta \quad \text{implies} \quad \alpha < \psi(x).$$

It is *lower semicontinuous on* X if it is lower semicontinuous at every point of X. Now equip $\mathbb{R}$ with its usual metric and prove the next proposition (which can be thought of as a real-valued semicontinuous version of Theorem 3.7).

(a) $\varphi: X \to \mathbb{R}$ is upper semicontinuous at x_0 if and only if

$$\limsup_n \varphi(x_n) \;\le\; \varphi(x_0)$$

for every X-valued sequence $\{x_n\}$ that converges in (X, d) to x_0. Similarly, $\psi: X \to \mathbb{R}$ is lower semicontinuous at x_0 if and only if

$$\liminf_n \psi(x_n) \;\ge\; \psi(x_0)$$

for every X-valued sequence $\{x_n\}$ that converges in (X, d) to x_0.

Hint: Take $\varepsilon > 0$ and x_0 arbitrary, and set $\beta = \varphi(x_0) + \varepsilon$. Suppose φ is upper semicontinuous at x_0 and show that there exists $\delta > 0$ such that

$$d(x, x_0) < \delta \quad \text{implies} \quad \varphi(x) < \varepsilon + \varphi(x_0).$$

If $x_n \to x_0$ in (X, d), then show that there is an integer $n_\delta > 0$ such that

$$n \ge n_\delta \quad \text{implies} \quad \varphi(x_n) < \varepsilon + \varphi(x_0).$$

Now conclude: $\limsup_n \varphi(x_n) \le \varphi(x_0)$. Conversely, if $\varphi: X \to \mathbb{R}$ is not upper semicontinuous at x_0, then verify that there exists $\beta > \varphi(x_0)$ such that for every $\delta > 0$ there exists $x_\delta \in X$ such that

$$d(x_\delta, x_0) < \delta \quad \text{and} \quad \varphi(x_\delta) \ge \beta.$$

Set $\delta_n = \frac{1}{n}$ and $x_n = x_{\delta_n}$ for each integer $n \ge 1$. Thus $x_n \to x_0$ in (X, d) while $\varphi(x_n) \ge \beta$ for all n, and hence $\limsup_n \varphi(x_n) > \varphi(x_0)$.

(b) Show that $\varphi: X \to \mathbb{R}$ is continuous if and only if it is both upper and lower semicontinuous on X.

Hint: Problem 3.13 and Theorem 3.7.

Problem 3.28. Show that the composition of uniformly continuous mappings is again uniformly continuous.

Problem 3.29. Let X, Y, and Z be metric spaces.

(a) If $F: X \to Y$ is continuous at x_0 and $G: Y \to Z$ is continuous at $F(x_0)$, then the composition $G \circ F: X \to Z$ is continuous at x_0. (*Hint*: Lemma 3.11.)

(b) Now use Corollary 3.13 to show by induction that if $F: X \to X$ is continuous, then so is its nth power $F^n: X \to X$ for every $n \ge 1$.

Problem 3.30. The restriction of a continuous mapping to a subspace is continuous. That is, if $F: X \to Y$ is a continuous mapping of a metric space X into a metric space Y, and if A is a subspace of X, then the restriction $F|_A: A \to Y$ of F to A is continuous.

Hint: $F|_A = F \circ J$ where $J: A \to X$ is the inclusion map — use Corollary 3.13.

Problem 3.31. Let X be an arbitrary set. The largest (or strongest) topology on X is the discrete topology $\wp(X)$ (where every subset of X is open), and the smallest (or weakest) topology on X is the indiscrete topology (where the only open subsets of X are the empty set and the whole space). The collection of all topologies on X is partially ordered by the inclusion ordering $\subseteq$. Recall that $\mathcal{T}_1 \subseteq \mathcal{T}_2$ (i.e., $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$ or, equivalently, $\mathcal{T}_2$ is stronger than $\mathcal{T}_1$) if every element of $\mathcal{T}_1$ also is an element of $\mathcal{T}_2$. Prove the following propositions.

(a) If $\mathcal{U} \subseteq \wp(X)$ is an arbitrary collection of subsets of X, then there exists a smallest (weakest) topology $\mathcal{T}$ on X such that $\mathcal{U} \subseteq \mathcal{T}$.

Hint: The power set $\wp(X)$ is a topology on X. The intersection of a nonempty collection of topologies on X is again a topology on X.

(b) Show that the collection of all topologies on X is a complete lattice in the inclusion ordering.

Problem 3.32. Let d_1 and d_2 be two metrics on a set X. Show that d_1 and d_2 are equivalent if and only if for each $x_0 \in X$ and each $\varepsilon > 0$ the following two conditions hold.

(i) There exists $\delta_1 > 0$ such that

$$d_1(x, x_0) < \delta_1 \quad \text{implies} \quad d_2(x, x_0) < \varepsilon.$$

(ii) There exists $\delta_2 > 0$ such that

$$d_2(x, x_0) < \delta_2 \quad \text{implies} \quad d_1(x, x_0) < \varepsilon.$$

Hint: $d_1 \sim d_2$ if and only if the identity $I\colon (X, d_1) \to (X, d_2)$ is a homeomorphism.

Problem 3.33. Let $\{(X_i, d_i)\}_{i=1}^n$ be a finite collection of metric spaces and let $Z = \prod_{i=1}^n X_i$ be the Cartesian product of their underlying sets. Consider the metrics d_p for each $p \geq 1$ and d_∞ on Z that were defined in Problem 3.9. Show that, for an arbitrary $p \geq 1$,

$$d_\infty(x, y) \;\leq\; d_p(x, y) \;\leq\; d_1(x, y) \;\leq\; n\, d_\infty(x, y)$$

for every $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $Z = \prod_{i=1}^n X_i$.

Hint: $d_p(x, y) \leq d_1(x, y)$ by the Jensen inequality (Problem 3.5).

Thus conclude that the metrics d_∞ and d_p for every $p \geq 1$ are all uniformly equivalent on Z, so that the product spaces $(\prod_{i=1}^n X_i,\, d_\infty)$ and $(\prod_{i=1}^n X_i,\, d_p)$ for every $p \geq 1$ are all uniformly homeomorphic.

Problem 3.34. Let (X, d) be a metric space and equip the real line $\mathbb{R}$ with its usual metric. Take $u, v \in X$ arbitrary and note that both functions

$$d(\cdot\,, u)\colon (X, d) \to \mathbb{R} \quad \text{and} \quad d(v, \cdot\,)\colon (X, d) \to \mathbb{R}$$

preserve convergence (Problem 3.14(a)), and hence they are continuous by Corollary 3.8. Now consider the Cartesian product $X \times X$ equipped with the metric d_1 of Problem 3.9: $d_1(z, w) = d(x, u) + d(y, v)$ for every $z = (x, y)$ and $w = (u, v)$ in $X \times X$.

(a) Show that $d(\cdot\,,\cdot): (X \times X, d_1) \to \mathbb{R}$ is continuous.

Hint: Verify that $x_n \to x$ and $y_n \to y$ in (X, d) whenever $(x_n, y_n) \to (x, y)$ in $(X \times X, d_1)$. Now use Problem 3.14(b) and Corollary 3.8.

Next let d' denote any of the metrics d_p (for an arbitrary $p \geq 1$) or the metric d_∞ on $X \times X$ as in Problem 3.9.

(b) Show that $d(\cdot\,,\cdot): (X \times X, d') \to \mathbb{R}$ is continuous.

Hint: According to Problem 3.33 and Corollary 3.19, it can be verified that the identity $I: (X \times X, d') \to (X \times X, d_1)$ is a (uniform) homeomorphism. Now use item (a) and Corollary 3.13.

Problem 3.35. If d and d' are equivalent metrics on a set X, then an X-valued sequence $\{x_n\}$ converges in (X, d) to $x \in X$ if and only if it converges in (X, d') to the same limit x (Corollary 3.19). If d and d' are not equivalent, then it may happen that an X-valued sequence $\{x_n\}$ converges to $x \in X$ in (X, d) but does not converge (to any point) in (X, d') (e.g., see Examples 3.F and 3.G). Can a sequence converge in (X, d) to a point $x \in X$ and also converge in (X, d') to a different point $x' \in X$? Yes, it can. We shall equip a set X with two metrics d and d', and exhibit an X-valued sequence $\{x_n\}$ and a pair of distinct points x and x' in X such that

$$x_n \to x \quad \text{in} \quad (X, d) \qquad \text{and} \qquad x_n \to x' \quad \text{in} \quad (X, d').$$

Consider the set $\mathbb{R}^2$ and let d denote the Euclidean metric on it (or any of the metrics on $\mathbb{R}^2$ introduced in Example 3.A, which are uniformly equivalent according to Problem 3.33). Set $v = (0, 1) \in \mathbb{R}^2$ and let V be the vertical axis joining v to the point $0 = (0, 0) \in \mathbb{R}^2$ (in the jargon of Chapter 2, set $V = \text{span}\{v\}$). Now consider a function $d': \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as follows. If x and y are both in $\mathbb{R}^2 \backslash V$ or both in V, then set $d'(x, y) = d(x, y)$. If one of them is in V but not the other, then $d'(x, y) = d(x + v, y)$ if $x \in V$ and $y \in \mathbb{R}^2 \backslash V$, or $d'(x, y) = d(x, y + v)$ if $x \in \mathbb{R}^2 \backslash V$ and $y \in V$.

(a) Show that d' is a metric on $\mathbb{R}^2$.

Hint: If $x, y \in V$, then $d'(x, y) = d(x + v, y + v)$.

Take the $\mathbb{R}^2$-valued sequence $\{x_n\}$ with $x_n = (\frac{1}{n}, 1)$ for each $n \geq 1$. Show that

(b) $$x_n \to v \quad \text{in} \quad (\mathbb{R}^2, d) \qquad \text{and} \qquad x_n \to 0 \quad \text{in} \quad (\mathbb{R}^2, d').$$

(This construction was communicated by Ivo Fernandez Lopez.)

Problem 3.36. Upper and lower semicontinuity were defined in Problem 3.27. Equip the real line with its usual metric, let X be a metric space, and consider the following statement.

(i) Let $\varphi: X \to \mathbb{R}$ be an upper semicontinuous function on X and let $\psi: X \to \mathbb{R}$ be a lower semicontinuous function on X. If $\varphi(x) \leq \psi(x)$ for every $x \in X$, then there is a continuous function $\eta: X \to \mathbb{R}$ such that $\varphi(x) \leq \eta(x) \leq \psi(x)$ for every $x \in X$.

This is the *Hahn Interpolation Theorem*. Use it to prove the *Tietze Extension Theorem*, which is stated below.

(ii) If $f: A \to \mathbb{R}$ is a bounded and continuous function on a nonempty and closed subset A of a metric space X, then f has a continuous extension $g: X \to \mathbb{R}$ over the whole space X such that $\inf_{x \in X} g(x) = \inf_{a \in A} f(a)$ and $\sup_{x \in X} g(x) = \sup_{a \in A} f(a)$.

Hint: Verify that the functions $\varphi: X \to \mathbb{R}$ and $\psi: X \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} f(x), & x \in A, \\ \inf_{a \in A} f(a), & x \in X \backslash A, \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} f(x), & x \in A, \\ \sup_{a \in A} f(a), & x \in X \backslash A, \end{cases}$$

extend $f: A \to \mathbb{R}$ over X and satisfy the hypothesis of the Hahn Interpolation Theorem. To show that φ is upper semicontinuous at an arbitrary point $a_0 \in A$, use the fact that f is continuous on A (take $\beta > f(a_0)$ and set $\varepsilon = \beta - f(a_0)$). To show that φ is upper semicontinuous at an arbitrary point $x_0 \in X \backslash A$, use the fact that $X \backslash A$ is open in X and so it includes an open ball $B_\delta(x_0)$ centered at x_0 for some $\delta > 0$ ($d(x, x_0) < \delta$ implies $\varphi(x) < \beta$ for every $\beta > \varphi(x_0)$).

Problem 3.37. We can define neighborhoods in a general topological space as we did in a metric space. Precisely, a *neighborhood* of a point x in a topological space X is any subset of X that includes an open subset which contains x.

(a) Show that a subset of a topological space X is open in X if and only if it is a neighborhood of each one of its points.

A topological space X is a *Hausdorff space* if for every pair of distinct points x and y in X there exist neighborhoods N_x and N_y of x and y, respectively, such that $N_x \cap N_y = \varnothing$. Prove the following propositions.

(b) Each singleton in a Hausdorff space is closed (i.e., $X \backslash \{x\}$ is open in X for every $x \in X$).

(c) Every metric space is a Hausdorff space (regarding the metric topology).

(d) For every pair of distinct points x and y in a metric space X there exist nonempty open balls $B_\varepsilon(x)$ and $B_\rho(y)$ centered at x and y, respectively, such that $B_\varepsilon(x) \cap B_\rho(y) = \varnothing$.

Problem 3.38. Let $(X, \mathcal{T}_X)$ be a topological space and let A be a subset of X. A set $U' \subseteq A$ is said to be *open relative* to A if $U' = A \cap U$ for some $U \in \mathcal{T}_X$. Prove the assertions (a) through (g).

(a) The collection $\mathcal{T}_A$ of all relatively open subsets of A is a topology on A.

$\mathcal{T}_A$ is called the *relative topology* on A. When a subset A of X is equipped with this relative topology it is called a *subspace* of X; that is, $(A, \mathcal{T}_A)$ is a subspace of $(X, \mathcal{T}_X)$. If a subspace A of X is an open subset of X, then it is called an *open subspace* of X. Similarly, if it is a closed subset of X, then it is called an *closed subspace* of X. If $(Y, \mathcal{T}_Y)$ is a topological space, and if $F: X \to Y$ is a mapping of a set X into Y, then the collection $\mathcal{F}^{-1}(\mathcal{T}_Y) = \{F^{-1}(U)\colon U \in \mathcal{T}_Y\}$ of all inverse images $F^{-1}(U)$ of open sets U in Y forms a topology on X. This is the topology *inversely induced* on X by F, which is the weakest topology on X that makes F continuous.

(b) The relative topology on A is the topology inversely induced on A by the inclusion map of A into X. (Recall that the inclusion map of A into X is the function $J: A \to X$ defined by $J(x) = x$ for every $x \in A$.)

Let (X, d) be a metric space and let $\mathcal{T}_X$ be the metric topology on X. Suppose A is a subset of X and consider the (metric) subspace (A, d) of the metric space (X, d). Let ${\mathcal{T}_A}'$ be the metric topology induced on A by the relative metric (i.e., let ${\mathcal{T}_A}'$ be the collection of all open sets in the metric space (A, d)).

(c) $U' \subseteq A$ is open in (A, d) if and only if $U' = A \cap U$ for some $U \subseteq X$ open in (X, d).

Thus the metric topology ${\mathcal{T}_A}'$ induced on A by the relative metric coincides with the relative topology $\mathcal{T}_A$ induced on A by the metric topology $\mathcal{T}_X$ on X; that is, ${\mathcal{T}_A}' = \mathcal{T}_A$, and hence the notion of *subspace* is unambiguously defined in a metric space. Let A be a subspace of a metric space X.

(d) $V' \subseteq A$ is closed in A (or *closed relative* to A) if and only if $V' = A \cap V$ for some closed subset V of X.

Hint: $A \backslash (A \cap B) = A \cap (X \backslash B)$ for arbitrary subsets A and B of X.

(e) Open subsets of an open subspace of X are open sets in X. Dually, closed subsets of a closed subspace of X are closed sets in X.

Let A be a subset of B $(A \subseteq B \subseteq X)$ and let A^- and B^- be the closures of A and B in X, respectively.

(f) $B \cap A^-$ coincides with the closure of A in the subspace B.

(g) A is dense in the subspace B if and only if $A^- = B^-$.

Problem 3.39. A point x in a metric space X is a *condensation point* of a subset A of X if the intersection of A with every nonempty open ball centered at x is uncountable. Let A' denote the set of all condensation points of an

arbitrary set A in X. Show that A' is closed in X and $A' \subseteq A^\star$, where $A^\star$ is the derived set of A.

Problem 3.40. Take an arbitrary real number $p \geq 1$ and consider the metric space $(R^p[0,1], d_p)$ of Example 3.E. Let $C[0,1]$ be the set of all scalar-valued continuous functions on the interval $[0,1]$ as in Example 3.D. Recall that the inclusion $C[0,1] \subset R^p[0,1]$ is interpreted in the following sense: if $x \in C[0,1]$, then $[x] \in R^p[0,1]$. Therefore, we are identifying $C[0,1]$ with the collection of all equivalence classes $[x] = \{x' \in r^p[0,1]\colon\ \delta_p(x', x) = 0\}$ that contain a continuous function $x \in C[0,1]$ (see Example 3.E). Use the Closed Set Theorem (Theorem 3.30) to show that $C[0,1]$ is neither closed nor open in $(R^p[0,1], d_p)$.

Hint: As usual, write x for $[x]$. Consider the $C[0,1]$-valued sequence $\{x_n\}_{n=1}^\infty$ and the $R^p[0,1]\backslash C[0,1]$-valued sequence $\{y_n\}_{n=1}^\infty$ defined by

$$x_n(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}], \\ n+1-2nt, & t \in [\frac{1}{2}, \frac{n+1}{2n}], \\ 0, & t \in [\frac{n+1}{2n}, 1], \end{cases} \qquad \text{and} \qquad y_n(t) = \begin{cases} 1, & t \in [0, \frac{1}{n}), \\ 0, & t \in [\frac{1}{n}, 1]. \end{cases}$$

Problem 3.41. Let A be a subset of a metric space X. The *boundary* of A is the set $\partial A = A^- \backslash A^\circ$. A point $x \in X$ is a *boundary point* of A if it belongs to ∂A. Prove the following propositions.

(a) $\partial A = A^- \cap (X\backslash A)^- = X\backslash(A^\circ \cup (X\backslash A)^\circ) = \partial(X\backslash A)$.

(b) $A^- = A^\circ \cup \partial A$, so that $A = A^-$ if and only if $\partial A \subseteq A$.

(c) ∂A is a closed subset of X (i.e., $(\partial A)^- = \partial A$).

(d) $A^\circ \cap \partial A = \varnothing$, so that $A = A^\circ$ if and only if $A \cap \partial A = \varnothing$.

(e) The collection $\{A^\circ, \partial A, (X\backslash A)^\circ\}$ is a partition of X.

(f) $\partial A \cap \partial B = \varnothing$ implies $(A \cup B)^\circ = A^\circ \cup B^\circ$ (for $B \subseteq X$).

Problem 3.42. Let (X, d) be a metric space.

(a) Show that a closed ball is a closed set.

Let $B_\rho(x)$ and $B_\rho[x]$ be arbitrary nonempty open and closed balls, respectively, both centered at the same point $x \in X$ and with the same radius $\rho > 0$. Prove the following propositions.

(b) $B_\rho(x) \subseteq B_\rho[x]^\circ$ and $\partial B_\rho[x] \subseteq \{y \in X\colon\ d(y,x) = \rho\}$.

(c) $B_\rho(x)^- \subseteq B_\rho[x]$ and $\partial B_\rho(x) \subseteq \{y \in X\colon\ d(y,x) = \rho\}$.

(d) Show that the above inclusions may be proper. Also show that, in general, $\partial B_\rho(x)$ and $\partial B_\rho[x]$ are not related by inclusion.

Hint: $X = [0,1] \cup [2,3]$, $x = 1$, and $\rho = 1$.

Problem 3.43. Let A be an arbitrary set in a metric space (X, d). Show that

(a) $\max\{\text{diam}(A^\circ), \text{diam}(\partial A)\} \leq \text{diam}(A) = \text{diam}(A^-)$,

(b) $d(x, A) = d(x, A^-)$ and $d(x, A) = 0$ if and only if $x \in A^-$.

Problem 3.44. For an arbitrary $p \geq 1$ let ℓ_+^p be the set of all scalar-valued p-summable sequences as in Example 3.B. Let ℓ_+^e be the set of all scalar-valued sequences $\{\xi_k\}_{k\in\mathbb{N}}$ for which there exist $\rho > 1$ and $\alpha \in (0, 1)$ such that $|\xi_k| \leq \rho\alpha^k$ for every $k \in \mathbb{N}$; and let ℓ_+^0 be the set of all scalar-valued sequences with a finite number of nonzero entries.

(a) Prove: If p and q are real numbers such that $1 \leq p < q$, then

$$\ell_+^0 \subset \ell_+^e \subset \ell_+^p \subset \ell_+^q,$$

where the above inclusions are all proper. (*Hint*: Problem 3.5.)

(b) Moreover, show that the sets ℓ_+^0, ℓ_+^e, and ℓ_+^p are all dense in the metric space (ℓ_+^q, d_q) of Example 3.B. (*Hint*: Example 3.P.)

Problem 3.45. Let A and B be subsets of a metric space X. Recall that $(A \cap B)^- \subseteq A^- \cap B^-$ and this inclusion may be proper. For instance, the sets in Example 3.M are disjoint while their closures are not. We shall now exhibit a pair of sets A and B with the following property.

$$A \cap B \neq \varnothing \quad \text{and} \quad (A \cap B)^- \neq A^- \cap B^-.$$

Consider the metric space (ℓ_+^2, d_2) as in Example 3.B. Set

$$A = \ell_+^1 \quad \text{and} \quad B = \big\{x_\beta \in \ell_+^2\colon\ x_\beta = \beta\{\tfrac{1}{k}\}_{k\geq 1} \text{ for some } \beta \in \mathbb{C}\big\}.$$

Recall that $\ell_+^1 \subset \ell_+^2$ (Problem 3.5) and $(\ell_+^1)^- = \ell_+^2$ in (ℓ_+^2, d_2) (i.e., the set ℓ_+^1 is dense in the metric space (ℓ_+^2, d_2) — Problem 3.44(b)). Show that $B = B^-$; that is, B is closed in (ℓ_+^2, d_2). (*Hint*: Theorem 3.30.) Thus conclude that $A \cap B = \{0\}$, $(A \cap B)^- = \{0\}$, and $A^- \cap B^- = B$.

Problem 3.46. Suppose $F\colon X \to Y$ is a continuous mapping of a metric space X into a metric space Y. Show that

(a) $$F(A^-) \subseteq F(A)^-$$

for every $A \subseteq X$. (*Hint*: Problem 1.2 and Theorem 3.23.) Now use the above result to conclude that, if $A \subseteq X$ and $C \subseteq Y$, then

(b) $$F(A) \subseteq C \quad \text{implies} \quad F(A^-) \subseteq C^-.$$

Finally, prove that

(c) $$A^- = B^- \quad \text{implies} \quad F(A)^- = F(B)^-$$

whenever $A \subseteq B \subseteq X$. (*Hint*: Proposition 3.32 and Corollary 3.8.) Thus, if A is dense in X and if F is continuous and has a dense range (in particular, if F is surjective), then $F(A)$ is dense in Y.

Problem 3.47. Consider the metric space (ℓ_+^p, d_p) for any $p \geq 1$. Take any scalar-valued bounded sequence $a = \{\alpha_n\}_{n\geq 1}$ from ℓ_+^∞, and consider the diagonal mapping $D_a\colon (\ell_+^p, d_p) \to (\ell_+^p, d_p)$ defined in Problem 3.22. Suppose the bounded sequence $\{\alpha_n\}_{n\geq 1}$ is such that $\alpha_n \neq 0$ for every $n \geq 1$. Let ℓ_+^0 be the set of all scalar-valued sequences with a finite number of nonzero entries.

(a) Show that $\mathcal{R}(D_a)^- = \ell_+^p$ (i.e., the range of D_a is dense in (ℓ_+^p, d_p)).

Hint: Verify that $\ell_+^0 \subseteq \mathcal{R}(D_a) \subseteq \ell_+^p$ (see Problem 3.44).

(b) Show that $D_a(\ell_+^0)^- = \ell_+^p$.

Hint: Problems 3.22(a), 3.44(b), and 3.46(c).

Problem 3.48. Prove the following results.

(a) If X is a separable metric space and $F\colon X \to Y$ is a continuous and surjective mapping of X onto a metric space Y, then Y is separable (i.e., a continuous mapping preserves separability).

Hint: Recall that, if there exists a surjective function of a set A onto a set B, then $\#B \leq \#A$. Use Problem 3.46.

(b) Separability is a topological invariant.

Problem 3.49. Verify the following propositions.

(a) A metric space X is separable if and only if there exists a countable subset A of X such that every nonempty open ball centered at each $x \in X$ meets A (i.e., if and only if there is a countable subset A of X such that every $x \in X$ is a point of adherence of A). (*Hint*: Propositions 3.27 and 3.32.)

(b) The product space $(\prod_{i=1}^n X_i, d)$ of a finite collection $\{(X_i, d_i)\}_{i=1}^n$ of separable metric space is a separable metric space. (Note: d is any of the metrics on $\prod_{i=1}^n X_i$ that were defined in Problem 3.9.)

(c) If $\ell_+^{c_o}$ is the set of all scalar-valued sequences that converge to zero, then

$$(\ell_+^{c_o}, d_\infty) \text{ is a separable metric space.}$$

Hint: According to Proposition 3.39(b), $(\ell_+^{c_o}, d_\infty)$ is a subspace of the nonseparable metric space $(\ell_+^\infty, d_\infty)$ of Example 3.Q. Show that the set of all rational-valued sequences with a finite number of nonzero entries is countable and dense in $(\ell_+^{c_o}, d_\infty)$ (but not dense in $(\ell_+^\infty, d_\infty)$). See Example

3.P. (Note: We say that a complex number is "rational" if its real and imaginary parts are rational numbers.)

(d) The metric space $(\ell_+^{c_o}, d_\infty)$ is not homeomorphic to $(\ell_+^{\infty}, d_\infty)$.

Problem 3.50. A subset A of a topological space X that is both closed and open is called *clopen* (or *closed-open*). A partition $\{A, B\}$ of X into the union of two nonempty disjoint clopen sets A and B is a *disconnection* of X (i.e., $\{A, B\}$ is a disconnection of $X = A \cup B$ if $A \cap B = \varnothing$ and A and B are both clopen subsets of X). If there exists a disconnection of X, then X is called *disconnected*. Otherwise X is called *connected*. In other words, *a topological space is connected if and only if the only clopen subsets of X are the whole space X and the empty set $\varnothing$*. A subset A of X is a *connected set* if, as a subspace of X, A is a connected topological space. A topological space is *totally disconnected* if there is no connected subset of it containing two distinct points. Prove the following propositions.

(a) X is disconnected if and only if it is the union of two disjoint nonempty open sets.

(b) X is disconnected if and only if it is the union of two disjoint nonempty closed sets.

(c) If A is a connected set in a topological space X, and if $A \subseteq B \subseteq A^-$, then B is connected. In particular, A^- is connected whenever A is connected.

Note: The closure A^- of a set A is defined in a topological space X in the same way we have defined it in a metric space: the smallest closed subset of X including A.

(d) The continuous image of a connected set is a connected set.

(e) Connectedness is a topological invariant.

Recall that a subset A of a topological space X is discrete if it consists entirely of isolated points (i.e., if every point in A does not belong to $(A\backslash\{x\})^-$, which means that in the subspace A every set is open in A). Suppose A is a discrete subset of X. If B is an arbitrary subset of A that contains more than one point, and if $x \in B$, then $B\backslash\{x\}$ and $\{x\}$ are both nonempty and open in A. Thus B is disconnected, and hence A is totally disconnected. Conclusion: *A discrete subset of a topological space is totally disconnected*.

(f) Show that the converse of the above italicized proposition fails.

Hint: Verify that $\mathbb{Q}$ is a totally disconnected subset of $\mathbb{R}$ that is dense in itself (i.e., it has no isolated point).

Problem 3.51. Prove the following proposition.

(a) $\{x_n\}$ is a Cauchy sequence in a metric space (X, d) if and only if

$$\lim_{n} \sup_{k \geq 1} d(x_{n+k}, x_n) \;=\; 0$$

(i.e., if and only if $\{d(x_{n+k}, x_n)\}$ converges to zero uniformly in k).

(b) Show that the real-valued sequence $\{x_n\}$ such that $x_n = \log n$ for each $n \geq 1$ has the following property.

$$\lim_{n} d(x_{n+k}, x_n) \;=\; 0$$

for every integer $k \geq 1$, where d is the usual metric on $\mathbb{R}$. However, $\{x_n\}$ is not a Cauchy sequence (it is not even bounded).

Hint: $\log\colon (0, \infty) \to \mathbb{R}$ is continuous. Use Corollary 3.8.

Problem 3.52. Let (X, d) be a metric space and let $\{x_n\}$ be an X-valued sequence. We say that $\{x_n\}$ is of *bounded variation* if

$$\sum_{n=1}^{\infty} d(x_{n+1}, x_n) \;<\; \infty.$$

If there exist real constants $\rho \geq 1$ and $\alpha \in (0, 1)$ such that

$$d(x_{n+1}, x_n) \;\leq\; \rho\, \alpha^n$$

for every n, then we say that $\{x_n\}$ has *exponentially decreasing increments*. Prove the following propositions.

(a) If a sequence in a metric space has exponentially decreasing increments, then it is of bounded variation.

(b) If a sequence in a metric space is of bounded variation, then it is a Cauchy sequence.

Thus, *if (X, d) is a complete metric space, then every sequence of bounded variation converges in (X, d), which implies that every sequence with exponentially decreasing increments converges in (X, d).*

(c) Every Cauchy sequence in a metric space has a subsequence with exponentially decreasing increments (and therefore every Cauchy sequence in a metric space has a subsequence of bounded variation).

Now prove the converse of the above italicized statement:

(d) If every sequence with exponentially decreasing increments in a metric space (X, d) converges in (X, d), then (X, d) is complete.

Hints: (a) Verify that $\sum_{n=0}^{m} \alpha^n \to \frac{1}{1-\alpha}$ as $m \to \infty$ for any $\alpha \in (0, 1)$. (b) Use the triangle inequality and Problems 3.10 and 3.11. (c) Show that for each

integer $k \geq 1$ there exists an integer $n_k \geq 1$ such that $d(x_n, x_{n_k}) \leq (\frac{1}{2})^k$ for every $n \geq n_k$ whenever $\{x_n\}$ is a Cauchy sequence. (d) Proposition 3.39(c).

Problem 3.53. Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a metric space (X, d) and consider the real sequence $\{d(x_n, y_n)\}$. Show that

(a) $\{d(x_n, y_n)\}$ converges in $\mathbb{R}$.

Hint: Use Problems 3.1(b) and 3.10(c) to show that $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

Moreover, if $\{x'_n\}$ and $\{y'_n\}$ are Cauchy sequences in a metric space (X, d) such that $\lim d(x_n, x'_n) = 0$ and $\lim d(y_n, y'_n) = 0$ (i.e., if $\{x'_n\}$ and $\{y'_n\}$ are Cauchy sequences in (X, d) equiconvergent with $\{x_n\}$ and $\{y_n\}$, respectively — see Problem 3.15), then

(b) $\lim d(x_n, y_n) = \lim d(x'_n, y'_n)$.

Hint: Set $\alpha_n = d(x_n, y_n)$, $\alpha'_n = d(x'_n, y'_n)$, $\alpha = \lim \alpha_n$, and $\alpha' = \lim \alpha'_n$. Use Problems 3.1(b) and 3.10(c) to show that $|\alpha_n - \alpha'_n| \to 0$. Now note that $0 \leq |\alpha - \alpha'| \leq |\alpha - \alpha_n| + |\alpha_n - \alpha'_n| + |\alpha'_n - \alpha'|$ for each n.

Problem 3.54. Suppose $\{x_n\}$ and $\{x'_n\}$ are two equiconvergent sequences in a metric space X — see Problem 3.15.

(a) Show that if one of them is a Cauchy sequence, then so is the other.

(b) A metric space X is complete whenever there exists a dense subset A of X such that every Cauchy sequence in A converges in X. Prove.

Problem 3.55. Let X be an arbitrary set. A function $d: X \times X \to \mathbb{R}$ is an *ultrametric* on X if it satisfies conditions (i), (ii), and (iii) in Definition 3.1 and also the *ultrametric inequality*, viz.,

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

for every x, y, and z in X. Clearly, the ultrametric inequality implies the triangle inequality, so that an ultrametric is a metric (example: the discrete metric is an ultrametric). Let d be an ultrametric on X and let x, y, and z be arbitrary points in X. Prove the following propositions.

(a) If $d(x, z) \neq d(z, y)$, then $d(x, y) = \max\{d(x, z), d(z, y)\}$.

(b) Every point in a nonempty open ball is a center of that ball. That is, if $\varepsilon > 0$ and $z \in B_\varepsilon(y)$, then $B_\varepsilon(y) = B_\varepsilon(z)$.

Hint: Suppose $z \in B_\varepsilon(y)$ and take any $x \in B_\varepsilon(y)$. If $d(x, z) = d(z, y)$, then $x \in B_\varepsilon(z)$. Use item (a) to show that if $d(x, z) \neq d(z, y)$, then $x \in B_\varepsilon(z)$. Thus conclude that $B_\varepsilon(y) \subseteq B_\varepsilon(z)$ whenever $z \in B_\varepsilon(y)$.

(c) If two nonempty open balls meet, then one of then is included in the other. In particular, if two nonempty open balls of the same radius meet, then they coincide with each other.

(d) Any nonempty open ball is a closed set. (*Hint*: Theorem 3.30.) Thus infer that the metric space (X, d) is totally disconnected. (*Hint*: Proposition 3.10 and Problem 3.50.)

(e) A sequence $\{x_n\}$ in (X, d) is a Cauchy sequence if and only if

$$\lim_n d(x_{n+1}, x_n) \,=\, 0.$$

Then conclude that $\{d(x_{n+k}, x_n)\}$ converges to zero uniformly in k if and only if it converges to zero for some integer $k \geq 1$; that is, verify that $\lim_n \sup_{k\geq 1} d(x_{n+k}, x_n) = 0$ if and only if $\lim_n d(x_{n+k}, x_n) = 0$ for some $k \geq 1$. (Compare with Problem 3.51.)

Problem 3.56. Let S be a nonempty set. Consider the collection $S^{\mathbb{N}}$ of all S-valued sequences. For any distinct sequences $x = \{x_k\}$ and $y = \{y_k\}$ in $S^{\mathbb{N}}$, set $k(x, y) = \min\{k \in \mathbb{N}\colon\ x_k \neq y_k\}$. Define a function $d\colon S^{\mathbb{N}} \times S^{\mathbb{N}} \to \mathbb{R}$ as follows.

$$d(x, y) \,=\, \begin{cases} 0, & x = y, \\ \frac{1}{k(x,y)}, & x \neq y. \end{cases}$$

(a) Show that d is an ultrametric on $S^{\mathbb{N}}$ and $(S^{\mathbb{N}}, d)$ is a complete metric space.

This metric d is called the *Baire metric* on $S^{\mathbb{N}}$. Set $S = \mathbb{F}$ (where $\mathbb{F}$ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$). Let ℓ_+^∞ be the set of all bounded scalar-valued sequences (i.e., $x = \{\xi_k\} \in \ell_+^\infty$ if and only if $\sup_k |\xi_k| < \infty$). Let d be the Baire metric on $\mathbb{F}^{\mathbb{N}}$ and consider the subspace (ℓ_+^∞, d) of the complete metric space $(\mathbb{F}^{\mathbb{N}}, d)$. Take the following ℓ_+^∞-valued sequence $\{x_n\}$: for each $n \in \mathbb{N}$, $x_n = \{\xi_{n(k)}\}_{k\in\mathbb{N}}$, where

$$\xi_{n(k)} \,=\, \begin{cases} n, & k = n, \\ 0, & k \neq n. \end{cases}$$

Let d_∞ be the sup-metric on ℓ_+^∞ and recall that $(\ell_+^\infty, d_\infty)$ is a complete metric space (Example 3.S(c)).

(b) Show that $\{x_n\}$ converges to 0 (the null sequence) in (ℓ_+^∞, d) but is unbounded in $(\ell_+^\infty, d_\infty)$.

(c) Show that the metric space (ℓ_+^∞, d) is not complete.

Hint: Consider the following ℓ_+^∞-valued sequence $\{y_n\}$: for each $n \in \mathbb{N}$, $y_n = \{\upsilon_{n(k)}\}_{k\in\mathbb{N}}$, where

$$\upsilon_{n(k)} = \begin{cases} k, & k \le n, \\ 0, & k > n. \end{cases}$$

Verify that $\{y_n\}$ converges in $(\mathbb{F}^{\mathbb{N}}, d)$ to $y = \{k\} \in \mathbb{F}^{\mathbb{N}} \backslash \ell_+^\infty$. Use item (a) and Theorems 3.30 and 3.40(a).

Problem 3.57. Recall that (ℓ_+^p, d_p) is a complete metric space for every $p \ge 1$ (Example 3.R(b)) and $\ell_+^0 \subset \ell_+^p \subset \ell_+^q$ whenever $1 \le p < q$ (Problem 3.44(b)). Consider the subspace (ℓ_+^p, d_q) of (ℓ_+^q, d_q) and show that

$$(\ell_+^p, d_q) \text{ is not a complete metric space.}$$

Now consider the subspace (ℓ_+^0, d_p) of (ℓ_+^p, d_p) and show that

$$(\ell_+^0, d_p) \text{ is not a complete metric space.}$$

Hint: Problem 3.44(b) and Theorem 3.40(a).

Problem 3.58. Take an arbitrary real number $p \ge 1$ and consider the metric space $(C[0,1], d_p)$ of Example 3.D. Prove that

$$(C[0,1], d_p) \text{ is not a complete metric space.}$$

Hint: Consider the $C[0,1]$-valued sequence $\{x_n\}$ defined in Problem 3.40. First take an arbitrary pair of integers m and n such that $1 \le m < n$ and show that $d_p(x_m, x_n)^p < \frac{1}{2m}$. Then infer that $\{x_n\}$ is a Cauchy sequence in $(C[0,1], d_p)$. Suppose there is a function x in $C[0,1]$ such that $d_p(x_n, x) \to 0$. Show that

$$\text{(i)} \quad \int_0^{\frac{1}{2}} |1 - x(t)|^p \, dt = 0 \qquad \text{and} \qquad \text{(ii)} \quad \int_{\frac{n+1}{2n}}^1 |x(t)|^p \, dt \to 0 \quad \text{as} \quad n \to \infty.$$

From (i) conclude that $x(t) = 1$ for all $t \in [0, \frac{1}{2}]$; in particular, $x(\frac{1}{2}) = 1$. From (ii) conclude that $x(t) = 0$ for all $t \in [\frac{n+1}{2n}, 1]$ and every $n \ge 1$; in particular, $x(\frac{n+1}{2n}) = 0$ for every $n \ge 1$ so that $x(\frac{1}{2}) = x(\lim \frac{n+1}{2n}) = \lim x(\frac{n+1}{2n}) = 0$ by Corollary 3.8. This leads to a contradiction (viz., $0 = 1$), and hence there is no function x in $C[0,1]$ such that $d_p(x_n, x) \to 0$. Thus the $C[0,1]$-valued Cauchy sequence $\{x_n\}$ does not converge in $(C[0,1], d_p)$.

Problem 3.59. Recall that $(\ell_+^\infty, d_\infty)$ is a complete metric space (Example 3.S). Let ℓ_+^c denote the set of all scalar-valued convergent sequences (i.e., $x = \{\xi_i\} \in \ell_+^c$ if and only if $|\xi_i - \xi| \to 0$ for some scalar ξ) and let $\ell_+^{c_o}$ denote the subset of ℓ_+^c consisting of all sequences that converge to zero. Since every convergent sequence is bounded (Proposition 3.39), it follows that

$$\ell_+^0 \subset \ell_+^p \subset \ell_+^{c_o} \subset \ell_+^c \subset \ell_+^\infty,$$

with the sets ℓ_+^p $(p \ge 1)$ and ℓ_+^0 defined as before (Problems 3.44 and 3.57). Use the Closed Set Theorem (Theorem 3.30) to verify the following propositions.

(a) $(\ell_+^{c_o}, d_\infty)$ and (ℓ_+^{c}, d_∞) are complete metric spaces.

(b) (ℓ_+^{0}, d_∞) and (ℓ_+^{p}, d_∞) are not complete metric spaces.

Hint: (a) To show that ℓ_+^c is closed in $(\ell_+^\infty, d_\infty)$, proceed as follows. Take any $\varepsilon > 0$. Let $\{x_n\}_{n\geq 1}$ be an ℓ_+^c-valued sequence so that each $x_n = \{\xi_{n(k)}\}_{k\geq 1}$ converges in $\mathbb{R}$. Thus, for each $n \geq 1$, there is an integer $k_{\varepsilon,n} \geq 1$ such that

$$|\xi_{n(k)} - \xi_{n(j)}| < \varepsilon$$

whenever $j, k \geq k_{\varepsilon,n}$. Now suppose $\{x_n\}_{n\geq 1}$ converges in $(\ell_+^\infty, d_\infty)$ to $x = \{\xi_{(k)}\}_{k\geq 1}$ in ℓ_+^∞ so that $\sup_k |\xi_{n(k)} - \xi_{(k)}| \to 0$ as $n \to \infty$. Thus there exists an integer $n_\varepsilon \geq 1$ such that

$$|\xi_{n(k)} - \xi_{(k)}| < \varepsilon$$

for every $k \geq 1$ whenever $n \geq n_\varepsilon$. Therefore,

$$|\xi_{(k)} - \xi_{(j)}| \leq |\xi_{(k)} - \xi_{n_\varepsilon(k)}| + |\xi_{n_\varepsilon(k)} - \xi_{n_\varepsilon(j)}| + |\xi_{n_\varepsilon(j)} - \xi_{(j)}| < 3\varepsilon$$

whenever $j, k \geq k_{\varepsilon,n_\varepsilon}$. Next conclude that x lies in ℓ_+^c. (b) To show that both sets ℓ_+^0 and ℓ_+^p are not closed in the metric space $(\ell_+^\infty, d_\infty)$, take the sequence $\{x_n\}_{n\geq 1}$ with $x_n = \{1, (\frac{1}{2})^{\frac{1}{p}}, \ldots, (\frac{1}{n})^{\frac{1}{p}}, 0, 0, 0, \ldots\} \in \ell_+^0$ for each $n \geq 1$, which converges in $(\ell_+^\infty, d_\infty)$ to $x = \{(\frac{1}{k})^{\frac{1}{p}}\}_{k\geq 1} \in \ell_+^{c_o} \backslash \ell_+^p$.

Remark: Note that ℓ_+^p is not dense in $(\ell_+^\infty, d_\infty)$. Indeed, if $y = \{\upsilon_k\}_{k\geq 1}$ is the constant sequence in ℓ_+^∞ with $\upsilon_k = 1$ for all $k \geq 1$, then $d_\infty(x, y) \geq 1$ for every x in $\ell_+^p \subset \ell_+^{c_o}$. Hence ℓ_+^0 is not dense in $(\ell_+^\infty, d_\infty)$ either.

Problem 3.60. Let $\{(X_i, d_i)\}_{i=1}^n$ be a finite collection of metric spaces and let $Z = \prod_{i=1}^n X_i$ be the Cartesian product of their underlying sets. Let d denote any of the metrics d_p (for an arbitrary $p \geq 1$) or d_∞ on $Z = \prod_{i=1}^n X_i$ as in Problem 3.9. Show that the product space $(\prod_{i=1}^n X_i, d)$ is complete if and only if (X_i, d_i) is a complete metric space for every $i = 1, \ldots, n$.

Hint: Consider the metric d_1 on Z (see Problem 3.9). Show that $(\prod_{i=1}^n X_i, d_1)$ is complete if and only if each (X_i, d_i) is complete. Now use Problem 3.33 and Lemma 3.43.

Problem 3.61. Take an arbitrary nondegenerate closed interval (of positive length) of the real line, say $I = [\alpha, \beta] \subset \mathbb{R}$ of length $\lambda = \beta - \alpha$. Consider a pair of closed intervals $\{[\alpha, \alpha + \frac{\lambda}{3}], [\beta - \frac{\lambda}{3}, \beta]\}$ consisting of the first and third closed subintervals of $I = [\alpha, \beta]$ of length $\frac{\lambda}{3}$, which will be referred to as the closed intervals *derived from I by removal of the central open third of I*. If $\mathcal{I} = \{I_i\}_{i=1}^n$ is a finite disjoint collection of nondegenerate closed intervals in $\mathbb{R}$, then let $\mathcal{I}' = \{I_i'\}_{i=1}^{2n}$ be the corresponding finite collection obtained by replacing each closed interval I_i in $\mathcal{I}$ with the pair of closed subintervals derived from I_i by removal of the central open third of I_i. Now consider the unit interval $[0, 1]$ and set

$$\mathcal{I}_0 = \big\{[0,1]\big\}.$$

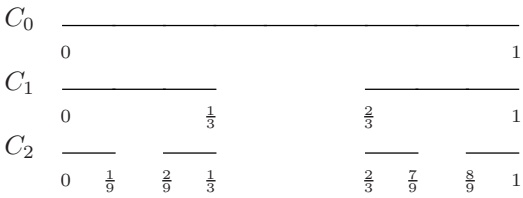
The derived intervals from $[0,1]$ by removal of the central open third are $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Set

$$\mathcal{I}_1 = \mathcal{I}_0' = \big\{[0, \tfrac{1}{3}], [\tfrac{2}{3}, 1]\big\}.$$

Similarly, replacing each closed interval in $\mathcal{I}_1$ by the pair of closed subintervals derived from it by removal of its central open third, set

$$\mathcal{I}_2 = \mathcal{I}_1' = \big\{[0, \tfrac{1}{9}], [\tfrac{2}{9}, \tfrac{1}{3}], [\tfrac{2}{3}, \tfrac{7}{9}], [\tfrac{8}{9}, 1]\big\}.$$

Take an arbitrary positive integer n. Suppose the collection of intervals $\mathcal{I}_k$ has already been defined for each $k = 0, 1, \ldots, n$, and set $\mathcal{I}_{n+1} = \mathcal{I}_n'$. This leads to an inductive construction of a disjoint collection $\mathcal{I}_n$ of 2^n nondegenerate closed subintervals of the unit interval $[0,1]$, each having length $\frac{1}{3^n}$, for every $n \in \mathbb{N}_0$. Set $C_n = \bigcup \mathcal{I}_n$ for each $n \in \mathbb{N}_0$ so that $\{C_n\}_{n\in\mathbb{N}_0}$ is a decreasing sequence of subsets of the unit interval (i.e., $C_{n+1} \subset C_n \subset C_0 = [0,1]$) such that each set C_n is the union of 2^n disjoint nondegenerate subintervals of length $\frac{1}{3^n}$.



The *Cantor set* is the set

$$C = \bigcap_{n\in\mathbb{N}_0} C_n.$$

Equip the real line $\mathbb{R}$ with its usual metric and prove the following assertions.

(a) The Cantor set is a *nonempty*, *closed*, and *bounded* subset of $\mathbb{R}$.

(b) The Cantor set has an *empty interior* and hence it is *nowhere dense*.

Hint: Recall that each set C_n consists of 2^n intervals of length $\frac{1}{3^n}$. Take an arbitrary point $\gamma \in C$ and an arbitrary $\varepsilon > 0$. Verify that there exists a positive integer n_ε such that the open ball $B_\varepsilon(\gamma)$ is not included in C_{n_ε}. Now conclude that the nonempty open ball $B_\varepsilon(\gamma)$ is not included in C, which means that γ is not an interior point of C.

(c) The Cantor set *has no isolated point* and hence it is a *perfect* subset of $\mathbb{R}$.

Hint: Consider the hint to item (b). Verify that there exists a positive integer n_ε such that the open ball $B_\varepsilon(\gamma)$ includes an end point of some of the closed intervals of C_{n_ε}.

(d) The Cantor set is *uncountable*.

Hint: See Example 3.W(b), and recall that $\mathbb{R}$ is a complete metric space.

(e) The Cantor set is *totally disconnected*.

Hint: If α and γ are two points of an arbitrary subset A of C such that $\alpha < \gamma$, and if n is a positive integer such that $\frac{1}{3^n} < \gamma - \alpha$, then both α and γ do not belong to any interval of length $\frac{1}{3^n}$. Thus α and γ must belong to different intervals in C_n, so that there exists a real number β such that $\alpha < \beta < \gamma$ and $\beta \notin C_n$. Verify that $\{A \cap (-\infty, \beta), A \cap (\beta, \infty)\}$ is a disconnection of the set A. See Problem 3.50.

Remark: Let $\mu(C_n)$ denote the length of each set C_n, which consists of 2^n *disjoint* intervals of length $\frac{1}{3^n}$. Therefore, $\mu(C_n) = (\frac{2}{3})^n$. If we agree that the length $\mu(A)$ of a subset A of the real line can be defined somehow as to bear the property that $0 \le \mu(A) \le \mu(B)$ whenever $A \subseteq B \subseteq \mathbb{R}$ (provided the lengths $\mu(A)$ and $\mu(B)$ are "well defined"), then the Cantor set C is such that $0 \le \mu(C) = \mu(\bigcap_n C_n) \le \mu(C_m)$ for all m (note: the length $\mu(\bigcap_n C_n)$ of $\bigcap_n C_n$ is "well defined" whenever the length $\mu(C_n)$ is "well defined" for every C_n). Hence $\mu(C) = 0$. That is, the Cantor set has *length zero*.

Problem 3.62. Consider the construction of the previous problem. Note that each set C_n (for $n \ge 1$) is obtained from C_{n-1} by removing 2^{n-1} central open subintervals, each of length $\frac{1}{3^n}$. Now, instead of removing at each iteration 2^{n-1} central open subintervals of length $\frac{1}{3^n}$, remove at each iteration 2^{n-1} central open subintervals of length $\frac{1}{4^n}$. Let $\{S_n\}_{n \in \mathbb{N}_0}$ be the resulting collection of closed subsets of the unit interval $S_0 = [0, 1]$. Note that the length of S_n is

$$\mu(S_n) \,=\, 1 - \sum_{i=0}^{n-1} \frac{2^i}{4^{i+1}} \,=\, \frac{1}{2} + \frac{1}{2^{n+1}}$$

for each $n \in \mathbb{N}$. Consider the sequence $\{x_n\}$ consisting of the characteristic functions of the subsets S_n of S_0; that is, for every $n \in \mathbb{N}$,

$$x_n(t) \,=\, \begin{cases} 1, & t \in S_n, \\ 0, & t \in S_0 \backslash S_n. \end{cases}$$

Each x_n belongs to $R^p[0, 1]$ for every $p \ge 1$ (see Example 3.E), so that $\{x_n\}$ is an $R^p[0, 1]$-valued sequence. Equip $R^p[0, 1]$ with its usual metric d_p.

(a) Show that $\{x_n\}$ is a Cauchy sequence in $(R^p[0, 1], d_p)$.

(b) Show that $\{x_n\}$ does not converge in $(R^p[0, 1], d_p)$.

Hint: Suppose there exists x in $R^p[0, 1]$ such that $d_p(x_n, x) \to 0$. Consider the null function $0 \in R^p[0, 1]$ and show that $d_p(x_n, 0) > (\frac{1}{2})^{\frac{1}{p}}$ for all n. Use the triangle inequality to infer that $d_p(x, 0) \ge (\frac{1}{2})^{\frac{1}{p}}$. On the other hand, set $S = \bigcap_n S_n$ so that $S_0 \backslash S = \bigcup_n (S_0 \backslash S_n)$. Take any $m \ge 0$ and show that

$$\int_{\bigcup_{n=1}^{m} (S_0 \backslash S_n)} |x(t)|^p \, dt \,\le \int_{S_0} |x(t) - x_k(t)|^p \, dt$$

for every $k \geq m$. Thus conclude that $\int_{S_0 \setminus S} |x(t)|^p \, dt = 0$, and hence the lower Riemann integral of $|x|^p$ is zero. Since $|x|^p$ is Riemann integrable, we get $\int_{S_0} |x(t)|^p \, dt = 0$. This contradicts the fact that $\int_{S_0} |x(t)|^p \, dt \geq \frac{1}{2}$.

From (a) and (b) it follows that, for any $p \geq 1$,

$$(R^p[0,1], d_p) \text{ is not a complete metric space.}$$

Remark: The failure of $R^p[0,1]$ to be complete when equipped with its usual metric d_p is regarded as one of the defects in the definition of the Riemann integral. A more general concept of integral, viz., the *Lebesgue integral*, corrects this and other drawbacks of the Riemann integral. Let $L^p[0,1]$ be the collection of all equivalence classes (as in Example 3.E) of scalar-valued functions x on $[0,1]$ such that $\int_0^1 |x(t)|^p \, dt < \infty$ (the integral now is the Lebesgue integral). Every Riemann integrable function is Lebesgue integrable (and if a function is Riemann integrable, then its Riemann and Lebesgue integrals coincide), and so $R^p[0,1] \subset L^p[0,1]$. Moreover, it can be shown that

$$R^p[0,1] \text{ is dense in } (L^p[0,1], d_p).$$

Therefore,

$$(L^p[0,1], d_p) \text{ is a completion of } (R^p[0,1], d_p).$$

In fact, let $\mathbb{F}$ stand either for $\mathbb{R}$ or $\mathbb{C}$, as usual, and consider the sets

$$P[0,1] \subset C[0,1] \subset R^p[0,1] \subset L^p[0,1]$$

of all $\mathbb{F}$-valued polynomials with coefficients in $\mathbb{F}$, of all $\mathbb{F}$-valued continuous functions, of all $\mathbb{F}$-valued Riemann p-integrable functions, and of all $\mathbb{F}$-valued Lebesgue p-integrable functions, respectively, on [0,1], as in Examples 3.D, 3.E, and 3.P, and Problem 3.58, where the above inclusions are to be interpreted in the sense of equivalence classes as in Problem 3.40. We saw in Example 3.P that $P[0,1]$ is dense in $(C[0,1], d_\infty)$, and so it is readily verified that

$$P[0,1] \text{ is dense in } (C[0,1], d_p).$$

Since it can also be shown that

$$C[0,1] \text{ is dense in } (R^p[0,1], d_p),$$

it follows that $(L^p[0,1], d_p)$ is also a completion of $(P[0,1], d_p)$ and $(C[0,1], d_p)$.

Problem 3.63. A metric space is complete if and only if every decreasing sequence $\{V_n\}_{n=1}^{\infty}$ of nonempty closed subsets of X for which $\operatorname{diam}(V_n) \to 0$ is such that $\bigcap_{n=1}^{\infty} V_n \neq \varnothing$.

Hint: This result, likewise Lemma 3.79, is also attributed to Cantor. Its proof follows closely the proof of Lemma 3.79. Consider the same X-valued sequence $\{v_n\}_{n \in \mathbb{N}}$ that was defined in part (a) of the proof of Lemma 3.79. Show that

$\{v_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if $\operatorname{diam}(V_n) \to 0$. Suppose X is complete, set $v = \lim v_n$, and verify that $v \in V_n$ for every $n \in \mathbb{N}$ so that $\bigcap_{m\in\mathbb{N}} V_m \neq \varnothing$. Conversely, let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary X-valued Cauchy sequence and consider the decreasing sequence $\{V_m^-\}_{m\in\mathbb{N}}$ of nonempty closed subsets of X that was defined in part (b) of the proof of Lemma 3.79. Show that $\operatorname{diam}(V_n^-) \to 0$. If $\bigcap_{m\in\mathbb{N}} V_m^- \neq \varnothing$, then there exists $v \in V_n^-$ for every $n \in \mathbb{N}$. Verify that $x_n \to v$ and conclude that X is complete.

Problem 3.64. Let $\{(X_i, d_i)\}_{i=1}^n$ be a finite collection of metric spaces and let $Z = \prod_{i=1}^n X_i$ be the Cartesian product of their underlying sets. Let d denote any of the metrics d_p (for any $p \geq 1$) or d_∞ on $Z = \prod_{i=1}^n X_i$ as in Problem 3.9.

(a) Show that the product space $(\prod_{i=1}^n X_i, d)$ is totally bounded if and only if (X_i, d_i) is totally bounded for every $i = 1, \ldots, n$.

Hint: Use Lemma 3.73 to show that $(\prod_{i=1}^n X_i, d_1)$ is totally bounded if and only if each (X_i, d_i) is totally bounded. Also apply Problem 3.33 and Corollary 3.81.

(b) Show that $(\prod_{i=1}^n X_i, d)$ is compact if and only if (X_i, d_i) is compact for every $i = 1, \ldots, n$.

Hint: Item (a), Problem 3.60, and Corollary 3.81.

Remark: Let $\{X_\gamma\}_{\gamma\in\Gamma}$ be an indexed family of nonempty topological spaces. Let $Z = \prod_{\gamma\in\Gamma} X_\gamma$ be the Cartesian product of the underlying sets $\{X_\gamma\}_{\gamma\in\Gamma}$. The *product topology* on Z is the topology inversely induced on Z by the family $\{\pi_\gamma\}_{\gamma\in\Gamma}$ of projections of X onto each X_γ (the weakest topology on Z that makes each projection $\pi_\gamma\colon Z \to X_\gamma$ continuous). Compactness in a topological space is defined as in Definition 3.60. The important *Tikhonov Theorem* says that $\prod_{\gamma\in\Gamma} X_\gamma$ *is compact if and only if* X_γ *is compact for every* $\gamma \in \Gamma$.

Problem 3.65. Every closed ball of positive radius in (ℓ_+^p, d_p) is not totally bounded (and hence not compact).

Hint: Consider the metric space (ℓ_+^p, d_p) of Example 3.B for any $p \geq 1$. Let $B_\rho[x_0]$ be a closed ball of radius $\rho > 0$ centered at an arbitrary $x_0 \in \ell_+^p$. Take the ℓ_+^p-valued sequence $\{e_i\}_{i\in\mathbb{N}}$ of Example 3.X and set $x_i = \rho e_i + x_0$ for each $i \geq 1$. Instead of following the approach of Example 3.X, show that $\{x_i\}$ is a $B_\rho[x_0]$-valued sequence that has no Cauchy subsequence. Apply Lemma 3.73.

Problem 3.66. Prove the following propositions.

(a) Every closed ball of positive radius in $(C[0,1], d_\infty)$ is not compact.

Hint: Consider the metric space $(C[0,1], d_\infty)$ of Example 3.D. Let $B_\rho[x_0]$ be the closed ball of radius $\rho > 0$ centered at an arbitrary $x_0 \in C[0,1]$. Consider the mapping $\varphi\colon B_\rho[x_0] \to \mathbb{R}$ defined by

$$\varphi(x) = \int_0^1 |x(t) - x_0(t)|\, dt - |x(0) - x_0(0)|$$

for every $x \in B_\rho[x_0]$. Equip $B_\rho[x_0]$ with the sup-metric d_∞ and the real line $\mathbb{R}$ with its usual metric. Show that φ is continuous, $\varphi(x) < \rho$ for all $x \in B_\rho[x_0]$, and $\sup_{x \in B_\rho[x_0]} \varphi(x) = \rho$. Now use the Weierstrass Theorem (Theorem 3.86) to verify that $B_\rho[x_0]$ is not compact.

(b) Every closed ball of positive radius in $(C[0,1], d_\infty)$ is a complete subspace.

Hint: Example 3.T, Problem 3.42(a), and Theorem 3.40(b).

(c) Every closed ball of positive radius in $(C[0,1], d_\infty)$ is not totally bounded.

Hint: Corollary 3.81 and item (a).

Problem 3.67. A topological space X is *locally compact* if every point of X has a compact neighborhood. Prove the following assertions.

(a) A metric space X is locally compact if and only if there exists a compact closed ball of positive radius centered at each point of X.

(b) $\mathbb{R}^n$ and $\mathbb{C}^n$ (equipped with any of their uniformly equivalent metrics of Example 3.A) are locally compact.

(c) (ℓ_+^p, d_p) and $(C[0,1], d_\infty)$ are not locally compact.

(d) Every open subspace and every closed subspace of a locally compact metric space is locally compact.

Problem 3.68. Consider the metric space (ℓ_+^p, d_p) for an arbitrary $p \geq 1$.

(a) Prove that a subset A of ℓ_+^p is totally bounded if and only if

$$\sup_{\{\xi_k\} \in A} \sum_{k=1}^{\infty} |\xi_k|^p < \infty \quad \text{and} \quad \lim_n \sup_{\{\xi_k\} \in A} \sum_{k=n}^{\infty} |\xi_k|^p = 0$$

(i.e., A is bounded and $\sum_{k=n}^{\infty} |\xi_k|^p \to 0$ as $n \to \infty$ uniformly on A).

(b) Show that a subset A of ℓ_+^p is compact if and only if it is closed and satifiles the above conditions.

(c) Verify that every closed ball of positive radius in ℓ_+^p is not compact.

Problem 3.69. Let $\{\xi_{k(0)}\}$ be an arbitrary point in ℓ_+^p and set

$$S_0 = \big\{\{\xi_k\} \in \ell_+^p\colon\ |\xi_k| \leq |\xi_{k(0)}| \ \text{ for every } \ k\big\}.$$

Use the preceding problem to show that S_0 is a compact subset of the metric space (ℓ_+^p, d_p). In particular, the set

$$S = \big\{\{\xi_k\}_{k\geq 1} \in \ell_+^2\colon\ |\xi_k| \leq \tfrac{1}{k} \text{ for every } k \geq 1\big\},$$

which is known as the *Hilbert cube*, is compact in (ℓ_+^2, d_2). Show that the Hilbert cube has an empty interior. (*Hint*: Verify that $(\ell_+^2 \backslash S)^- = \ell_+^2$.) Then conclude that the Hilbert cube is nowhere dense.

Problem 3.70. Suppose X is a compact metric space, let Y be any metric space, and consider the metric space $(C[X,Y], d_\infty)$ of Example 3.Y. Take an arbitrary real number $\gamma > 0$ and let $C_\gamma[X,Y]$ denote the subset of $C[X,Y]$ consisting of all Lipschitzian mappings of X into Y that have a Lipschitz constant less than or equal to γ.

(a) Show that $C_\gamma[X,Y]$ is equicontinuous and closed in $(C[X,Y], d_\infty)$.

Hint: Set $\delta = \frac{\varepsilon}{\gamma}$ for equicontinuity. Then apply the Closed Set Theorem: if $f_n \in C_\gamma[X,Y]$ and if $f_n \to f \in C[X,Y]$, then by the triangle inequality $d_Y(f(x), f(y)) \leq d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(y)) + d_Y(f_n(y), f(y)) \leq 2\, d_\infty(f_n, f) + \gamma d_X(x,y)$.

From now on suppose the space Y is compact. Thus Y is complete (Corollary 3.81), and hence $(C[X,Y], d_\infty)$ is complete (Example 3.Y).

(b) Show that $C_\gamma[X,Y]$ is pointwise totally bounded and then conclude that $C_\gamma[X,Y]$ is a compact subset of the metric space $(C[X,Y], d_\infty)$.

Particular case ($\gamma = 1$): The set $C_1[X,Y]$ of all contractions of a compact metric space X into a compact metric space Y is a compact subset of $(C[X,Y], d_\infty)$. Let $J[X,Y]$ denote the set of all isometries of a compact metric space X into a compact metric space Y so that $J[X,Y] \subset C_1[X,Y]$.

(c) Show that $J[X,Y]$ is closed in $(C[X,Y], d_\infty)$ and then conclude that $J[X,Y]$ is a compact subset of the metric space $(C[X,Y], d_\infty)$.

Hint: Again, apply the Closed Set Theorem: if $\{f_n\}$ is a $J[X,Y]$-valued sequence that converges to $f \in C[X,Y]$, then (cf. Problem 3.1(b)) verify that $|d_X(x,y) - d_Y(f(x), f(y))| = |d_Y(f_n(x), f_n(y)) - d_Y(f(x), f(y))| \leq d_Y(f_n(x), f(x)) + d_Y(f_n(y), f(y)) \leq 2\, d_\infty(f_n, f)$.

Problem 3.71. Let m, g, and T be positive constants and let $u\colon [0,T] \to \mathbb{R}$ be a continuous function (i.e., $u \in C[0,T] = C([0,T], \mathbb{R})$). Consider the ordinary differential equation

$$m\, \frac{d^2x}{dt^2}(t) = u(t) - m\, g \quad \text{for every} \quad t \in [0,T],$$

with

$$x(0) = \frac{dx}{dt}(0) = \frac{d^2x}{d\, t^2}(0) = \frac{dx}{dt}(T) = 0.$$

(a) Show that

$$\frac{d^2x}{dt^2}(0) = 0 \quad \Longleftrightarrow \quad u(0) = mg,$$

$$\frac{dx}{dt}(0) = \frac{dx}{dt}(T) \quad \Longleftrightarrow \quad \int_0^T u(t)\,dt = mgT,$$

$$x(0) = \frac{dx}{dt}(0) = 0 \quad \Longrightarrow \quad x(T) = \frac{1}{m}\int_0^T \Big(\int_0^t u(s)\,ds\Big)\,dt - \frac{gT^2}{2}.$$

Hint: Verify that the solution $x\colon [0,T] \to \mathbb{R}$ to the above differential equation (which is a twice continuously differentiable function) under the initial conditions $x(0) = \frac{dx}{dt}(0) = 0$ is given by

$$\frac{dx}{dt}(t) \;=\; \frac{1}{m}\int_0^t u(s)\,ds - gt \quad \text{for every} \quad t \in [0,T],$$

$$x(t) \;=\; \frac{1}{m}\int_0^t \Big(\int_0^\tau u(s)\,ds\Big)\,d\tau - \frac{gt^2}{2} \quad \text{for every} \quad t \in [0,T].$$

Let M and K be positive constants such that $mg < M$. Consider the following subsets of $C[0,T]$.

$$\begin{aligned}
U_1 &= \big\{u \in C[0,T]\colon\ 0 \le u(t) \text{ for every } t \in [0,T]\big\},\\
U_2 &= \big\{u \in C[0,T]\colon\ u(t) \le M \text{ for every } t \in [0,T]\big\},\\
U_3 &= \big\{u \in C[0,T]\colon\ |u(t) - u(s)| \le K|t-s| \text{ for every } t,s \in [0,T]\big\},\\
U_4 &= \big\{u \in C[0,T]\colon\ u(0) = mg\big\},\\
U_5 &= \big\{u \in C[0,T]\colon\ \textstyle\int_0^T u(t)\,dt = mgT\big\}.
\end{aligned}$$

Set $U = \bigcup_{i=1}^5 U_i \subset C[0,T]$ and consider the functional $f\colon U \to \mathbb{R}$ defined by

$$f(u) \;=\; x(T) \quad \text{for every} \quad u \in U.$$

This comprises the proper setup for a classical optimal control problem. If $u(t)$ stands for the downward thrust of a rocket of mass m burning over the time interval $[0,T]$, then the above differential equation is a model for the rocket altitude $x(t)$ — Newton's Second Law where g is the gravitational constant. The constraints on the admissible thrusts are described in the set U. Indeed, U_1 and U_2 say that the thrust is nonnegative and has a maximum value, U_3 affirms that the changes of thrust are limited, and U_4 and U_5 take into account conditions on the trust that ensure null initial acceleration in U_4 and coincident initial and final velocities in U_5 — which actually are null velocities when the final altitude $x(T)$ is given as in (a). An optimal control problem is that of selecting $u \in U$ that maximizes the final altitude $x(T)$ in (a). Many optimal control techniques start by assuming that an optimal solution $u^* \in U$

exists, and then proceed to build up algorithms to determine it. Next we shall use the Arzelà–Ascoli Theorem (Example 3.Z) to prove the existence of u^*. Consider the metric spaces $(\mathbb{R}, d)$ and $(C[0,T], d_\infty)$ where d is the usual metric on $\mathbb{R}$ (Example 3.A) and d_∞ is the sup-metric on $C[0,T]$ (Example 3.D).

(b) Show that U is nonempty and that $f\colon (U, d_\infty) \to (\mathbb{R}, d)$ is continuous.

Hint: Verify that f is Lipschitzian with Lipschitz constant $\frac{T}{m}$.

(c) Show that U is closed in $(C[0,T], d_\infty)$.

Hint: Show that each U_i is closed and use Theorem 3.22(b).

(d) Show that $U \subset C[0,T]$ is pointwise totally bounded.

Hint: Verify that the closure of $U_t = \{u(t) \in \mathbb{R}\colon\ u \in \mathcal{U}\}$ is bounded in $(\mathbb{R}, d)$ for each $t \in [0,T]$, and so compact (Theorem 3.83), and hence totally bounded in $(\mathbb{R}, d)$ (Corollary 3.81). Conclude by using Example 3.Z(i).

(e) Show that $U \subset C[0,T]$ is equicontinuous on $[0,T]$.

Hint: If $u \in \mathcal{U}$, then $u \in U_3$. Thus for any $\varepsilon > 0$ there is a $\delta = \frac{\varepsilon}{K}$ such that $|t-s| \le \delta$ implies $|u(t)-u(s)| \le K|t-s| \le \varepsilon$. Then $U \subset (C([0,T], \mathbb{R}), d_\infty)$ is (uniformly) equicontinuous on $[0,1]$ (cf. Example 3.Z(ii)).

(f) Show that U is compact in $(C[0,T], d_\infty)$.

Hint: $([0,T], d)$ is a compact metric space and $(\mathbb{R}, d)$ is a complete metric space (aren't they?). Apply (c), (d), and (e) to the Arzelà–Ascoli Theorem in Example 3.Z(d) and conclude that U is compact in $(C([0,T], \mathbb{R}), d_\infty)$.

(g) Show that there exists $u^* \in U$ such that $f(u^*) = \max_{u \in U} f(u)$.

Hint: Use (b) and (f) together with Theorem 3.86.

4

Banach Spaces

Our purpose now is to put algebra and topology to work together. For instance, from algebra we get the notion of finite sums (either ordinary or direct sums of vectors, linear manifolds, or linear transformations), and from topology the notion of convergent sequences. If algebraic and topological structures are suitably laid on the same underlying set, then we may consider the concept of infinite sums and convergent series. More importantly, as continuity plays a central role in the theory of topological spaces, and linear transformation plays a central role in the theory of linear spaces, when algebra and topology are properly combined they yield the concept of *continuous linear transformation*; the very central theme of this book.

4.1 Normed Spaces

To begin with let us point out, once and for all, that throughout this chapter $\mathbb{F}$ will denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, both equipped with their usual topologies induced by their usual metrics.

If we intend to combine algebra and topology so that a given set is endowed with both algebraic and topological structures, then we might simply equip a linear space with some metric, and hence it would become a linear space that is also a metric space. However, an arbitrary metric on a linear space may induce a topological structure that has nothing to do with the algebraic structure (i.e., these structures may live apart on the same underlying set). A richer and more useful structure is obtained when the metric recognizes the operations of vector addition and scalar multiplication that come with the linear space, and incorporates these operations in its own definition. With this in mind, let us first define a couple of concepts. A metric (or a pseudometric) d on a linear space $\mathcal{X}$ over $\mathbb{F}$ is said to be *additively invariant* if

$$d(x, y) = d(x + z, y + z)$$

for every x, y, and z in $\mathcal{X}$. This means that the translation mapping $\mathcal{X} \to \mathcal{X}$ defined by $x \mapsto x + z$ for any $z \in \mathcal{X}$ is an isometry. If d is such that

C.S. Kubrusly, *The Elements of Operator Theory*, DOI 10.1007/978-0-8176-4998-2_4,

© Springer Science+Business Media, LLC 2011

$$d(\alpha x, \alpha y) \;=\; |\alpha|\, d(x, y)$$

for every x and y in $\mathcal{X}$ and every α in $\mathbb{F}$, then the metric d is called *absolutely homogeneous*. A program for equipping a linear space with a metric that has the above "linear-like" properties goes as follows. Let $p\colon \mathcal{X} \to \mathbb{R}$ be a real-valued functional on a linear space over $\mathbb{F}$ (recall: $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$ so that $\mathbb{R}$ is always embedded in $\mathbb{F}$). It is *nonnegative homogeneous* if

$$p(\alpha x) \;=\; \alpha p(x)$$

for every $x \in \mathcal{X}$ and every *nonnegative* (real) scalar $\alpha \in \mathbb{F}$, and *subadditive* if

$$p(x + y) \;\le\; p(x) + p(y)$$

for every x and y in $\mathcal{X}$. If p is both nonnegative homogeneous and subadditive, then it is called a *sublinear* functional. If

$$p(\alpha x) \;=\; |\alpha|\, p(x)$$

for every x in $\mathcal{X}$ and α in $\mathbb{F}$, then p is *absolutely homogeneous*. A subadditive absolutely homogeneous functional is a *convex* functional. (Note that this includes the classical definition of a convex functional, namely, if $p\colon \mathcal{X} \to \mathbb{R}$ is convex, then $p(\alpha x + \beta x) \le \alpha p(x) + \beta p(x)$ for every $x, y \in \mathcal{X}$ and every $\alpha \in [0, 1]$ with $\beta = 1 - \alpha$.) If

$$p(x) \;\ge\; 0$$

for all x in $\mathcal{X}$, then p is *nonnegative*. A nonnegative convex functional is a *seminorm* (or a *pseudonorm* — i.e., a nonnegative absolutely homogeneous subadditive functional). If

$$p(x) > 0 \quad \text{whenever} \quad x \neq 0,$$

then p is called *positive*. A positive seminorm is a *norm* (i.e., a positive absolutely homogeneous subadditive functional). Summing up: A norm is a real-valued functional on a linear space with the following four properties, called the *norm axioms*.

Definition 4.1. Let $\mathcal{X}$ be a linear space over $\mathbb{F}$. A real-valued function

$$\|\ \|\colon \mathcal{X} \to \mathbb{R}$$

is a *norm* on $\mathcal{X}$ if the following conditions are satisfied for all vectors x and y in $\mathcal{X}$ and all scalars α in $\mathbb{F}$.

(i)	$\|x\| \ge 0$	(*nonnegativeness*),
(ii)	$\|x\| > 0$ if $x \neq 0$	(*positiveness*),
(iii)	$\|\alpha x\| = \|\alpha\|\, \|x\|$	(*absolute homogeneity*),
(iv)	$\|x + y\| \le \|x\| + \|y\|$	(*subadditivity – triangle inequality*).

A linear space $\mathcal{X}$ equipped with a norm on it is a *normed space* (synonyms: *normed linear space* or *normed vector space*). If $\mathcal{X}$ is a real or complex linear space (so that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) equipped with a norm on it, then it is referred to as a *real* or *complex normed space*, respectively.

Note that these are not independent axioms. For instance, axiom (i) can be derived from axioms (ii) and (iii): an absolutely homogeneous positive functional is necessarily nonnegative. Indeed, for $\alpha = 0$ in (iii) we get $\|0\| = 0$ and, conversely, $x = 0$ whenever $\|x\| = 0$ by positiveness in (ii). Therefore, if $\| \| : \mathcal{X} \to \mathbb{R}$ is a norm, then

$$\|x\| = 0 \quad \text{if and only if} \quad x = 0.$$

Proposition 4.2. *If* $\| \| : \mathcal{X} \to \mathbb{R}$ *is a norm on a linear space* $\mathcal{X}$*, then the function* $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$*, defined by*

$$d(x, y) \;=\; \|x - y\|$$

for every $x, y \in \mathcal{X}$*, is a metric on* $\mathcal{X}$*.*

Proof. From (i) and (ii) in Definition 4.1 we get the metric axiom (i) of Definition 3.1. Positiveness (ii) and absolute homogeneity (iii) of Definition 4.1 imply positiveness (ii) and symmetry (iii) of Definition 3.1, respectively. Finally, the triangle inequality (iv) of Definition 3.1 follows from the triangle inequality (iv) and absolute homogeneity (iii) of Definition 4.1. □

A word on notation and terminology. According to Definition 4.1 a normed space actually is an ordered pair $(\mathcal{X}, \| \|)$, where $\mathcal{X}$ is a linear space and $\| \|$ is a norm on $\mathcal{X}$. As in the case of a metric space, we shall refer to a normed space in several ways. We may speak of $\mathcal{X}$ itself as a normed space when the norm $\| \|$ is either clear in the context or immaterial and, in this case, we shall simply say "$\mathcal{X}$ is a normed space". However, in order to avoid confusion among different normed spaces, we may occasionally insert a subscript on the norms (e.g., $(\mathcal{X}, \| \|_X)$ and $(\mathcal{Y}, \| \|_Y)$). If a linear space $\mathcal{X}$ can be equipped with more than one norm, say $\| \|_1$ and $\| \|_2$, then $(\mathcal{X}, \| \|_1)$ and $(\mathcal{X}, \| \|_2)$ will represent *different* normed spaces with the same linear space $\mathcal{X}$. The metric d of Proposition 4.2 is the *metric generated* by the norm $\| \|$, and so a normed space is a special kind of linear metric space. Whenever we refer to the topological (metric) structure of a normed space $(\mathcal{X}, \| \|)$, it will always be understood that such a topology on $\mathcal{X}$ is that induced by the metric d generated by the norm $\| \|$. This is called the *norm topology*. Note that the norm $\|x\|$ of every vector x in a normed space $(\mathcal{X}, \| \|)$ is precisely the distance (in the norm topology) between x and the origin 0 of the linear space $\mathcal{X}$ (i.e., $\|x\| = d(x, 0)$, where d is the metric generated by the norm $\| \|$). Proposition 4.2 says that every norm on a linear space generates a metric, but an arbitrary metric on a linear space may not be generated by any norm on it. The next proposition tells us when a metric on a linear space is generated by a norm.

Proposition 4.3. *Let $\mathcal{X}$ be a linear space. A metric on $\mathcal{X}$ is generated by a norm on $\mathcal{X}$ if and only if it is additively invariant and absolutely homogeneous. Moreover, for each additively invariant and absolutely homogeneous metric on $\mathcal{X}$ there exists a unique norm on $\mathcal{X}$ that generates it.*

Proof. If $d_{\|\|}$ is a metric on a normed space $\mathcal{X}$ generated by a norm $\|\ \|$ on $\mathcal{X}$, then it is additively invariant ($d_{\|\|}(x,y) = \|x-y\| = \|x+z-(y+z)\| = d_{\|\|}(x+z,y+z)$ for every x, y and z in $\mathcal{X}$) and also absolutely homogeneous ($d_{\|\|}(\alpha x,\alpha y) = \|\alpha x-\alpha y\| = \|\alpha(x-y)\| = |\alpha|\,\|x-y\| = |\alpha|\,d_{\|\|}(x,y)$ for every x, y in $\mathcal{X}$ and every scalar α). Conversely, if d is an additively invariant and absolutely homogeneous metric on a linear space $\mathcal{X}$, then the function $\|\ \|_d\colon \mathcal{X} \to \mathbb{R}$ defined by $\|x\|_d = d(x,0)$ for every x in $\mathcal{X}$ is a norm on $\mathcal{X}$. Indeed, properties (i) and (ii) of Definition 4.1 are trivially verified by the first two metric axioms in Definition 3.1. Properties (iii) and (iv) of Definition 4.1 follow from absolute homogeneity ($\|\alpha x\|_d = d(\alpha x,0) = |\alpha|\,d(x,0) = |\alpha|\,\|x\|_d$ for every x in $\mathcal{X}$ and every scalar α) and from additive invariance ($\|x+y\|_d = d(x+y,0) = d(x,-y) \le d(x,0)+d(0,-y) = d(x,0)+d(0,y) = \|x\|+\|y\|$ for every x, y in $\mathcal{X}$). This norm $\|\ \|_d$ generates the metric d since $\|x-y\|_d = d(x-y,0) = d(x,y)$ for every x, y in $\mathcal{X}$, and uniqueness is straightforward: if $\|\ \|_1$ and $\|\ \|_2$ generate d, then $\|x\|_1 = d(x,0) = \|x\|_2$ for all x in $\mathcal{X}$. □

Let $(\mathcal{X}, \|\ \|)$ be a normed space. By Proposition 4.2 and Problem 3.1 it follows at once that

$$\big|\|x\| - \|y\|\big| \le \|x-y\|$$

for every $x, y \in \mathcal{X}$. Thus *the norm* $\|\ \|\colon \mathcal{X} \to \mathbb{R}$ *is a continuous mapping* with respect to the norm topology of $\mathcal{X}$ (see Problem 3.34). In fact, the preceding inequality says that every norm is a contraction (thus Lipschitzian and hence uniformly continuous). Therefore (cf. Corollary 3.8 and Lemma 3.43), a norm preserves convergence: *if $x_n \to x$ in the norm topology of $\mathcal{X}$, then $\|x_n\| \to \|x\|$ in $\mathbb{R}$*; and also preserves Cauchy sequences: *if $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$ with respect to the metric generated by the norm on $\mathcal{X}$, then $\{\|x_n\|\}$ is a Cauchy sequence in $\mathbb{R}$.*

A *Banach space* is a complete normed space. Obviously, completeness refers to the norm topology: a Banach space is a normed space that is complete as a metric space with respect to the metric generated by the norm. A *real* or *complex Banach space* is a complete real or complex normed space.

Let $\mathcal{X}$ be a linear space and let $\{x_n\}$ be an $\mathcal{X}$-valued sequence indexed by $\mathbb{N}$ (or by $\mathbb{N}_0$). For each $n \ge 1$ set

$$y_n = \sum_{i=1}^{n} x_i$$

in $\mathcal{X}$ so that $\{y_n\}_{n=1}^{\infty}$ is again an $\mathcal{X}$-valued sequence. This is called the *sequence of partial sums* of $\{x_n\}_{n=1}^{\infty}$. Now equip $\mathcal{X}$ with a norm $\|\ \|$. If the sequence of

partial sums $\{y_n\}_{n=1}^{\infty}$ converges in the normed space $\mathcal{X}$ to a point y in $\mathcal{X}$ (i.e., if $\|y_n - y\| \to 0$), then we say that $\{x_n\}_{n=1}^{\infty}$ is a *summable sequence* (or that the *infinite series* $\sum_{k=1}^{\infty} x_k$ *converges* in $\mathcal{X}$ to y — notation: $y = \sum_{k=1}^{\infty} x_k$). If the real-valued sequence $\{\|x_n\|\}_{n=1}^{\infty}$ is summable (i.e., if the infinite series $\sum_{k=1}^{\infty} \|x_k\|$ converges in $\mathbb{R}$ or, equivalently, if $\sum_{k=1}^{\infty} \|x_k\| < \infty$ — see Problem 3.11), then we say that $\{x_n\}_{n=1}^{\infty}$ is an *absolutely summable sequence* (or that the infinite series $\sum_{k=1}^{\infty} x_k$ is *absolutely convergent*).

Proposition 4.4. *A normed space is a Banach space if and only if every absolutely summable sequence is summable.*

Proof. Let $(\mathcal{X}, \| \|)$ be a normed space and let $\{x_n\}_{n=0}^{\infty}$ be an arbitrary $\mathcal{X}$-valued sequence.

(a) Consider the sequence $\{y_n\}_{n=0}^{\infty}$ of partial sums of $\{x_n\}_{n=0}^{\infty}$,

$$y_n \;=\; \sum_{i=0}^{n} x_i$$

in $\mathcal{X}$ for each $n \geq 0$. It is readily verified by induction (with a little help from the triangle inequality) that

$$\|y_{n+k} - y_n\| \;=\; \Big\| \sum_{i=n+1}^{n+k} x_i \Big\| \;\leq\; \sum_{i=n+1}^{n+k} \|x_i\|$$

for every pair of integers $n \geq 0$ and $k \geq 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is an absolutely summable sequence (i.e., $\sum_{j=0}^{\infty} \|x_j\| < \infty$) so that

$$0 \;\leq\; \sup_{k \geq 1} \|y_{n+k} - y_n\| \;\leq\; \sum_{j=n+1}^{\infty} \|x_j\| \to 0 \quad \text{as} \quad n \to \infty,$$

and hence $\lim_n \sup_{k \geq 1} \|y_{n+k} - y_n\| = 0$ (Problems 3.10(c) and 3.11). Equivalently, $\{y_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathcal{X}$ (Problem 3.51). Therefore, if $\mathcal{X}$ is a Banach space, then $\{y_n\}_{n=0}^{\infty}$ converges in $\mathcal{X}$, which means that $\{x_n\}_{n=0}^{\infty}$ is a summable sequence. Conclusion: An arbitrary absolutely summable sequence in a Banach space is summable.

(b) Conversely, suppose $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. By Problem 3.52(c) $\{x_n\}_{n=0}^{\infty}$ has a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ of bounded variation. Set $z_0 = x_{n_0}$ and

$$z_{k+1} \;=\; x_{n_{k+1}} - x_{n_k}$$

in $\mathcal{X}$, so that $x_{n_{k+1}} = x_{n_k} + z_{k+1}$, for every $k \geq 0$. Thus

$$x_{n_k} \;=\; \sum_{i=0}^{k} z_i$$

for every $k \geq 0$ (cf. Problem 2.19). Since $\{x_{n_k}\}_{k=0}^{\infty}$ is of bounded variation (i.e., $\sum_{k=0}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| < \infty$), it follows that $\{z_k\}_{k=0}^{\infty}$ is an absolutely summable sequence in $\mathcal{X}$. If every absolutely summable sequence in $\mathcal{X}$ is summable, then $\{z_k\}_{k=0}^{\infty}$ is a summable sequence, and so the subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ converges in $\mathcal{X}$. Thus (see Proposition 3.39(c)) the Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ converges in $\mathcal{X}$. Conclusion: Every Cauchy sequence in $\mathcal{X}$ converges in $\mathcal{X}$, which means that the normed space $\mathcal{X}$ is complete (i.e., $\mathcal{X}$ is a Banach space). □

4.2 Examples

Many of the examples of metric spaces exhibited in Chapter 3 are in fact examples of normed spaces: linear spaces equipped with an additively invariant and absolutely homogeneous metric.

Example 4.A. Let $\mathbb{F}^n$ be the linear space over $\mathbb{F}$ of Example 2.D (with either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$). Consider the functions $\| \ \|_p : \mathbb{F}^n \to \mathbb{R}$ (for each real number $p \geq 1$) and $\| \ \|_\infty : \mathbb{F}^n \to \mathbb{R}$ defined by

$$\|x\|_p = \Big(\sum_{i=1}^{n} |\xi_i|^p \Big)^{\frac{1}{p}} \quad \text{and} \quad \|x\|_\infty = \max_{1 \leq i \leq n} |\xi_i|$$

for every $x = (\xi_1, \ldots, \xi_n)$ in $\mathbb{F}^n$. It is easy to verify that these are norms on $\mathbb{F}^n$ (the triangle inequality comes from the Minkowski inequality of Problem 3.4) and also that the metrics generated by each of them are precisely the metrics d_p (for $p \geq 1$) and d_∞ of Example 3.A. Since $\mathbb{F}^n$, when equipped with any of these metrics, is a complete metric space (Example 3.R(a)), it follows that

$$\mathbb{F}^n \text{ is a Banach space}$$

when equipped with any of the norms $\| \ \|_p$ or $\| \ \|_\infty$. In particular, for $n = 1$ all of these norms are reduced to the absolute value function $| \ | : \mathbb{F} \to \mathbb{R}$, which is the *usual norm* on $\mathbb{F}$. The norm $\| \ \|_2$ plays a special role. On $\mathbb{R}^n$ it is the *Euclidean norm*, and the real Banach space $(\mathbb{R}^n, \| \ \|_2)$ is the n-dimensional *Euclidean space*. The complex Banach space $(\mathbb{C}^n, \| \ \|_2)$ is sometimes referred to as the n-dimensional *unitary space*.

Example 4.B. According to Example 2.E the set $\mathbb{F}^{\mathbb{N}}$ (or $\mathbb{F}^{\mathbb{N}_0}$) of all scalar-valued sequences is a linear space over $\mathbb{F}$. Now consider the subsets ℓ_+^p and ℓ_+^∞ of $\mathbb{F}^{\mathbb{N}}$ defined as in Example 3.B. These are linear manifolds of $\mathbb{F}^{\mathbb{N}}$ (vector addition and scalar multiplication — pointwise defined — of p-summable or bounded sequences are again p-summable or bounded sequences, respectively, by the Minkowski inequality of Problem 3.4), and hence ℓ_+^p and ℓ_+^∞ are linear spaces over $\mathbb{F}$. For each $p \geq 1$ consider function $\| \ \|_p : \ell_+^p \to \mathbb{R}$ defined by

$$\|x\|_p = \Big(\sum_{k=1}^{\infty} |\xi_k|^p \Big)^{\frac{1}{p}}$$

for every $x = \{\xi_k\}_{k\in\mathbb{N}}$ in ℓ_+^p (i.e., $\|x\|_p = \sup_{n\in\mathbb{N}}\sum_{k=1}^n |\xi_k|^p$ — see Example 3.B), and the function $\|\ \|_\infty : \ell_+^\infty \to \mathbb{R}$ given by

$$\|x\|_\infty = \sup_{k\in\mathbb{N}} |\xi_k|$$

for every $x = \{\xi_k\}_{k\in\mathbb{N}}$ in ℓ_+^∞. It is readily verified that $\|\ \|_p$ is a norm on ℓ_+^p and $\|\ \|_\infty$ is a norm on ℓ_+^∞ (as before, the Minkowski inequality leads to the triangle inequality). Moreover, the norm $\|\ \|_p$ generates the metric d_p and the norm $\|\ \|_\infty$ generates the metric d_∞ of Example 3.B. These are the *usual norms* on ℓ_+^p and ℓ_+^∞. Since (ℓ_+^p, d_p) is a complete metric space, and since $(\ell_+^\infty, d_\infty)$ also is a complete metric space (see Examples 3.R(b) and 3.S), it follows that

$$(\ell_+^p, \|\ \|_p) \text{ and } (\ell_+^\infty, \|\ \|_\infty) \text{ are Banach spaces.}$$

Similarly (see Examples 2.E, 3.B, 3.R(b) and 3.S again),

$$(\ell^p, \|\ \|_p) \text{ and } (\ell^\infty, \|\ \|_\infty) \text{ are Banach spaces,}$$

where the functions $\|\ \|_p : \ell^p \to \mathbb{R}$ and $\|\ \|_\infty : \ell^\infty \to \mathbb{R}$, defined by

$$\|x\|_p = \Big(\sum_{k=-\infty}^{\infty} |\xi_k|^p\Big)^{\frac{1}{p}} \quad \text{and} \quad \|x\|_\infty = \sup_{k\in\mathbb{Z}} |\xi_k|$$

for $x = \{\xi_k\}_{k\in\mathbb{Z}}$ in ℓ^p or in ℓ^∞, respectively, are the *usual norms* on the linear manifolds ℓ^p and ℓ^∞ of the linear space $\mathbb{F}^{\mathbb{Z}}$.

Let $\mathcal{X}$ be a linear space. A real-valued function $\|\ \| : \mathcal{X} \to \mathbb{R}$ is a *seminorm* (or a *pseudonorm*) on $\mathcal{X}$ if it satisfies the three axioms (i), (iii), and (iv) of Definition 4.1. It is worth noticing that the inequality $\big|\|x\| - \|y\|\big| \le \|x - y\|$ for every x, y in $\mathcal{X}$ still holds for a seminorm. The difference between a norm and a seminorm is that a seminorm does not necessarily satisfy the axiom (ii) of Definition 4.1 (i.e., a seminorm surely vanishes at the origin but it may also vanish at a nonzero vector). A seminorm generates a pseudometric as in Proposition 4.2: if $\|\ \|$ is a seminorm on $\mathcal{X}$, then $d(x, y) = \|x - y\|$ for every $x, y \in \mathcal{X}$ defines an additively invariant and absolutely homogeneous pseudometric on $\mathcal{X}$. Moreover, if a pseudometric is additively invariant and absolutely homogeneous, then it is generated by a seminorm as in Proposition 4.3: if d is an additively invariant and absolutely homogeneous pseudometric on $\mathcal{X}$, then $\|x\| = d(x, 0)$ for every $x \in \mathcal{X}$ defines a seminorm on $\mathcal{X}$ such that $d(x, y) = \|x - y\|$ for every $x, y \in \mathcal{X}$.

Proposition 4.5. *Let $\|\ \|$ be a seminorm on a linear space $\mathcal{X}$. The set $\mathcal{N}$ of all vectors x in $\mathcal{X}$ for which $\|x\| = 0$,*

$$\mathcal{N} = \big\{x \in \mathcal{X}\colon\ \|x\| = 0\big\},$$

is a linear manifold of $\mathcal{X}$. Consider the quotient space $\mathcal{X}/\mathcal{N}$ and set

$$\|[x]\|_{\sim} = \|x\|$$

for every coset $[x]$ *in* $\mathcal{X}/\mathcal{N}$, *where* x *is an arbitrary vector in* $[x]$. *This defines a norm on the linear space* $\mathcal{X}/\mathcal{N}$ *so that* $(\mathcal{X}/\mathcal{N}, \|\ \|_{\sim})$ *is a normed space.*

Proof. Indeed, $\mathcal{N}$ is a linear manifold of $\mathcal{X}$ (if $u, v \in \mathcal{N}$, then $0 \le \|u+v\| \le \|u\| + \|v\| = 0$ and $\|\alpha u\| = |\alpha|\|u\| = 0$ so that $u+v \in \mathcal{N}$ and $\alpha u \in \mathcal{N}$ for every scalar α). Consider the quotient space $\mathcal{X}/\mathcal{N}$ of $\mathcal{X}$ modulo $\mathcal{N}$ (Example 2.H), which is a linear space over the same scalar field of $\mathcal{X}$. Take an arbitrary coset

$$[x] = x + \mathcal{N} = \{x' \in \mathcal{X}:\ x' = x + z \text{ for some } z \in \mathcal{N}\}$$

in $\mathcal{X}/\mathcal{N}$ and note that $\|u\| = \|v\|$ for every u and v in $[x]$ (if $u, v \in [x]$, then $u - v \in \mathcal{N}$ and $0 \le \big|\|u\| - \|v\|\big| \le \|u - v\| = 0$). Thus set

$$\|[x]\|_{\sim} = \|x\|$$

for an arbitrary $x \in [x]$ (i.e., for an arbitrary representative of the equivalence class $[x]$), which defines a function from $\mathcal{X}/\mathcal{N}$ to $\mathbb{R}$,

$$\|\ \|_{\sim} : \mathcal{X}/\mathcal{N} \to \mathbb{R}.$$

It is clear that $\|[x]\|_{\sim} \ge 0$. If $\|[x]\|_{\sim} = 0$, then $[x] = \mathcal{N} = [0]$, the origin of the linear space $\mathcal{X}/\mathcal{N}$ (reason: $\|[x]\|_{\sim} = 0$ implies that every x' in $[x]$ belongs to $\mathcal{N}$ and also that every u in $\mathcal{N}$ belongs to $[x]$). Moreover,

$$\|\alpha[x]\|_{\sim} = \|[\alpha x]\|_{\sim} = \|\alpha x\| = |\alpha|\|x\| = |\alpha|\|[x]\|_{\sim},$$

$$\|[x] + [y]\|_{\sim} = \|[x+y]\|_{\sim} = \|x+y\| \le \|x\| + \|y\| = \|[x]\|_{\sim} + \|[y]\|_{\sim},$$

for every $[x], [y] \in \mathcal{X}/\mathcal{N}$ and every scalar α (Example 2.H). Therefore (Definition 4.1), $\|\ \|_{\sim}$ is a norm on the linear space $\mathcal{X}/\mathcal{N}$. □

Remark: Note that the relation $\sim$ on $\mathcal{X}$ defined by

$$x' \sim x \quad \text{if} \quad \|x' - x\| = 0$$

or, equivalently,

$$x' \sim x \quad \text{if} \quad x' - x \in \mathcal{N}$$

is a linear equivalence relation on the linear space $\mathcal{X}$ in the sense of Example 2.G. Consider the quotient space $\mathcal{X}/\sim$ of $\mathcal{X}$ modulo $\sim$. If vector addition and scalar multiplication are defined in $\mathcal{X}/\sim$ as in Example 2.H, then $X/\sim$ is a linear space that coincides with $X/\mathcal{N}$. In this case (i.e., when $\mathcal{X}$ is a linear space and $\|\ \|$ is a seminorm on $\mathcal{X}$), the metric $d_{\sim}$ on $X/\mathcal{N}$ generated by the norm $\|\ \|_{\sim}$ is precisely the metric $\widetilde{d}$ on $\mathcal{X}/\sim$ of Proposition 3.3 obtained by the pseudometric d on $\mathcal{X}$ generated by the seminorm $\|\ \|$.

Example 4.C. Take an arbitrary real number $p \ge 1$ and consider the setup of Example 3.E: $r^p(S)$ is the set of all $\mathbb{F}$-valued Riemann p-integrable functions

on a nondegenerate interval S of the real line $\mathbb{R}$. This is a linear manifold of the linear space $\mathbb{F}^S$ (see Example 2.E), and hence $r^p(S)$ is a linear space over $\mathbb{F}$. Indeed, addition and scalar multiplication of Riemann p-integrable functions on S are again Riemann p-integrable functions on S (Minkowski inequality). It is clear that the pseudometric δ_p on $r^p(S)$ defined in Example 3.E is additively invariant and absolutely homogeneous. Thus δ_p is generated by a seminorm $|\ |_p$ on $r^p(S)$,

$$|x|_p \;=\; \delta_p(x,0) \;=\; \Big(\int_S |x(s)|^p\,ds\Big)^{\frac{1}{p}}$$

for every $x \in r^p(S)$, so that $\delta_p(x,y) = |x-y|_p$ for every $x, y \in r^p(S)$. Consider the linear manifold $\mathcal{N} = \{x \in r^p(S)\colon\ |x|_p = 0\}$ and set $R^p(S) = r^p(S)/\mathcal{N}$, the quotient space of $r^p(S)$ modulo $\mathcal{N}$ (i.e., the collection of all equivalence classes $[x] = \{x' \in r^p(S)\colon\ |x'-x|_p = 0\}$ for every $x \in r^p(S)$). By Proposition 4.5 the function $\|\ \|_p\colon R^p(S) \to \mathbb{R}$ defined by $\|[x]\|_p = |x|_p$ for every $[x]$ in $R^p(S)$ is a norm on $R^p(S)$ (where x is an arbitrary representative of the equivalence class $[x]$).This is the *usual norm* on $R^p(S)$. Note that $R^p(S)$ is precisely the quotient space $r^p(S)/\!\sim$ of Example 3.E (see the remark that follows Proposition 4.5). Moreover, $\|\ \|_p$ is the norm on $R^p(S)$ that generates the usual metric d_p of Example 3.E:

$$d_p([x],[y]) \;=\; \|[x]-[y]\|_p \;=\; \|[x-y]\|_p \;=\; |x-y|_p \;=\; \delta_p(x,y)$$

for every $[x], [y] \in R^p(S)$, where x and y are arbitrary vectors in $[x]$ and $[y]$, respectively. According to common usage, we shall write $x \in R^p(S)$ instead of $[x] \in R^p(S)$, and also

$$\|x\|_p \;=\; d_p(x,0) \;=\; \Big(\int_S |x(s)|^p\,ds\Big)^{\frac{1}{p}}$$

for every $x \in R^p(S)$ to represent the norm $\|\ \|_p$ on $R^p(S)$. Therefore,

$$(R^p(S), \|\ \|_p) \text{ is a normed space}$$

but not a Banach space (Problem 3.62). Its completion, of course, is:

$$(L^p(S), \|\ \|_p) \text{ is a Banach space.}$$

(We shall discuss the completion of a normed space in Section 4.7.)

Example 4.D. Let $C[0,1]$ be the set of all $\mathbb{F}$-valued continuous functions on the interval $[0,1]$ as in Example 3.D. Again, this is a linear manifold of the linear space $\mathbb{F}^{[0,1]}$ of Example 2.E (addition and scalar multiplication of continuous functions are continuous functions), and hence $C[0,1]$ is a linear space over $\mathbb{F}$. In fact, $C[0,1]$ is a linear manifold of the linear space $r^p[0,1]$ of the previous example (every continuous function on $[0,1]$ is Riemann p-integrable). For each $p \geq 1$ consider the function $\|\ \|_p\colon C[0,1] \to \mathbb{R}$ defined by

$$\|x\|_p \,=\, \Big(\int_0^1 |x(t)|^p\,dt\Big)^{\frac{1}{p}}$$

for every $x \in C[0,1]$. This is the norm on $C[0,1]$ that generates the metric d_p of Example 3.D so that

$$(C[0,1], \|\ \|_p) \text{ is a normed space.}$$

According to Problem 3.58, $(C[0,1], \|\ \|_p)$ is not a Banach space. Recall that $C[0,1]$ can be viewed as a subset of $R^p[0,1]$ (in the sense of Problem 3.40) and, as such, it can be shown that $C[0,1]$ is dense in $(R^p[0,1], \|\ \|_p)$. Therefore, since $R^p[0,1]$ is dense in $(L^p[0,1], \|\ \|_p)$, it follows that the Banach space $(L^p[0,1], \|\ \|_p)$ is a completion of $(C[0,1], \|\ \|_p)$ — see Problem 3.38(g) and the remark that follows Problem 3.62.

Let $\{\mathcal{X}_\gamma\}_{\gamma\in\Gamma}$ be an indexed family of linear spaces over the same field $\mathbb{F}$. The set $\bigoplus_{\gamma\in\Gamma}\mathcal{X}_\gamma$ of all indexed families $\{x_\gamma\}_{\gamma\in\Gamma}$, where $x_\gamma \in \mathcal{X}_\gamma$ for each index $\gamma \in \Gamma$, becomes a linear space over $\mathbb{F}$ if vector addition and scalar multiplication are defined on $\bigoplus_{\gamma\in\Gamma}\mathcal{X}_\gamma$ as

$$\{x_\gamma\}_{\gamma\in\Gamma} \oplus \{y_\gamma\}_{\gamma\in\Gamma} = \{x_\gamma + y_\gamma\}_{\gamma\in\Gamma} \quad \text{and} \quad \alpha\{x_\gamma\}_{\gamma\in\Gamma} = \{\alpha x_\gamma\}_{\gamma\in\Gamma}$$

for every $\{x_\gamma\}_{\gamma\in\Gamma}$ and $\{y_\gamma\}_{\gamma\in\Gamma}$ in $\bigoplus_{\gamma\in\Gamma}\mathcal{X}_\gamma$ and every α in $\mathbb{F}$. This is the *direct sum* (or the *full direct sum*) of the family $\{\mathcal{X}_\gamma\}_{\gamma\in\Gamma}$. The underlying set of the linear space $\bigoplus_{\gamma\in\Gamma}\mathcal{X}_\gamma$ is the Cartesian product $\prod_{\gamma\in\Gamma}\mathcal{X}_\gamma$ of the underlying sets of each linear space $\mathcal{X}_\gamma$ (cf. Section 2.8).

Example 4.E. Let $\{(\mathcal{X}_i, \|\ \|_i)\}_{i=1}^n$ be a finite collection of normed spaces, where the linear spaces $\mathcal{X}_i$ are all over the same field $\mathbb{F}$, and let $\bigoplus_{i=1}^n\mathcal{X}_i$ be the direct sum of the family $\{\mathcal{X}_i\}_{i=1}^n$. Consider the functions $\|\ \|_p\colon \bigoplus_{i=1}^n\mathcal{X}_i \to \mathbb{R}$ (for each real number $p \geq 1$) and $\|\ \|_\infty\colon \bigoplus_{i=1}^n\mathcal{X}_i \to \mathbb{R}$ defined by

$$\|x\|_p = \Big(\sum_{i=1}^n \|x_i\|_i^p\Big)^{\frac{1}{p}} \quad \text{and} \quad \|x\|_\infty = \max_{1\leq i\leq n} \|x_i\|_i$$

for every $x = (x_1, \dots, x_n)$ in $\bigoplus_{i=1}^n\mathcal{X}_i$. It is easy to verify that these are norms on the direct sum $\bigoplus_{i=1}^n\mathcal{X}_i$. For instance, the triangle inequality for the norm $\|\ \|_p$ comes from the Minkowski inequality (Problem 3.4): for every $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\bigoplus_{i=1}^n\mathcal{X}_i$,

$$\begin{aligned}\|x \oplus y\|_p \,&=\, \Big(\sum_{i=1}^n \|x_i + y_i\|_i^p\Big)^{\frac{1}{p}} \,\leq\, \Big(\sum_{i=1}^n \big(\|x_i\|_i + \|y_i\|_i\big)^p\Big)^{\frac{1}{p}} \\ &\leq\, \Big(\sum_{i=1}^n \|x_i\|_i^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^n \|y_i\|_i^p\Big)^{\frac{1}{p}} \,=\, \|x\|_p + \|y\|_p.\end{aligned}$$

Moreover, these norms generate the metrics d_p and d_∞ of Problem 3.9 (recall: the underlying set of the linear space $\bigoplus_{i=1}^n\mathcal{X}_i$ is the Cartesian product $\prod_{i=1}^n\mathcal{X}_i$ of the underlying sets of each linear space $\mathcal{X}_i$), and therefore (Problem 3.60)

$$\big(\textstyle\bigoplus_{i=1}^{n}\mathcal{X}_i, \|\ \|_p\big) \text{ and } \big(\textstyle\bigoplus_{i=1}^{n}\mathcal{X}_i, \|\ \|_\infty\big) \text{ are Banach spaces}$$

if and only if each $(\mathcal{X}_i, \|\ \|_i)$ is a Banach space. If the normed spaces $(\mathcal{X}_i, \|\ \|_i)$ coincide with a fixed normed space $(\mathcal{X}, \|\ \|)$, then $\bigoplus_{i=1}^{n}\mathcal{X}$ is the direct sum of n copies of $\mathcal{X}$ (a linear space whose underlying set is the Cartesian product $\prod_{i=1}^{n}\mathcal{X} = \mathcal{X}^n$ of n copies of the underlying set of the linear space $\mathcal{X}$ — Section 1.7). We can identify $\bigoplus_{i=1}^{n}\mathcal{X}$ with $\mathcal{X}^n$ (where the linear operations on $\mathcal{X}^n$ are defined coordinatewise) so that

$$(\mathcal{X}^n, \|\ \|_p) \text{ and } (\mathcal{X}^n, \|\ \|_\infty) \text{ are Banach spaces}$$

whenever $(\mathcal{X}, \|\ \|)$ is a Banach space. Note that this generalizes the Banach spaces of Example 4.A.

Example 4.F. Let $\{(\mathcal{X}_k, \|\ \|_k)\}$ be a countably infinite collection of normed spaces, where the linear spaces $\mathcal{X}_k$ are over the same field $\mathbb{F}$, and let $\bigoplus_k\mathcal{X}_k$ be the direct sum of $\{\mathcal{X}_k\}$. For each $p \geq 1$ consider the subset $\big[\bigoplus_k\mathcal{X}_k\big]_p$ of $\bigoplus_k\mathcal{X}_k$ consisting of all *p-summable families* $\{x_k\}$ in $\bigoplus_k\mathcal{X}_k$. That is, $\{x_k\} \in \big[\bigoplus_k\mathcal{X}_k\big]_p$ if and only if $\sum_k\|x_k\|_k^p < \infty$, where $\sum_k\|x_k\|_k^p$ is the supremum of the set of all finite sums of positive numbers from $\{\|x_k\|_k^p\}$. This is a linear manifold of the linear space $\bigoplus_k\mathcal{X}_k$ and so $\big[\bigoplus_k\mathcal{X}_k\big]_p$ is a linear space over $\mathbb{F}$. As in Example 4.E, it is readily verified that the function $\|\ \|_p\colon \big[\bigoplus_k\mathcal{X}_k\big]_p \to \mathbb{R}$, defined by

$$\|x\|_p \;=\; \Big(\sum_k \|x_k\|_k^p\Big)^{\frac{1}{p}}$$

for every $x = \{x_k\} \in \big[\bigoplus_k\mathcal{X}_k\big]_p$, is a norm on $\big[\bigoplus_k\mathcal{X}_k\big]_p$. Now consider the subset $\big[\bigoplus_k\mathcal{X}_k\big]_\infty$ of $\bigoplus_k\mathcal{X}_k$ consisting of all *bounded families* $\{x_k\}$ in $\bigoplus_k\mathcal{X}_k$ (i.e., $\{x_k\} \in \big[\bigoplus_k\mathcal{X}_k\big]_\infty$ if and only if $\sup_k\|x_k\|_k < \infty$). This again is a linear manifold of the linear space $\bigoplus_k\mathcal{X}_k$ so that $\big[\bigoplus_k\mathcal{X}_k\big]_\infty$ is itself a linear space over $\mathbb{F}$. It is easy to show that the function $\|\ \|_\infty\colon \big[\bigoplus_k\mathcal{X}_k\big]_\infty \to \mathbb{R}$, defined by

$$\|x\|_\infty \;=\; \sup_k \|x_k\|_k$$

for every $x = \{x_k\} \in \big[\bigoplus_k\mathcal{X}_k\big]_\infty$, is a norm on $\big[\bigoplus_k\mathcal{X}_k\big]_\infty$. Moreover, it can also be shown that

$$\big(\big[\textstyle\bigoplus_k\mathcal{X}_k\big]_p, \|\ \|_p\big) \text{ and } \big(\big[\textstyle\bigoplus_k\mathcal{X}_k\big]_\infty, \|\ \|_\infty\big) \text{ are Banach spaces}$$

if and only if each $(\mathcal{X}_k, \|\ \|_k)$ is a Banach space (*hint*: Example 3.R(b)). Again, if the normed spaces $(\mathcal{X}_k, \|\ \|_k)$ coincide with a fixed normed space $(\mathcal{X}, \|\ \|)$, then $\bigoplus_k\mathcal{X}$ is the direct sum of countably infinite copies of $\mathcal{X}$. (Recall that $\bigoplus_k\mathcal{X}$ is a linear space whose underlying set is the Cartesian product $\prod_k\mathcal{X}$ of countably infinite copies of the underlying set of the linear space $\mathcal{X}$, which coincide with $\mathcal{X}^{\mathbb{N}}$, $\mathcal{X}^{\mathbb{N}_0}$, or $\mathcal{X}^{\mathbb{Z}}$ if the indices k run over $\mathbb{N}$, $\mathbb{N}_0$, or $\mathbb{Z}$, respectively — Section 1.7.) As before, we shall adopt the identifications $\bigoplus_{k\in\mathbb{N}}\mathcal{X} = \mathcal{X}^{\mathbb{N}}$,

$\bigoplus_{k\in\mathbb{N}_0}\mathcal{X}=\mathcal{X}^{\mathbb{N}_0}$, and $\bigoplus_{k\in\mathbb{Z}}\mathcal{X}=\mathcal{X}^{\mathbb{Z}}$ (the linear operations on $\mathcal{X}^{\mathbb{N}}$, $\mathcal{X}^{\mathbb{N}_0}$, and $\mathcal{X}^{\mathbb{Z}}$ are defined as in Example 2.F). It is usual to denote $\big[\bigoplus_k\mathcal{X}\big]_p$ in $\mathcal{X}^{\mathbb{N}}$ (or in $\mathcal{X}^{\mathbb{N}_0}$) by $\ell_+^p(\mathcal{X})$: the linear manifold of $\mathcal{X}^{\mathbb{N}}$ consisting of all p-summable $\mathcal{X}$-valued sequences; and $\big[\bigoplus_k\mathcal{X}\big]_\infty$ in $\mathcal{X}^{\mathbb{N}}$ (or in $\mathcal{X}^{\mathbb{N}_0}$) by $\ell_+^\infty(\mathcal{X})$: the linear manifold of $\mathcal{X}^{\mathbb{N}}$ consisting of all bounded $\mathcal{X}$-valued sequences. That is,

$$\ell_+^p(\mathcal{X}) \,=\, \big\{\{x_k\}\in\mathcal{X}^{\mathbb{N}}\colon\ \textstyle\sum_{k=1}^\infty\|x_k\|^p<\infty\big\},$$

$$\ell_+^\infty(\mathcal{X}) \,=\, \big\{\{x_k\}\in\mathcal{X}^{\mathbb{N}}\colon\ \sup_{k\in\mathbb{N}}\|x_k\|<\infty\big\}.$$

The norms $\|x\|_p=\big(\sum_{k=1}^\infty\|x_k\|^p\big)^{\frac{1}{p}}$ and $\|x\|_\infty=\sup_{k\in\mathbb{N}}\|x_k\|$, for every $x=\{x_k\}$ either in $\ell_+^p(\mathcal{X})$ or in $\ell_+^\infty(\mathcal{X})$, are the *usual norms* on $\ell_+^p(\mathcal{X})$ and $\ell_+^\infty(\mathcal{X})$, respectively, and

$$(\ell_+^p(\mathcal{X}),\|\ \|_p) \text{ and } (\ell_+^\infty(\mathcal{X}),\|\ \|_\infty) \text{ are Banach spaces}$$

whenever $(\mathcal{X},\|\ \|)$ is a Banach space. Similarly, $\big[\bigoplus_k\mathcal{X}\big]_p$ in $\mathcal{X}^{\mathbb{Z}}$ is denoted by $\ell^p(\mathcal{X})$ and $\big[\bigoplus_k\mathcal{X}\big]_\infty$ in $\mathcal{X}^{\mathbb{Z}}$ is denoted by $\ell^\infty(\mathcal{X})$ and, when equipped with their *usual norms* $\|\ \|_p$ and $\|\ \|_\infty$,

$$(\ell^p(\mathcal{X}),\|\ \|_p) \text{ and } (\ell^\infty(\mathcal{X}),\|\ \|_\infty) \text{ are Banach spaces}$$

whenever $(\mathcal{X},\|\ \|)$ is a Banach space. This example generalizes the Banach spaces of Example 4.B.

4.3 Subspaces and Quotient Spaces

If $(\mathcal{X},\|\ \|)$ is a normed space, and if $\mathcal{M}$ is a linear manifold of the linear space $\mathcal{X}$, then it is easy to show that the restriction $\|\ \|_M\colon\mathcal{M}\to\mathbb{R}$ of the norm $\|\ \|\colon\mathcal{X}\to\mathbb{R}$ to $\mathcal{M}$ is a norm on $\mathcal{M}$ so that $(\mathcal{M},\|\ \|_M)$ is a normed space. Moreover, the metric $d_M\colon\mathcal{M}\times\mathcal{M}\to\mathbb{R}$ generated by the norm $\|\ \|_M$ on $\mathcal{M}$ coincides with the restriction to $\mathcal{M}\times\mathcal{M}$ of the metric $d\colon\mathcal{X}\times\mathcal{X}\to\mathbb{R}$ generated by the norm $\|\ \|$ on $\mathcal{X}$. Thus $(\mathcal{M},d_M)$ is a subspace of the metric space $(\mathcal{X},d)$. If a linear manifold of a normed space is regarded as a normed space, then it will be understood that the norm on it is the restricted norm $\|\ \|_M$. We shall drop the subscripts and write $(\mathcal{M},\|\ \|)$ and $(\mathcal{M},d)$ instead of $(\mathcal{M},\|\ \|_M)$ and $(\mathcal{M},d_M)$, respectively, and often refer to the normed space $(\mathcal{M},\|\ \|)$ by simply saying that "$\mathcal{M}$ is a linear manifold of $\mathcal{X}$".

Proposition 4.6. *Let $\mathcal{M}$ be a linear manifold of a normed space $\mathcal{X}$. If $\mathcal{M}$ is open in $\mathcal{X}$, then $\mathcal{M}=\mathcal{X}$.*

Proof. Since $\mathcal{M}$ is a linear manifold of the linear space $\mathcal{X}$, the origin 0 of $\mathcal{X}$ lies in $\mathcal{M}$. If $\mathcal{M}$ is open in $\mathcal{X}$ (in the norm topology, of course), then $\mathcal{M}$ includes a nonempty open ball with center at the origin: $B_\varepsilon(0)=\{y\in\mathcal{X}\colon\ \|y\|<\varepsilon\}\subset\mathcal{M}$ for some $\varepsilon>0$. Take an arbitrary $x\neq0$ in $\mathcal{X}$ and set $z=\varepsilon(2\|x\|)^{-1}x$ in $\mathcal{X}$

so that $\|z\| = \frac{\varepsilon}{2}$, and hence $z \in B_\varepsilon(0) \subset \mathcal{M}$. Thus $x = (2\|x\|)\varepsilon^{-1}z$ lies in $\mathcal{M}$ (since $\mathcal{M}$ is a linear space). Therefore, every nonzero vector in $\mathcal{X}$ also lies in $\mathcal{M}$. Conclusion: $\mathcal{X} \subseteq \mathcal{M}$ so that $\mathcal{M} = \mathcal{X}$ (because $\mathcal{M} \subseteq \mathcal{X}$). □

This shows that a normed space $\mathcal{X}$ is itself the only open linear manifold of $\mathcal{X}$. On the other hand, the closed linear manifolds of a normed space are far more interesting. In fact, they are so important that we give then a name. A *closed* linear manifold of a normed space is called a *subspace* of $\mathcal{X}$. (*Warning*: A subspace of a metric space (X, d) is simply a subset of X equipped with the "same" metric d, while a subspace of a normed space $(\mathcal{X}, \|\ \|)$ is a linear manifold of $\mathcal{X}$ equipped with the "same" norm $\|\ \|$ that is *closed* in $(\mathcal{X}, \|\ \|)$.)

Proposition 4.7. *A linear manifold of a Banach space $\mathcal{X}$ is itself a Banach space if and only if it is a subspace of $\mathcal{X}$.*

Proof. Corollary 3.41. □

Example 4.G. As usual, let Y^S denote the collection of all functions of a nonempty set S into a set Y. If $\mathcal{Y}$ is a linear space over $\mathbb{F}$, then $\mathcal{Y}^S$ is a linear space over $\mathbb{F}$ (Example 2.F). Suppose $\mathcal{Y}$ is a normed space and consider the subset $B[S, \mathcal{Y}]$ of $\mathcal{Y}^S$ consisting of all bounded mappings of S into $\mathcal{Y}$. This is a linear manifold of $\mathcal{Y}^S$ (vector addition and scalar multiplication of bounded mappings are bounded mappings), and hence $B[S, \mathcal{Y}]$ is a linear space over $\mathbb{F}$. Now consider the function $\|\ \|_\infty : B[S, \mathcal{Y}] \to \mathbb{R}$ defined by

$$\|f\|_\infty = \sup_{s \in S} \|f(s)\|$$

for every $f \in B[S, \mathcal{Y}]$, where $\|\ \| : \mathcal{Y} \to \mathbb{R}$ is the norm on $\mathcal{Y}$. It is easy to verify that $(B[S, \mathcal{Y}], \|\ \|_\infty)$ is a normed space. The norm $\|\ \|_\infty$ on $B[S, \mathcal{Y}]$, which is referred to as the *sup-norm*, generates the sup-metric d_∞ of Example 3.C. Thus, according to Example 3.S,

$(B[S, \mathcal{Y}], \|\ \|_\infty)$ is a Banach space if and only if $\mathcal{Y}$ is a Banach space.

Moreover, if X is a nonempty metric space and if $BC[X, \mathcal{Y}]$ is the subset of $B[X, \mathcal{Y}]$ made up of all continuous mappings from $B[X, \mathcal{Y}]$, then $BC[X, \mathcal{Y}]$ is a linear manifold of the linear space $B[X, \mathcal{Y}]$ (addition and scalar multiplication of continuous functions are continuous functions), and so $BC[X, \mathcal{Y}]$ is itself a linear space over $\mathbb{F}$. According to Example 3.N, the linear manifold $BC[X, \mathcal{Y}]$ is closed in $(B[X, \mathcal{Y}], \|\ \|_\infty)$, and so $(BC[X, \mathcal{Y}], \|\ \|_\infty)$ is a subspace of the normed space $(B[X, \mathcal{Y}], \|\ \|_\infty)$. Thus (cf. Example 3.T and Proposition 4.7),

$(BC[X, \mathcal{Y}], \|\ \|_\infty)$ is a Banach space if and only if $\mathcal{Y}$ is a Banach space.

If the metric space X is compact, then $C[X, \mathcal{Y}] = BC[X, \mathcal{Y}]$, where $C[X, \mathcal{Y}]$ stands for the set of all continuous mappings of the compact metric space X into the normed space $\mathcal{Y}$ (see Example 3.Y). Therefore,

$(C[X, \mathcal{Y}], \| \ \|_\infty)$ is a Banach space if X is compact and $\mathcal{Y}$ is Banach.

By setting $X = [0, 1]$ equipped with the usual metric on $\mathbb{R}$ (which is compact — Heine–Borel Theorem), it follows that

$$(C[0, 1], \| \ \|_\infty) \text{ is a Banach space,}$$

where $C[0, 1] = C([0, 1], \mathbb{F})$ is the linear space over $\mathbb{F}$ of all $\mathbb{F}$-valued continuous functions defined on the closed interval $[0, 1]$ (recall: $(\mathbb{F}, | \ |)$ is a Banach space — Example 4.A), and

$$\|x\|_\infty = \sup_{0 \le t \le 1} |x(t)| = \max_{0 \le t \le 1} |x(t)| \quad \text{for every} \quad x \in C[0, 1]$$

is the *sup-norm* on $C[0, 1]$ (cf. Examples 3.T and 3.Y).

In any normed space $\mathcal{X}$ the zero linear manifold $\{0\}$ and the whole space $\mathcal{X}$ are subspaces of $\mathcal{X}$. If a subspace $\mathcal{M}$ is a proper subset of $\mathcal{X}$, then it is said to be a *proper subspace*. A *nontrivial subspace* $\mathcal{M}$ of a normed space $\mathcal{X}$ is a nonzero proper subspace of it ($\{0\} \ne \mathcal{M} \ne \mathcal{X}$). The next proposition shows that, if the dimension of the linear space $\mathcal{X}$ is greater than 1, then there are many nontrivial subspaces of the normed space $\mathcal{X}$.

Proposition 4.8. *Let $\mathcal{X}$ be a normed space.*

(a) *The closure $\mathcal{M}^-$ of a linear manifold $\mathcal{M}$ of $\mathcal{X}$ is a subspace of $\mathcal{X}$.*

(b) *The intersection of an arbitrary nonempty collection of subspaces of $\mathcal{X}$ is again a subspace of $\mathcal{X}$.*

Proof. (a) The closure $\mathcal{M}^-$ of a linear manifold $\mathcal{M}$ of a normed space $\mathcal{X}$ is clearly a closed subset of $\mathcal{X}$. What is left to be shown is that this closed subset of $\mathcal{X}$ is also a linear manifold of $\mathcal{X}$. Take two arbitrary points x and y in $\mathcal{M}^-$. According to Proposition 3.27, there exist $\mathcal{M}$-valued sequences $\{x_n\}$ and $\{y_n\}$ that converge to x and y, respectively, in the norm topology of $\mathcal{X}$. Therefore (cf. Problem 4.1), the $\mathcal{M}$-valued sequence $\{x_n + y_n\}$ converges in $\mathcal{X}$ to $x + y$, and hence $x + y \in \mathcal{M}^-$ (see Proposition 3.27 again). Similarly, $\alpha x \in \mathcal{M}^-$ for every scalar α. Thus $\mathcal{M}^-$ is a linear manifold of $\mathcal{X}$.

(b) The intersection of an arbitrary nonempty collection of linear manifolds of a linear space is a linear manifold (cf. Section 2.2), and the intersection of an arbitrary collection of closed subsets of a metric space is a closed set (cf. Theorem 3.22). Thus the intersection of an arbitrary nonempty collection of closed linear manifolds of a normed space is a closed linear manifold. □

Let A be a subset of a normed space $\mathcal{X}$. The (linear) span of A, $\operatorname{span} A$, was defined in Section 2.2 as the intersection of all linear manifolds of $\mathcal{X}$ that include A, which coincides with the smallest linear manifold of $\mathcal{X}$ that includes A. Recall that the smallest closed subset of $\mathcal{X}$ that includes $\operatorname{span} A$ is precisely

its closure $(\operatorname{span} A)^-$ in $\mathcal{X}$ (by the very definition of closure). According to Proposition 4.8(a) $(\operatorname{span} A)^-$ is a subspace of $\mathcal{X}$. This is the smallest closed linear manifold of $\mathcal{X}$ that includes A. Set

$$\bigvee A = \big(\operatorname{span} A\big)^-,$$

which is called the *subspace spanned* by A. If a subspace $\mathcal{M}$ of $\mathcal{X}$ (which may be $\mathcal{X}$ itself) is such that $\mathcal{M} = \bigvee A$ for some subset A of $\mathcal{X}$, then we say that A *spans* $\mathcal{M}$ or that A is a *spanning set* for $\mathcal{M}$ (note: the same terminology of Section 2.2 but now with a different meaning); or still that A is a *total set* for $\mathcal{M}$. Note that the intersection of all subspaces of $\mathcal{X}$ that include A is the smallest subspace of $\mathcal{X}$ (see Proposition 4.8(b)) that includes A. Summing up: $\bigvee A$ *is the smallest subspace of* $\mathcal{X}$ *that includes* A, which coincides with *the intersection of all subspaces of* $\mathcal{X}$ *that include* A. It is readily verified that $\bigvee \varnothing = \{0\}$, $\bigvee \mathcal{M} = \mathcal{M}^-$ for every linear manifold $\mathcal{M}$ of $\mathcal{X}$, and $A \subseteq \bigvee A = \bigvee\big(\bigvee A\big)$ for every subset A of $\mathcal{X}$. Moreover, if A and B are subsets of $\mathcal{X}$, then

$$A \subseteq B \quad \text{implies} \quad \bigvee A \subseteq \bigvee B.$$

Proposition 4.9. *Let* $\mathcal{X}$ *be a normed space.*

(a) *A set* A *spans* $\mathcal{X}$ *if and only if every linear manifold of* $\mathcal{X}$ *that includes* A *is dense in* $\mathcal{X}$.

(b) $\mathcal{X}$ *is separable if and only if it is spanned by a countable set.*

Proof. (a) Let A be a subset of a normed space $\mathcal{X}$. Take an arbitrary linear manifold $\mathcal{M}$ of $\mathcal{X}$ such that $A \subseteq \mathcal{M}$. Recall: $\bigvee A \subseteq \bigvee \mathcal{M} = \mathcal{M}^-$. Thus $\bigvee A = \mathcal{X}$ implies $\mathcal{M}^- = \mathcal{X}$. Conversely, note that $\operatorname{span} A$ is a linear manifold of $\mathcal{X}$ that includes A. If every linear manifold of $\mathcal{X}$ that includes A is dense in $\mathcal{X}$, then $\bigvee A = (\operatorname{span} A)^- = \mathcal{X}$.

(b) Let $\mathcal{X}$ be a normed space. If $\mathcal{X}$ is separable as a metric space, then (by definition) there exists a countable set, say $A \subseteq \mathcal{X}$, such that $A^- = \mathcal{X}$. Since $A \subseteq \bigvee A = \big(\bigvee A\big)^-$, it follows that $\bigvee A = \mathcal{X}$. Conversely, suppose $(\operatorname{span} A)^- = \mathcal{X}$ for some countable subset A of $\mathcal{X}$. Recall that $\operatorname{span} A$ is the set of all (finite) linear combinations of vectors in A (Proposition 2.2). Let $\mathcal{M}$ denote the set of all (finite) linear combinations of vectors in A with rational coefficients (note: we say that a complex number is "rational" if its real and imaginary parts are rational numbers). Since $\mathbb{Q}^- = \mathbb{R}$, it follows that $\mathcal{M}^- = (\operatorname{span} A)^-$ (Example 3.P), and hence $\mathcal{M}^- = \mathcal{X}$. Moreover, it is readily verified that $\mathcal{M}$ is a linear space over the rational field $\mathbb{Q}$ and so $\mathcal{M} = \operatorname{span} A$ (over $\mathbb{Q}$). Thus $\mathcal{M}$ has a Hamel basis included in A (Theorem 2.6) so that $\dim \mathcal{M} \leq \# A \leq \aleph_0$, and therefore $\# \mathcal{M} = \max\{\# \mathbb{Q}, \dim \mathcal{M}\} = \aleph_0$ (Problem 2.8). Conclusion: $\mathcal{M}$ is a countable dense subset of $\mathcal{X}$. Outcome: $\mathcal{X}$ is separable. □

In Section 2.2 we considered the subcollection $\mathcal{L}at(\mathcal{X})$ of the power set $\wp(\mathcal{X})$ of a linear space $\mathcal{X}$ consisting of all linear manifolds of $\mathcal{X}$. $\mathcal{L}at(\mathcal{X})$ was

shown to be a complete lattice (in the inclusion ordering of $\wp(\mathcal{X})$): if $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ is an arbitrary nonempty subcollection of $\mathcal{L}at(\mathcal{X})$ (i.e., an arbitrary nonempty indexed family of linear manifolds of $\mathcal{X}$), then

$$\inf\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma} = \bigcap_{\gamma\in\Gamma} \mathcal{M}_\gamma,$$

$$\sup\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma} = \operatorname{span}\Big(\bigcup_{\gamma\in\Gamma} \mathcal{M}_\gamma\Big) = \sum_{\gamma\in\Gamma} \mathcal{M}_\gamma,$$

both in $\mathcal{L}at(\mathcal{X})$. In particular, if $\{\mathcal{M},\mathcal{N}\}$ is a pair of linear manifolds of $\mathcal{X}$, then $\mathcal{M}\wedge\mathcal{N} = \mathcal{M}\cap\mathcal{N}$ and $\mathcal{M}\vee\mathcal{N} = \operatorname{span}(\mathcal{M}\cup\mathcal{N}) = \mathcal{M}+\mathcal{N}$ are both linear manifolds of $\mathcal{X}$.

Now shift from linear manifolds to subspaces. Let $\operatorname{Lat}(\mathcal{X})$ denote the subcollection of the power set $\wp(\mathcal{X})$ of a normed space $\mathcal{X}$ made up of all *subspaces* of $\mathcal{X}$. Clearly, $\operatorname{Lat}(\mathcal{X}) \subseteq \mathcal{L}at(\mathcal{X})$. If $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ is an arbitrary nonempty subcollection of $\operatorname{Lat}(\mathcal{X})$ (i.e., an arbitrary nonempty indexed family of subspaces of the normed space $\mathcal{X}$), then $\bigcap_{\gamma\in\Gamma}\mathcal{M}_\gamma \in \operatorname{Lat}(\mathcal{X})$ (by Proposition 4.8(b)) and $\bigcap_{\gamma\in\Gamma}\mathcal{M}_\gamma \subseteq \mathcal{M}_\alpha$ for all $\mathcal{M}_\alpha \in \{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$, so that $\bigcap_{\gamma\in\Gamma}\mathcal{M}_\gamma$ is a lower bound for $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$. Moreover, if $\mathcal{V}$ in $\operatorname{Lat}(\mathcal{X})$ is a lower bound for $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ (i.e., if $\mathcal{V}\subseteq\mathcal{M}_\gamma$ for all $\gamma\in\Gamma$), then $\mathcal{V}\subseteq\bigcap_{\gamma\in\Gamma}\mathcal{M}_\gamma$. Thus $\bigcap_{\gamma\in\Gamma}\mathcal{M}_\gamma = \inf\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$. If we adopt the usual notation $\bigwedge_{\gamma\in\Gamma}\mathcal{M}_\gamma = \inf\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$, then

$$\bigwedge_{\gamma\in\Gamma} \mathcal{M}_\gamma = \bigcap_{\gamma\in\Gamma} \mathcal{M}_\gamma.$$

Similarly, $\mathcal{M}_\alpha \subseteq \bigvee\big(\bigcup_{\gamma\in\Gamma}\mathcal{M}_\gamma\big) \in \operatorname{Lat}(\mathcal{X})$ for all $\mathcal{M}_\alpha \in \{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ and, if $\mathcal{U}$ in $\operatorname{Lat}(\mathcal{X})$ is an upper bound for $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ (i.e., if $\mathcal{M}_\gamma\subseteq\mathcal{U}$ for all $\gamma\in\Gamma$ so that $\bigcup_{\gamma\in\Gamma}\mathcal{M}_\gamma\subseteq\mathcal{U}$), then $\bigvee\big(\bigcup_{\gamma\in\Gamma}\mathcal{M}_\gamma\big)\subseteq\mathcal{U}$. Thus $\bigvee\big(\bigcup_{\gamma\in\Gamma}\mathcal{M}_\gamma\big) = \sup\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$. Again, if we take up the usual notation $\bigvee_{\gamma\in\Gamma}\mathcal{M}_\gamma = \sup\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$, then

$$\bigvee_{\gamma\in\Gamma} \mathcal{M}_\gamma = \bigvee\Big(\bigcup_{\gamma\in\Gamma} \mathcal{M}_\gamma\Big) = \Big(\sum_{\gamma\in\Gamma} \mathcal{M}_\gamma\Big)^-.$$

This is the *topological sum* of $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$. Conclusion: $\operatorname{Lat}(\mathcal{X})$ is a complete lattice. *The collection of all subspaces of a normed space is a complete lattice in the inclusion ordering.* If $\mathcal{M}$ and $\mathcal{N}$ are subspaces of $\mathcal{X}$, then $\mathcal{M}\wedge\mathcal{N} = \mathcal{M}\cap\mathcal{N}$ and $\mathcal{M}\vee\mathcal{N} = \bigvee(\mathcal{M}\cup\mathcal{N}) = (\mathcal{M}+\mathcal{N})^-$ lie in $\operatorname{Lat}(\mathcal{X})$. However (and this is rather important), it may happen that $\mathcal{M}+\mathcal{N} \neq (\mathcal{M}+\mathcal{N})^-$: the (algebraic) sum of subspaces is not necessarily a subspace (it is a linear manifold but not necessarily a *closed* linear manifold). We shall see later (next chapter) an example of a pair of subspaces (of a Banach space) whose sum is not closed.

Remark: Let $\{\mathcal{M}_i\}_{i=1}^n$ be a finite collection of linear manifolds of a normed space $\mathcal{X}$. Although it may happen that $\sum_{i=1}^n \mathcal{M}_i^- \subsetneqq \big(\sum_{i=1}^n \mathcal{M}_i^-\big)^-$, we have

$$\Big(\sum_{i=1}^{n} \mathcal{M}_i^-\Big)^- \;=\; \Big(\sum_{i=1}^{n} \mathcal{M}_i\Big)^-.$$

Indeed, take $x \in \big(\sum_{i=1}^n \mathcal{M}_i^-\big)^-$ and $\varepsilon > 0$ arbitrary. Thus $\|x - \overline{u}\| < \varepsilon$ for some $\overline{u} = \sum_{i=1}^n \overline{u}_i$ in $\sum_{i=1}^n \mathcal{M}_i^-$ with each $\overline{u}_i$ in $\mathcal{M}_i^-$, and so $\|\overline{u}_i - u_i\| < \varepsilon$ for some u_i in $\mathcal{M}_i$, for every integer i. Set $u = \sum_{i=1}^n u_i$ in $\sum_{i=1}^n \mathcal{M}_i$. Thus $\|x - u\| = \|x - \sum_{i=1}^n u_\gamma\| = \|x - \sum_{i=1}^n \overline{u}_i + \sum_{i=1}^n (u_i - \overline{u}_i)\| = \|x - \overline{u} + \sum_{i=1}^n (u_i - \overline{u}_i)\| \le \|x - \overline{u}\| + \sum_{i=1}^n \|u_i - \overline{u}_1\| \le (n+1)\varepsilon$. Then (since n does not depend on ε), for every $\delta > 0$ set $\varepsilon = \frac{\delta}{n+1}$ so that $\|x - u\| < \delta$, and hence $x \in \big(\sum_{i=1}^n \mathcal{M}_i\big)^-$. Therefore, $\big(\sum_{i=1}^n \mathcal{M}_i^-\big)^- \subseteq \big(\sum_{i=1}^n \mathcal{M}_i\big)^-$. The converse inclusion is trivial.

Next we consider the quotient space of a normed space modulo a subspace. Suppose $\mathcal{M}$ is a subspace of a normed space $(\mathcal{X}, \|\ \|_X)$. Let

$$[x] \;=\; x + \mathcal{M} \;=\; \big\{x' \in \mathcal{X}\colon\ x' = x + u \text{ for some } u \in \mathcal{M}\big\}$$

be the coset (of x modulo $\mathcal{M}$) in the quotient space $\mathcal{X}/\mathcal{M}$ (of $\mathcal{X}$ modulo $\mathcal{M}$), which is a linear space over the same field of the linear space $\mathcal{X}$ (cf. Example 2.H). Consider the function $\|\ \|\colon \mathcal{X}/\mathcal{M} \to \mathbb{R}$ given by

$$\|[x]\| \;=\; \inf_{x' \in [x]} \|x'\|_X \;=\; \inf_{u \in \mathcal{M}} \|x + u\|_X$$

for every $[x]$ in $\mathcal{X}/\mathcal{M}$, where x is any vector of $\mathcal{X}$ in $[x]$. This in fact defines a function from $\mathcal{X}/\mathcal{M}$ to $\mathbb{R}$, for $\|[x]\|$ depends only on the coset $[x]$ and not on a particular representative vector x in $[x]$. Note that $\inf_{u \in \mathcal{M}} \|x + u\|_X = \inf_{u \in \mathcal{M}} \|x - u\|_X$ because $\mathcal{M}$ is a linear manifold. Thus, with d_X denoting the metric on $\mathcal{X}$ generated by the norm $\|\ \|_X$,

$$\|[x]\| \;=\; \inf_{u \in \mathcal{M}} d_X(x, u) \;=\; d_X(x, \mathcal{M})$$

for every $[x] \in \mathcal{X}/\mathcal{M}$, where x is any vector of $\mathcal{X}$ in $[x]$. We claim that $\|\ \|$ is a norm on the quotient space $\mathcal{X}/\mathcal{M}$. Indeed, nonnegativeness,

$$\|[x]\| \;\ge\; 0$$

for every $[x] \in \mathcal{X}/\mathcal{M}$, is trivially verified. To verify positiveness, proceed as follows. Take an arbitrary $x \in [x]$. If $\|[x]\| = 0$, then $d_X(x, \mathcal{M}) = 0$ so that $x \in \mathcal{M}^- = \mathcal{M}$ (cf. Problem 3.43(b) — recall: $\mathcal{M}$ is closed in $\mathcal{X}$). This means that $x \in [0]$ (reason: the origin $[0]$ of $\mathcal{X}/\mathcal{M}$ is $\mathcal{M}$ — see Example 2.H), and hence $[x] = [0]$. Equivalently,

$$\|[x]\| > 0 \quad \text{ whenever } \quad [x] \ne 0.$$

Absolute homogeneity and subadditivity (i.e., the triangle inequality) are also easily verified. Recall that (Example 2.H)

$$\alpha[x] = [\alpha x] \quad \text{ and } \quad [x] + [y] = [x + y]$$

for every $[x], [y]$ in $\mathcal{X}/\mathcal{M}$ and every scalar α. Since $\mathcal{M}$ is a linear manifold, it follows by absolute homogeneity and subadditivity of the norm $\|\ \|_X$ that

$$\begin{aligned}\|\alpha[x]\| &= \|[\alpha x]\| = \inf_{u\in\mathcal{M}} \|\alpha x + u\|_X = \inf_{u\in\mathcal{M}} \|\alpha x + \alpha u\|_X \\ &= |\alpha| \inf_{u\in\mathcal{M}} \|x + u\|_X = |\alpha| \, \|[x]\|,\end{aligned}$$

$$\begin{aligned}\|[x]+[y]\| &= \|[x+y]\| = \inf_{u\in\mathcal{M}} \|x+y+u\|_X = \inf_{u\in\mathcal{M}} \|x+u+y+u\|_X \\ &\le \inf_{u\in\mathcal{M}} \|x+u\|_X + \inf_{u\in\mathcal{M}} \|y+u\|_X = \|[x]\| + \|[y]\|,\end{aligned}$$

for every $[x], [y]$ in $\mathcal{X}/\mathcal{M}$ and every scalar α. Such a norm is referred to as the *quotient norm*. When a quotient space $\mathcal{X}/\mathcal{M}$ (of a *normed* space $\mathcal{X}$ modulo a *subspace* $\mathcal{M}$) is regarded as a normed space, it is this quotient norm that is assumed to equip it. Since $\inf_{u\in\mathcal{M}}\|x+u\|_X \le \|x\|_X + \inf_{u\in\mathcal{M}}\|u\|_X = \|x\|_X$,

$$\|[x]\| \le \|x\|_X$$

for every $[x] \in \mathcal{X}/\mathcal{M}$ and every $x \in [x]$. Thus

$$\|[x]-[y]\| = \|[x-y]\| \le \|x-y\|_X$$

for every $x, y \in \mathcal{X}$. Thus the natural mapping π of $\mathcal{X}$ onto $\mathcal{X}/\mathcal{M}$ (defined by $\pi(x) = [x]$ for every $x \in \mathcal{X}$) is uniformly continuous (a contraction, actually), and so it preserves convergence and Cauchy sequences (Corollary 3.8 and Lemma 3.43). *If* $\{x_n\}$ *converges in* $\mathcal{X}$ *to* $x \in \mathcal{X}$*, then* $\{[x_n]\}$ *converges in* $\mathcal{X}/\mathcal{M}$ *to* $[x] \in \mathcal{X}/\mathcal{M}$, and *if* $\{x_n\}$ *is a Cauchy sequence in* $\mathcal{X}$*, then* $\{[x_n]\}$ *is a Cauchy sequence in* $\mathcal{X}/\mathcal{M}$.

Proposition 4.10. *If* $\mathcal{M}$ *is a subspace of a Banach space* $\mathcal{X}$*, then the quotient space* $\mathcal{X}/\mathcal{M}$ *is a Banach space.*

Proof. Let $\mathcal{M}$ be a subspace of a normed space $(\mathcal{X}, \|\ \|_X)$. Consider the quotient space $\mathcal{X}/\mathcal{M}$ equipped with the quotient norm $\|\ \|$ and let $\{[x_n]\}_{n=0}^{\infty}$ be an arbitrary Cauchy sequence in $\mathcal{X}/\mathcal{M}$. According to Problem 3.52(c), $\{[x_n]\}_{n=0}^{\infty}$ has a subsequence $\{[x_{n_k}]\}_{k=0}^{\infty}$ of bounded variation so that

$$\sum_{k=0}^{\infty} \|[x_{n_{k+1}} - x_{n_k}]\| = \sum_{k=0}^{\infty} \|[x_{n_{k+1}}] - [x_{n_k}]\| < \infty.$$

If $x \in \mathcal{X}$, then there exists an $\varepsilon > 0$ and a vector $u_{\varepsilon,x} \in \mathcal{M}$ such that

$$\|x + u_{\varepsilon,x}\|_X \le \inf_{u\in\mathcal{M}} \|x+u\|_X + \varepsilon = \|[x]\| + \varepsilon.$$

In particular, for each integer $k \ge 0$ there exists a vector $u_k \in \mathcal{M}$ such that

$$\|(x_{n_{k+1}} - x_{n_k}) + u_k\|_X \le \|[x_{n_{k+1}} - x_{n_k}]\| + (\tfrac{1}{2})^k.$$

Therefore, as $\sum_{k=0}^{\infty}(\frac{1}{2})^k < \infty$,

$$\sum_{k=0}^{\infty} \|(x_{n_{k+1}} - x_{n_k}) + u_k\|_X < \infty.$$

In words, the $\mathcal{X}$-valued sequence $\{(x_{n_{k+1}} - x_{n_k}) + u_k\}_{k=0}^{\infty}$ is absolutely summable. Take the sequence $\{y_k\}_{k=0}^{\infty}$ in $\mathcal{X}$ of partial sums of $\{(x_{n_{k+1}} - x_{n_k}) + u_k\}_{k=0}^{\infty}$. A trivial induction shows that

$$y_k = \sum_{i=0}^{k} (x_{n_{i+1}} - x_{n_i}) + u_i = (x_{n_{k+1}} - x_{n_0}) + \sum_{i=0}^{k} u_i$$

for each $k \geq 0$. If $\mathcal{X}$ is a Banach space, then the absolutely summable sequence $\{(x_{n_{k+1}} - x_{n_k}) + u_k\}_{k=0}^{\infty}$ is summable (cf. Proposition 4.4), which means that $\{y_k\}_{k=0}^{\infty}$ converges in $\mathcal{X}$. Thus $\{[y_k]\}_{k=0}^{\infty}$ converges in $\mathcal{X}/\mathcal{M}$ (since the natural mapping of $\mathcal{X}$ onto $\mathcal{X}/\mathcal{M}$, $x \mapsto [x]$, is continuous). But $[x_{n_{k+1}}] - [x_{n_0}] = [x_{n_{k+1}} - x_{n_0}] = [y_k]$ for each $k \geq 0$ (because $\sum_{i=0}^{k} u_i$ lies in $\mathcal{M}$ for every $k \geq 0$). Then the subsequence $\{[x_{n_k}]\}_{k=0}^{\infty}$ converges in $\mathcal{X}/\mathcal{M}$, and so the Cauchy sequence $\{[x_n]\}_{n=0}^{\infty}$ converges in $\mathcal{X}/\mathcal{M}$ (Proposition 3.39(c)). Conclusion: Every Cauchy sequence in $\mathcal{X}/\mathcal{M}$ converges in $\mathcal{X}/\mathcal{M}$; that is, $\mathcal{X}/\mathcal{M}$ is Banach. □

4.4 Bounded Linear Transformations

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. A *continuous linear transformation* of $\mathcal{X}$ into $\mathcal{Y}$ is a linear transformation of the linear space $\mathcal{X}$ into the linear space $\mathcal{Y}$ that is continuous with respect to the norm topologies of $\mathcal{X}$ and $\mathcal{Y}$. (Note that $\mathcal{X}$ and $\mathcal{Y}$ are necessarily linear spaces over the same scalar field.) This is the most important concept that results from the combination of algebraic and topological structures.

Definition 4.11. A linear transformation T of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ is *bounded* if there exists a constant $\beta \geq 0$ such that

$$\|Tx\| \leq \beta \|x\|$$

for every $x \in \mathcal{X}$. (The norm on the left-hand side is the norm on $\mathcal{Y}$ and that on the right-hand side is the norm on $\mathcal{X}$.)

Proposition 4.12. *A linear transformation of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ is bounded if and only if it maps bounded subsets of $\mathcal{X}$ into bounded subsets of $\mathcal{Y}$.*

Proof. Let A be a bounded subset of $\mathcal{X}$ so that $\sup_{a \in A} \|a\| < \infty$ (Problem 4.5). If T is bounded, then there exists a real number $\beta > 0$ such that $\|Tx\| \leq \beta \|x\|$ for every $x \in \mathcal{X}$. Therefore

$$\sup_{y \in T(A)} \|y\| = \sup_{a \in A} \|Ta\| \leq \beta \sup_{a \in A} \|a\| < \infty,$$

and so $T(A)$ is bounded in $\mathcal{Y}$. Conversely, suppose $T(A)$ is bounded in $\mathcal{Y}$ for every bounded set A in $\mathcal{X}$. In particular, $T(B_1[0])$ is bounded in $\mathcal{Y}$: the closed unit ball centered at the origin of $\mathcal{X}$, $B_1[0]$, is certainly bounded in $\mathcal{X}$. Thus

$$\sup_{\|b\|\le 1} \|Tb\| = \sup_{b\in B_1[0]} \|Tb\| = \sup_{y\in T(B_1[0])} \|y\| < \infty.$$

Take an arbitrary nonzero x in $\mathcal{X}$. Since $\|\frac{x}{\|x\|}\| = 1$, it follows that

$$\|Tx\| = \|x\| \big\|T\big(\tfrac{x}{\|x\|}\big)\big\| \le \sup_{\|b\|\le 1} \|Tb\| \, \|x\|$$

for every $0 \ne x \in \mathcal{X}$, and hence T is bounded (since the inequality $\|Tx\| \le \sup_{\|b\|\le 1}\|Tb\|\,\|x\|$ holds trivially for $x = 0$). □

The next elementary result is extremely important.

Proposition 4.13. *Let T be a bounded linear transformation of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$. The null space $\mathcal{N}(T)$ of T is a subspace of $\mathcal{X}$.*

Proof. The null space (or kernel) of T,

$$\mathcal{N}(T) = \big\{x \in \mathcal{X}\colon\ Tx = 0\big\} = T^{-1}(\{0\}),$$

is a linear manifold of $\mathcal{X}$ (Problem 2.10). The Closed Set Theorem (Theorem 3.30) ensures that it is closed in $\mathcal{X}$. Indeed, if $\{x_n\}$ is an $\mathcal{N}(T)$-valued sequence (i.e., $Tx_n = 0$ for every n) that converges in $\mathcal{X}$ to $x \in \mathcal{X}$, then

$$0 \le \|Tx\| = \|Tx_n - Tx\| = \|T(x_n - x)\| \le \beta\|x_n - x\| \to 0$$

as $n \to \infty$ (because T is linear and bounded), and hence $x \in \mathcal{N}(T)$. □

Theorem 4.14. *Let T be a linear transformation of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$. The following assertions are pairwise equivalent.*

(a) *T is bounded.*

(b) *T is Lipschitzian.*

(c) *T is uniformly continuous.*

(d) *T is continuous.*

(e) *T is continuous at some point x_0 of $\mathcal{X}$.*

(f) *T is continuous at the origin $0 \in \mathcal{X}$.*

Proof. Let $T\colon \mathcal{X} \to \mathcal{Y}$ be a linear transformation. If T is bounded, then there exists $\beta \ge 0$ such that, for every $x_1, x_2 \in \mathcal{X}$,

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \le \beta\|x_1 - x_2\|,$$

and hence (a)⇒(b). Recall that (b)⇒(c)⇒(d)⇒(e) trivially. Now suppose T is continuous at $x_0 \in \mathcal{X}$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|Tx - T0\| = \|Tx\| = \|T(x + x_0) - Tx_0\| < \varepsilon$ whenever $\|x - 0\| = \|x\| = \|(x + x_0) - x_0\| < \delta$, and so T is continuous at $0 \in \mathcal{X}$. Therefore (e)⇒(f). Next suppose T is continuous at $0 \in \mathcal{X}$. Thus (for $\varepsilon = 1$) there exists a $\delta > 0$ such that $\|Tx\| = \|Tx - T0\| < 1$ whenever $\|x\| = \|x - 0\| < \delta$. Since $\|\frac{\delta}{2\|x\|}x\| < \delta$ for every nonzero x in $\mathcal{X}$,

$$\|Tx\| = \tfrac{2\|x\|}{\delta}\Big\|\tfrac{\delta}{2\|x\|}Tx\Big\| = \tfrac{2\|x\|}{\delta}\Big\|T\big(\tfrac{\delta}{2\|x\|}x\big)\Big\| < \tfrac{2}{\delta}\|x\|$$

for every $0 \neq x \in \mathcal{X}$, and hence T is bounded. Thus (f)⇒(a). □

Observe from Theorem 4.14 that the terms "*bounded linear transformation*" and "*continuous linear transformation*" are synonyms, and as such we shall use them interchangeably. If $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces (over the same field $\mathbb{F}$), then the collection of all bounded linear transformations of $\mathcal{X}$ into $\mathcal{Y}$ will be denoted by $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$. It is easy to verify (triangle inequality and homogeneity for the norm on $\mathcal{Y}$) that $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is a linear manifold of the linear space $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$ of all linear transformations of $\mathcal{X}$ into $\mathcal{Y}$ (see Section 2.5), and so $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ *is a linear space over* $\mathbb{F}$. The origin of $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is the *null transformation*, denoted by O ($Ox = 0$ for every $x \in \mathcal{X}$). For each $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ set

$$\|T\| = \inf\big\{\beta \geq 0\colon\ \|Tx\| \leq \beta\|x\| \text{ for every } x \in \mathcal{X}\big\}.$$

(Recall: If $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$, then there is a $\beta \geq 0$ such that $\|Tx\| \leq \beta\|x\|$ for every $x \in \mathcal{X}$, and hence the nonnegative number $\|T\|$ exists for every $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$.) If x is any vector in $\mathcal{X}$, then $\|Tx\| \leq (\|T\| + \varepsilon)\|x\|$ for every $\varepsilon > 0$ so that $\|Tx\| \leq \inf_{\varepsilon > 0}(\|T\| + \varepsilon)\|x\|$. Thus

$$\|Tx\| \leq \|T\|\,\|x\| \quad \text{for every} \quad x \in \mathcal{X},$$

and hence

$$\|T\| = \min\big\{\beta \geq 0\colon\ \|Tx\| \leq \beta\|x\| \text{ for every } x \in \mathcal{X}\big\}.$$

Therefore

$$\|T\| \geq 0 \quad \text{and} \quad \|T\| = 0 \text{ if and only if } T = O.$$

Moreover, for any α in $\mathbb{F}$ and any S in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ we get $\|(\alpha T)x\| = \|\alpha(Tx)\| = |\alpha|\|Tx\|$ and $\|(T + S)x\| = \|Tx + Sx\| \leq \|Tx\| + \|Sx\| \leq (\|T\| + \|S\|)\|x\|$, for every $x \in \mathcal{X}$. This implies that

$$\|\alpha T\| = |\alpha|\|T\| \quad \text{and} \quad \|T + S\| \leq \|T\| + \|S\|$$

for every $T, S \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ and every scalar $\alpha \in \mathbb{F}$. Conclusion: The function $\mathcal{B}[\mathcal{X}, \mathcal{Y}] \to \mathbb{R}$ defined by $T \mapsto \|T\|$ is a norm on the linear space $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$. Thus $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ *is a normed space*, and $\|T\|$ is referred to as the *norm* of $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$,

or the *usual norm* on $\mathcal{B}[\mathcal{X},\mathcal{Y}]$, or yet as the *induced uniform norm* on $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ — the norm $\|T\|$ on $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ is induced by the norm $\|Tx\|$ on $\mathcal{Y}$ and the norm $\|x\|$ on $\mathcal{X}$. This is the norm that will be assumed to equip $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ whenever $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ is regarded as a normed space. The reader is invited to show that

$$\|T\| = \sup_{\|x\|\le 1} \|Tx\| = \sup_{\|x\|<1} \|Tx\| = \sup_{\|x\|=1} \|Tx\| = \sup_{x\neq 0} \tfrac{\|Tx\|}{\|x\|},$$

where the last two expressions make sense only if $\mathcal{X} \neq \{0\}$.

A *contraction* in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ is a bounded linear transformation $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ such that $\|T\| \le 1$. Clearly,

$$\|T\| \le 1 \quad \Longleftrightarrow \quad \|Tx\| \le \|x\| \ \text{ for every } \ x \in \mathcal{X}.$$

If $\mathcal{X} \neq \{0\}$, then a transformation $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ is a contraction if and only if $\sup_{x\neq 0}(\|Tx\|/\|x\|) \le 1$. A *strict contraction* in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ is a bounded linear transformation $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ such that $\|T\| < 1$. It is obvious that every strict contraction is a contraction. If $\mathcal{X} \neq \{0\}$, then a transformation $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ is a strict contraction if and only if $\sup_{x\neq 0}(\|Tx\|/\|x\|) < 1$. Carefully note that, if $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ and $\mathcal{X} \neq \{0\}$, then

$$\|T\| < 1 \quad \Longrightarrow \quad \|Tx\| < \|x\| \ \text{ for every } \ 0 \neq x \in \mathcal{X} \quad \Longrightarrow \quad \|T\| \le 1.$$

A transformation that satisfies the middle inequality is called a *proper contraction*. Every strict contraction is a proper contraction, which in turn is a (plain) contraction. The converses to the above one-way implications fail in general. It is clear that there exist contractions that are not proper (the identity, for instance). The next example exhibits a proper contraction that is not strict. A *nonstrict contraction* is precisely a bounded linear transformation $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ with norm 1 (i.e., $\|T\| \le 1$ and $\|T\| \not< 1$ if and only if $\|T\| = 1$).

Example 4.H. Let ℓ^p_+ be the Banach space of Example 4.B for some $p \ge 1$. We shall drop the subscript "p" from the norm $\|\ \|_p$ on ℓ^p_+ and write simply $\|\ \|$. Now consider the *diagonal mapping* $D_a\colon \ell^p_+ \to \ell^p_+$ of Problem 3.22 for some bounded sequence $a = \{\alpha_k\}_{k=0}^{\infty} \in \ell^\infty_+$:

$$D_a x = \{\alpha_k \xi_k\}_{k=0}^{\infty} \quad \text{for every} \quad x = \{\xi_k\}_{k=0}^{\infty} \in \ell^p_+$$

(i.e., $D_a(\xi_0, \xi_1, \xi_2, \ldots) = (\alpha_0\xi_0, \alpha_1\xi_1, \alpha_2\xi_2, \ldots)$). Common notation for a diagonal mapping of ℓ^p_+ into itself includes the following usual representation as an *infinite diagonal matrix*:

$$D_a = \operatorname{diag}(\{\alpha_k\}_{k=0}^{\infty}) = \operatorname{diag}(\alpha_0, \alpha_1, \alpha_2, \ldots) = \begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \alpha_2 & \\ & & & \ddots \end{pmatrix}.$$

This bounded linear transformation is called a *diagonal operator*. Linearity is trivially verified. Boundedness (i.e., continuity) follows from Problem 3.22(a). Indeed, if $a = \{\alpha_k\}_{k=0}^{\infty}$ is a bounded sequence (i.e., a sequence in ℓ^∞_+), then

$$\|D_a x\| \;=\; \Big(\sum_{k=0}^{\infty} |\alpha_k \xi_k|^p\Big)^{\frac{1}{p}} \;\le\; \sup_k |\alpha_k| \Big(\sum_{k=0}^{\infty} |\xi_k|^p\Big)^{\frac{1}{p}} \;=\; \sup_k |\alpha_k|\, \|x\|$$

for every $x = \{\xi_k\}_{k=0}^{\infty} \in \ell_+^p$, so that $\|D_a\| \le \sup_k |\alpha_k| = \|a\|_\infty$. On the other hand, consider the ℓ_+^p-valued sequence $\{e_j\}_{j=0}^{\infty}$ where, for each $j \ge 0$, e_j is a scalar-valued sequence with just one nonzero entry (equal to 1) at the jth position (i.e., $e_j = \{\delta_{jk}\}_{k=0}^{\infty}$ for every $j \ge 0$, with $\delta_{jk} = 1$ if $k = j$ and $\delta_{jk} = 0$ if $k \ne j$). Since $D_a e_j = \alpha_j e_j$ and $\|e_j\| = 1$, it follows that $\|D_a e_j\| = |\alpha_j|$ and hence $\|D_a\| = \sup_{\|x\|=1} \|D_a x\| \ge \|D_a e_j\| = |\alpha_j|$, for every $j \ge 0$. Thus $\|D_a\| \ge \sup_j |\alpha_j| = \|a\|_\infty$. Therefore,

$$\|D_a\| \;=\; \|a\|_\infty.$$

If $a = \{\alpha_k\}_{k=0}^{\infty}$ is a constant sequence, say $\alpha_k = \alpha$ for all $k \ge 0$, then $D_a = \operatorname{diag}(\alpha, \alpha, \alpha, \ldots) = \alpha I$, a multiple of the identity I on ℓ_+^p. In this case D_a is called a *scalar operator* (see Problem 4.19). It is clear that $\|\alpha I x\| = |\alpha|\, \|x\|$ for every $x \in \ell_+^p$ and $\|\alpha I\| = |\alpha|$. If $a = \{\alpha_k\}_{k=0}^{\infty}$ is the increasing sequence with $\alpha_k = \frac{k+1}{k+2}$ for every $k \ge 0$, then $D_a = \operatorname{diag}\big(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\big)$ is a nonstrict proper contraction: $\|D_a x\| < \|x\|$ for every $0 \ne x \in \ell_+^p$ and $\|D_a\| = 1$.

Proposition 4.15. *$\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is a Banach space if $\mathcal{Y}$ is a Banach space.*

Proof. Let $\{T_n\}$ be an arbitrary Cauchy sequence in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$. Thus, for each $x \in \mathcal{X}$, the sequence $\{T_n x\}$ is a Cauchy sequence in $\mathcal{Y}$ (reason: $\|T_m x - T_n x\| \le \|T_m - T_n\|\, \|x\|$ for every $x \in \mathcal{X}$ and every pair of indices m and n). Since $\mathcal{Y}$ is complete, $\{T_n x\}$ converges in $\mathcal{Y}$. Then for each $x \in \mathcal{X}$ there is a vector $y_x \in \mathcal{Y}$ such that $T_n x \to y_x$ in $\mathcal{Y}$. Let T be the mapping that assigns to each x in $\mathcal{X}$ this (unique) limit vector y_x in $\mathcal{Y}$; that is, $Tx = \lim_n T_n x$ for every $x \in \mathcal{X}$.

Claim 1. $T\colon \mathcal{X} \to \mathcal{Y}$ is linear.

Proof. Since T_n is linear for each n, and since the linear operations are continuous (Problem 4.1), it follows that

$$\begin{aligned} T(\alpha_1 x_1 + \alpha_2 x_2) \;&=\; \lim_n T_n(\alpha_1 x_1 + \alpha_2 x_2) \;=\; \lim_n (T_n \alpha_1 x_1 + T_n \alpha_2 x_2) \\ &=\; \alpha_1 \lim_n T_n x_1 + \alpha_2 \lim_n T_n x_2 \;=\; \alpha_1 T x_1 + \alpha_2 T x_2 \end{aligned}$$

for every $x_1, x_2 \in \mathcal{X}$ and every pair of scalars α_1 and α_2. □

Claim 2. $T\colon \mathcal{X} \to \mathcal{Y}$ is bounded.

Proof. Take an arbitrary real number $\varepsilon > 0$. Since $\{T_n\}$ is a Cauchy sequence in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$, it follows that there exists an integer $n_\varepsilon \ge 1$ such that $\|T_n - T_m\| < \varepsilon$ whenever $m, n \ge n_\varepsilon$. Thus $\|T_n x - T_m x\| \le \|T_n - T_m\|\, \|x\| < \varepsilon \|x\|$ for every $x \in \mathcal{X}$ and every $m, n \ge n_\varepsilon$. Therefore,

$$\begin{aligned} \|(T_n - T)x\| \;&=\; \|T_n x - Tx\| \;=\; \big\|T_n x - \lim_m T_m x\big\| \\ &=\; \big\|\lim_m (T_n x - T_m x)\big\| \;=\; \lim_m \big\|T_n x - T_m x\big\| \;\le\; \varepsilon \|x\| \end{aligned}$$

for every $x \in \mathcal{X}$ and every $n \geq n_\varepsilon$ (reason: every norm is continuous — see Corollary 3.8). Hence the linear transformation $T_n - T$ is bounded for each $n \geq n_\varepsilon$, and so is $T = T_n - (T_n - T)$. □

Claim 3. $T_n \to T$ in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$.

Proof. For every $\varepsilon > 0$ there exists an integer $n_\varepsilon \geq 1$ such that

$$n \geq n_\varepsilon \quad \text{implies} \quad \|T_n - T\| = \sup_{\|x\| \leq 1} \|(T_n - T)x\| < \varepsilon$$

according to the above displayed inequality. □

Conclusion: Every Cauchy sequence $\{T_n\}$ in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ converges in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ to $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$, which means that the normed space $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is complete (with respect to the induced uniform norm topology of $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$). That is, $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is a Banach space. □

Comparing the preceding proposition with Example 4.G, we might expect that, if $\mathcal{X} \neq \{0\}$, then $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is a Banach space if and only if $\mathcal{Y}$ is a Banach space. This indeed is the case, and we shall prove the converse of Proposition 4.15 in Section 4.10. Recall that bounded linear transformations can be multiplied (where multiplication means composition). The resulting transformation is linear as well as bounded.

Proposition 4.16. *Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be normed spaces over the same scalar field. If $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ and $S \in \mathcal{B}[\mathcal{Y}, \mathcal{Z}]$, then $ST \in \mathcal{B}[\mathcal{X}, \mathcal{Z}]$ and*

$$\|ST\| \leq \|S\| \, \|T\|.$$

Proof. The composition $ST\colon \mathcal{X} \to \mathcal{Z}$ is linear (Problem 2.15) and $\|STx\| \leq \|S\| \|Tx\| \leq \|S\| \|T\| \|x\|$ for every x in $\mathcal{X}$. □

The above inequality is a rather important additional property shared by the (induced uniform) norm of bounded linear transformations. We shall refer to it as the *operator norm property*.

Let $\mathcal{X}$ be a normed space and set $\mathcal{B}[\mathcal{X}] = \mathcal{B}[\mathcal{X}, \mathcal{X}]$ for short. The elements of $\mathcal{B}[\mathcal{X}]$ are called *operators*. In other words, by an *operator* (or a *bounded linear operator*) we simply mean a bounded linear transformation of a normed space $\mathcal{X}$ into itself, so that $\mathcal{B}[\mathcal{X}]$ is the normed space of all operators on $\mathcal{X}$.

Example 4.I. Let $\{(\mathcal{X}_k, \| \ \|_k)\}_{k \in \mathbb{I}}$ be a countable (either finite or infinite) indexed family of normed spaces (over the same scalar field), and consider the direct sum $\bigoplus_{k \in \mathbb{I}} \mathcal{X}_k$ of the linear spaces $\{\mathcal{X}_k\}_{k \in \mathbb{I}}$. (To avoid trivialities, assume that each $\mathcal{X}_k$ is nonzero.) Equip the linear space $\bigoplus_{k \in \mathbb{I}} \mathcal{X}_k$ with any of the norms $\| \ \|_\infty$ or $\| \ \|_p$ (for any $p \geq 1$) as in Examples 4.E and 4.F. If the index set $\mathbb{I}$ is infinite, then write $\big(\bigoplus_{k \in \mathbb{I}} \mathcal{X}_k, \| \ \|_\infty\big)$ for the normed space $\big(\big[\bigoplus_{k \in \mathbb{I}} \mathcal{X}_k\big]_\infty, \| \ \|_\infty\big)$

and $\big(\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k, \|\ \|_p\big)$ for the normed space $\big(\big[\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k\big]_p, \|\ \|_p\big)$. For each $i \in \mathbb{I}$ define a mapping $P_i\colon \bigoplus_{k\in\mathbb{I}}\mathcal{X}_k \to \bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$ by

$$P_i x \;=\; x_i \quad \text{for every} \quad x = \{x_k\}_{k\in\mathbb{I}} \in \bigoplus\nolimits_{k\in\mathbb{I}}\mathcal{X}_k,$$

where we are identifying each vector $x_i \in \mathcal{X}_i$ with the indexed family $\{x_{i(k)}\}_{k\in\mathbb{I}}$ in $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$ that has just one nonzero entry (equal to x_i) at the ith position (i.e., set $x_{i(k)} = \delta_{ik}x_k$ so that $x_{i(k)} = 0_k \in \mathcal{X}_k$, the origin of $\mathcal{X}_k$, if $k \neq i$ and $x_{i(i)} = x_i$). Briefly, we are writing x_i for $(0_1, \ldots, 0_{i-1}, x_i, 0_{i+1}, \ldots)$. Each P_i is a linear transformation. Indeed,

$$\begin{aligned} P_i(\alpha u \oplus \beta v) \;&=\; P_i\big(\alpha\{u_k\}_{k\in\mathbb{I}} \oplus \beta\{v_k\}_{k\in\mathbb{I}}\big) \\ &=\; P_i\big(\{\alpha u_k + \beta v_k\}_{k\in\mathbb{I}}\big) \;=\; \alpha u_i + \beta v_i \;=\; \alpha P_i u \oplus \beta P_i v \end{aligned}$$

for every $u = \{u_k\}_{k\in\mathbb{I}}$ and $v = \{v_k\}_{k\in\mathbb{I}}$ in $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$ and every pair of scalars $\{\alpha, \beta\}$. Moreover,

$$\mathcal{R}(P_i) \;=\; \mathcal{X}_i$$

if we identify each normed space $\mathcal{X}_i$ with the linear manifold $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_{i(k)}$ of $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$, such that $\mathcal{X}_{i(k)} = \{0_k\}$ for $k \neq i$ and $\mathcal{X}_{i(i)} = \mathcal{X}_i$, equipped with any of the norms $\|\ \|_\infty$ or $\|\ \|_p$. (To identify $\mathcal{X}_i$ with $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_{i(k)}$ simply means that these normed spaces are *isometrically isomorphic*, a concept that will be introduced in Section 4.7.) Note that $\{x_{i(k)}\}_{k\in\mathbb{I}}$ lies in $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_{i(k)}$ for every x_i in $\mathcal{X}_i$ and $\|x_i\|_i = \|\{x_{i(k)}\}_{k\in\mathbb{I}}\|_\infty = \|\{x_{i(k)}\}_{k\in\mathbb{I}}\|_p$ for any $p \geq 1$. Also note that each P_i is a projection (i.e., an idempotent linear transformation):

$$P_i^2 x \;=\; P_i\{x_{i(k)}\}_{k\in\mathbb{I}} \;=\; \{x_{i(k)}\}_{k\in\mathbb{I}} \;=\; P_i x$$

for every $x = \{x_k\}_{k\in\mathbb{I}}$ in $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$ so that $P_i = P_i^2$ for each $i \in \mathbb{I}$. In fact, each P_i is a continuous projection. To show that $P_i\colon \bigoplus_{k\in\mathbb{I}}\mathcal{X}_k \to \bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$ is continuous, when $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$ is equipped with any of the norms $\|\ \|_\infty$ or $\|\ \|_p$, note that

$$\|P_i x\|_\infty \;=\; \|\{x_{i(k)}\}_{k\in\mathbb{I}}\|_\infty \;=\; \|x_i\|_i \;\leq\; \sup_{k\in\mathbb{I}} \|x_k\|_i \;=\; \|x\|_\infty$$

for every $x = \{x_k\}_{i\in\mathbb{I}}$ in $\big(\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k, \|\ \|_\infty\big)$, and

$$\|P_i x\|_p^p \;=\; \|\{x_{i(k)}\}_{k\in\mathbb{I}}\|_p^p \;=\; \|x_i\|_i^p \;\leq\; \sum_{k\in\mathbb{I}} \|x_k\|_k^p \;=\; \|x\|_p^p$$

for every $x = \{x_k\}_{k\in\mathbb{I}}$ in $\big(\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k, \|\ \|_p\big)$. Thus P_i is a bounded linear transformation, and hence continuous (Theorem 4.14). In other words, each P_i is a *projection operator* in $\mathcal{B}[\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k]$. Actually, if we use the same symbols $\|\ \|_\infty$ and $\|\ \|_p$ to denote the induced uniform norms on $\mathcal{B}[\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k]$ whenever $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$ is equipped with either $\|\ \|_\infty$ or $\|\ \|_p$, respectively, then the above inequalities ensure that $\|P_i\|_\infty \leq 1$ and $\|P_i\|_p \leq 1$. Thus each P_i is a *contraction* with respect to any of the norms $\|\ \|_\infty$ or $\|\ \|_p$. On the other hand, since $P_i = P_i^2$, it

follows that $\|P_i\|_\infty = \|P_i^2\|_\infty \le \|P_i\|_\infty^2$ and $\|P_i\|_p = \|P_i^2\|_p \le \|P_i\|_p^2$ (operator norm property). Hence $1 \le \|P_i\|_\infty$ and $1 \le \|P_i\|_p$ (since $P_i \ne O$) so that

$$\|P_i\|_\infty \,=\, \|P_i\|_p \,=\, 1.$$

Summing up: If $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$ is equipped with any of the norms $\|\ \|_\infty$ or $\|\ \|_p$ (for any $p \ge 1$), then $\{P_i\}_{i\in\mathbb{I}}$ is a family of projections in $\mathcal{B}[\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k]$ with unit norm. When we identify $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_{i(k)} = \mathcal{R}(P_i)$ with $\mathcal{X}_i$, as we actually did, then each mapping $P_i\colon \bigoplus_{k\in\mathbb{I}}\mathcal{X}_k \to \bigoplus_{k\in\mathbb{I}}\mathcal{X}_{i(k)} \subseteq \bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$ can be viewed as a function from $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$ onto $\mathcal{X}_i$, and hence we write $P_i\colon \bigoplus_{k\in\mathbb{I}}\mathcal{X}_k \to \mathcal{X}_i$. This is called the *natural projection* of $\bigoplus_{k\in\mathbb{I}}\mathcal{X}_k$ onto $\mathcal{X}_i$.

Definition 4.17. A normed space $\mathcal{A}$ that is also an algebra with respect to a product operation $\mathcal{A}\times\mathcal{A} \to \mathcal{A}$ (see Problem 2.30), say $(A, B) \mapsto AB$, such that $\|AB\| \le \|A\|\,\|B\|$ is called a *normed algebra*. A *Banach algebra* is a normed algebra that is complete as a normed space. If a normed algebra possesses an identity I such that $\|I\| = 1$, then it is a *unital normed algebra* (or a *normed algebra with identity*). If a unital normed algebra is complete as a normed space, then it is a *unital Banach algebra* (or a *Banach algebra with identity*).

Let $\mathcal{L}[\mathcal{X}] = \mathcal{L}[\mathcal{X}, \mathcal{X}]$ denote the linear space of all linear transformations on $\mathcal{X}$ (i.e., the linear space of all linear transformations of a linear space $\mathcal{X}$ into itself — see Section 2.5). We have already seen that $\mathcal{B}[\mathcal{X}]$ is a linear manifold of $\mathcal{L}[\mathcal{X}]$. If $\mathcal{X} \ne \{0\}$, then $\mathcal{L}[\mathcal{X}]$ is much more than a mere linear space; it is an algebra with identity, where the product in $\mathcal{L}[\mathcal{X}]$ is interpreted as composition (Problem 2.30). The operator norm property ensures that $\mathcal{B}[\mathcal{X}]$ is a subalgebra (in the algebraic sense) of $\mathcal{L}[\mathcal{X}]$, and hence an algebra in its own right. Moreover, the operator norm property also ensures that $\mathcal{B}[\mathcal{X}]$ in fact is a normed algebra. Let I denote the *identity operator* in $\mathcal{B}[\mathcal{X}]$ (i.e., $Ix = x$ for every $x \in \mathcal{X}$), which is the identity of the algebra $\mathcal{B}[\mathcal{X}]$. Of course, $\|I\| = 1$ by the very definition of norm (the induced uniform norm, that is) on $\mathcal{B}[\mathcal{X}]$. Thus $\mathcal{B}[\mathcal{X}]$ *is a normed algebra with identity* and, according to Proposition 4.15,

$\mathcal{B}[\mathcal{X}]$ is a Banach algebra with identity if $\mathcal{X} \ne \{0\}$ is a Banach space.

We know from Problem 4.1 that addition and scalar multiplication in $\mathcal{B}[\mathcal{X}]$ are continuous (as they are in any normed space). The operator norm property allows us to conclude that multiplication is continuous too. Indeed, if $\{S_n\}$ and $\{T_n\}$ are $\mathcal{B}[\mathcal{X}]$-valued sequences that converge in $\mathcal{B}[\mathcal{X}]$ to $S \in \mathcal{B}[\mathcal{X}]$ and to $T \in \mathcal{B}[\mathcal{X}]$, respectively, then

$$\|S_nT_n - ST\| = \|S_n(T_n - T) + (S_n - S)T\| \le \|S_n\|\,\|T_n - T\| + \|S_n - S\|\,\|T\|$$

for every n. Since $\sup_n\|S_n\| < \infty$ (every convergent sequence is bounded), it follows that $\{S_nT_n\}$ converges in $\mathcal{B}[\mathcal{X}]$ to ST. Thus the product operation $\mathcal{B}[\mathcal{X}]\times\mathcal{B}[\mathcal{X}] \to \mathcal{B}[\mathcal{X}]$ given by $(S, T) \mapsto ST$ is continuous (cf. Corollary 3.8), where the topology on $\mathcal{B}[\mathcal{X}]\times\mathcal{B}[\mathcal{X}]$ is that induced by any of the equivalent metrics of Problems 3.9 and 3.33.

4.5 The Open Mapping Theorem and Continuous Inverse

Recall that a function $F\colon X \to Y$ from a set X to a set Y has an inverse on its range $\mathcal{R}(F)$ if there exists a (unique) function $F^{-1}\colon \mathcal{R}(F) \to X$ (called the *inverse of* F *on* $\mathcal{R}(F)$) such that $F^{-1}F = I_X$, where I_X stands for the identity on X. Moreover, F has an inverse on its range if and only if it is injective. If F is injective and surjective, then it is called *invertible* (in the set-theoretic sense) and $F^{-1}\colon Y \to X$ is the *inverse* of F. Furthermore, $F\colon X \to Y$ is invertible if and only if there exists a unique function $F^{-1}\colon Y \to X$ (the inverse of it) such that $F^{-1}F = I_X$ and $FF^{-1} = I_Y$, where I_Y stands for the identity on Y. (See Section 1.3 and Problems 1.5 through 1.8.) Now let T be a linear transformation of a linear space $\mathcal{X}$ into a linear space $\mathcal{Y}$. Theorems 2.8 and 2.10 say that $\mathcal{N}(T) = \{0\}$ if and only if T has a linear inverse on its range $\mathcal{R}(T)$. That is,

$$\mathcal{N}(T) = \{0\} \iff T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}] \text{ is injective} \iff T^{-1} \in \mathcal{L}[\mathcal{R}(T), \mathcal{X}] \text{ exists.}$$

Clearly, $\mathcal{N}(T) = \{0\}$ if and only if $0 < \|Tx\|$ for every nonzero $x \in \mathcal{X}$ (whenever $\mathcal{Y}$ is a normed space). This leads to the following proposition.

Proposition 4.18. *Take $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$, where $\mathcal{X}$ is a linear space and $\mathcal{Y}$ is a normed space. The following assertions are equivalent.*

(a) *There exists $T^{-1} \in \mathcal{L}[\mathcal{R}(T), \mathcal{X}]$* (i.e., *$T$ has an inverse on its range*).

(b) *$0 < \|Tx\|$ for every nonzero x in $\mathcal{X}$.*

Definition 4.19. A linear transformation T of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ is *bounded below* if there exists a constant $\alpha > 0$ such that

$$\alpha\|x\| \le \|Tx\|$$

for every $x \in \mathcal{X}$ (where the norm on the left-hand side is the norm on $\mathcal{X}$ and that on the right-hand side is the norm on $\mathcal{Y}$).

Note that $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ actually means that $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is *bounded above*. The next result is a continuous version of the previous proposition.

Proposition 4.20. *Take $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$, where $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces. The following assertions are equivalent.*

(a) *There exists $T^{-1} \in \mathcal{B}[\mathcal{R}(T), \mathcal{X}]$* (i.e., *$T$ has a continuous inverse on its range*).

(b) *T is bounded below.*

Moreover, if $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ and if $\mathcal{X}$ is a Banach space, then each of the above equivalent assertions implies that $\mathcal{R}(T)^- = \mathcal{R}(T)$ (i.e., *$\mathcal{R}(T)$ is closed in $\mathcal{Y}$*).

Proof. Suppose T is a linear transformation between normed spaces $\mathcal{X}$ and $\mathcal{Y}$. Recall that, if T has an inverse, then its inverse is linear (Theorem 2.10).

Proof of (a)⇒(b). If (a) holds, then there exists a constant $\beta > 0$ such that $\|T^{-1}y\| \le \beta\|y\|$ for every $y \in \mathcal{R}(T)$ (Definition 4.11). Take an arbitrary $x \in \mathcal{X}$ so that $Tx \in \mathcal{R}(T)$. Thus $\|x\| = \|T^{-1}Tx\| \le \beta\|Tx\|$, and hence $\frac{1}{\beta}\|x\| \le \|Tx\|$.

Proof of (b)⇒(a). If (b) holds, then $0 < \|Tx\|$ for every nonzero x in $\mathcal{X}$, which implies that there exists $T^{-1} \in \mathcal{L}[\mathcal{R}(T), \mathcal{X}]$ by Proposition 4.18. Take any y in $\mathcal{R}(T)$ so that $y = Tx$ for some $x \in \mathcal{X}$. Thus $\|T^{-1}y\| = \|T^{-1}Tx\| = \|x\| \le \frac{1}{\alpha}\|Tx\| = \frac{1}{\alpha}\|y\|$ if (b) holds, and so T^{-1} is bounded (Definition 4.11).

Now take an arbitrary $\mathcal{R}(T)$-valued convergent sequence $\{y_n\}$. Since each y_n is in $\mathcal{R}(T)$, there exists an $\mathcal{X}$-valued sequence $\{x_n\}$ such that $y_n = Tx_n$ for each n. Thus $\{Tx_n\}$ converges in $\mathcal{Y}$, and hence $\{Tx_n\}$ is a Cauchy sequence in $\mathcal{Y}$. If T is bounded below, then there exists $\alpha > 0$ such that

$$0 \le \alpha\|x_m - x_n\| \le \|T(x_m - x_n)\| = \|Tx_m - Tx_n\|$$

for every pair of indices m and n (recall: T is linear) so that $\{x_n\}$ also is a Cauchy sequence (in $\mathcal{X}$), and so it converges in $\mathcal{X}$ to, say, $x \in \mathcal{X}$ (whenever $\mathcal{X}$ is a Banach space). But T is continuous and so $y_n = Tx_n \to Tx$ (Corollary 3.8), which implies that the (unique) limit of $\{y_n\}$ lies in $\mathcal{R}(T)$. Conclusion: $\mathcal{R}(T)$ is closed in $\mathcal{Y}$ by the Closed Set Theorem (Theorem 3.30). □

Example 4.J. Take a sequence $a = \{\alpha_k\}_{k=0}^{\infty} \in \ell_+^\infty$ and consider the diagonal operator $D_a \in \mathcal{B}[\ell_+^p]$ (for some $p \ge 1$) of Example 4.H. If $\alpha_k \ne 0$ for every $k \ge 0$ and $D_a x = 0$ for some $x = \{\xi_k\}_{k=0}^{\infty} \in \ell_+^p$, then $x = 0$ (i.e., if $\alpha_k \ne 0$ and $\alpha_k \xi_k = 0$ for every $k \ge 0$, then $\xi_k = 0$ for every $k \ge 0$). Thus $\alpha_k \ne 0$ for every $k \ge 0$ implies $\mathcal{N}(D_a) = \{0\}$. Conversely, if $\mathcal{N}(D_a) = \{0\}$, then $D_a x \ne 0$ for every nonzero $x \in \ell_+^p$. In particular, $D_a e_j = \alpha_j e_j \ne 0$ so that $\|D_a e_j\| = |\alpha_j| \ne 0$, and hence $\alpha_j \ne 0$, for every $j \ge 0$ (see Example 4.H). Conclusion:

$$\mathcal{N}(D_a) = \{0\} \text{ if and only if } \alpha_k \ne 0 \text{ for every } k \ge 0.$$

Thus, since a linear transformation has a (linear) inverse on its range if and only if its null space is zero (i.e., if and only if it is injective), we may conclude:

$$\text{There exists } D_a^{-1} \in \mathcal{L}[\mathcal{R}(D_a), \ell_+^p] \text{ if and only if } \alpha_k \ne 0 \text{ for every } k \ge 0.$$

In this case, the linear (but not necessarily bounded) transformation D_a^{-1} is again a diagonal mapping,

$$D_a^{-1} y = \{\alpha_k^{-1} \upsilon_k\}_{k=0}^{\infty} \quad \text{for every} \quad y = \{\upsilon_k\}_{k=0}^{\infty} \in \mathcal{R}(D_a) \subseteq \ell_+^p,$$

whose domain $\mathcal{D}(D_a^{-1})$ is precisely the range $\mathcal{R}(D_a)$ of D_a (reason: this mapping is a diagonal, as diagonals were defined in Problem 3.22, and $D_a^{-1} D_a$ is the identity on ℓ_+^p). Recall that $\mathcal{D}(D_a^{-1}) = \ell_+^p$ if $\sup_k |\alpha_k^{-1}| < \infty$ (see Problem 3.22(a)), and hence $\mathcal{R}(D_a) = \ell_+^p$ if $\inf_k |\alpha_k| > 0$ (because $\inf_k |\alpha_k| > 0$ implies $\sup_k |\alpha_k^{-1}| < \infty$). Moreover,

$$\|D_a x\| = \Big(\sum_{k=0}^{\infty} |\alpha_k \xi_k|^p\Big)^{\frac{1}{p}} \geq \inf_k |\alpha_k| \Big(\sum_{k=0}^{\infty} |\xi_k|^p\Big)^{\frac{1}{p}} = \inf_k |\alpha_k| \, \|x\|$$

for every $x = \{\xi_k\}_{k=0}^{\infty} \in \ell_+^p$, so that D_a is bounded below if $\inf_k |\alpha_k| > 0$. Conversely, if D_a is bounded below, then there exists $\alpha > 0$ such that $\alpha \leq \|D_a e_j\| = |\alpha_j|$ for every $j \geq 0$ so that $\inf_j |\alpha_j| \geq \alpha > 0$. Outcome (cf. Proposition 4.20):

$$\text{There exists } D_a^{-1} \in \mathcal{B}[\ell_+^p] \text{ if and only if } \inf_k |\alpha_k| > 0.$$

That is, the inverse of a diagonal operator D_a in $B[\ell_+^p]$ exists and is itself a diagonal operator in $B[\ell_+^p]$ if and only if the bounded sequence $a = \{\alpha_k\}_{k=0}^{\infty}$ is *bounded away from zero* (see Problem 4.5).

If $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$, if $\mathcal{X}$ is a Banach space, and if there exists $T^{-1} \in \mathcal{B}[\mathcal{Y}, \mathcal{X}]$, then $\mathcal{Y}$ is a Banach space. Proof: Theorem 4.14, Lemma 3.43, and Corollary 3.8 (note: this supplies another proof for part of Proposition 4.20). Question: Does the converse hold? That is, if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, and if $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is invertible in the set-theoretic sense (i.e., injective and surjective), does it follow that $T^{-1} \in \mathcal{B}[\mathcal{Y}, \mathcal{X}]$? Yes, it does. This is the Inverse Mapping Theorem. To prove it we need a fundamental result in the theory of bounded linear transformations between Banach spaces, viz., the Open Mapping Theorem. Recall that a function $F: X \to Y$ between metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is an open mapping if $F(U)$ is open in Y whenever U is open in X (Section 3.4). The Open Mapping Theorem says that *every continuous linear transformation of a Banach space onto a Banach space is an open mapping*. That is, *a surjective bounded linear transformation between Banach spaces maps open sets onto open sets*.

Theorem 4.21. (The Open Mapping Theorem). *If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is surjective, then T is an open mapping.*

Proof. Let T be a surjective bounded linear transformation of a Banach space $\mathcal{X}$ onto a Banach space $\mathcal{Y}$. The nonempty open ball with center at the origin of $\mathcal{X}$ and radius ε will be denoted by X_ε, and the nonempty open ball with center at the origin of $\mathcal{Y}$ and radius δ will be denoted by Y_δ.

Claim 1. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $Y_\delta \subseteq T(X_\varepsilon)^-$ (i.e., $T(X_\varepsilon)^-$ is a neighborhood of 0 in $\mathcal{Y}$).

Proof. Since each x in $\mathcal{X}$ belongs to X_n for every integer $n > \|x\|$, it follows that the sequence of open balls $\{X_n\}$ covers $\mathcal{X}$. That is, $\mathcal{X} = \bigcup_{n=1}^{\infty} X_n$, and hence $T(\mathcal{X}) = \bigcup_{n=1}^{\infty} T(X_n)$ (see Problem 1.2(e)). Thus the countable family $\{T(X_n)\}$ of subsets of $\mathcal{Y}$ covers $T(\mathcal{X})$. If $T(\mathcal{X}) = \mathcal{Y}$ and $\mathcal{Y}$ is complete, then the Baire Category Theorem (Theorem 3.58) ensures the existence of a positive integer m such that $T(X_m)^-$ has nonempty interior. Since T is linear, $T(X_\varepsilon) = \frac{\varepsilon}{m} T(X_m)$ and so $T(X_\varepsilon)^- = \frac{\varepsilon}{m} T(X_m)^-$ has nonempty interior for every $\varepsilon > 0$ (because multiplication by a nonzero scalar is a homeomorphism). Take an arbitrary $\varepsilon > 0$ and an arbitrary $y_0 \in [T(X_{\frac{\varepsilon}{2}})^-]^\circ$ so that there exists a nonempty open ball with center at y_0, say $B_\delta(y_0)$, such that $B_\delta(y_0) \subseteq T(X_{\frac{\varepsilon}{2}})^-$.

If y is an arbitrary point of Y_δ, then $\|y + y_0 - y_0\| = \|y\| < \delta$, and therefore $y + y_0 \in B_\delta(y_0)$. Thus both $y + y_0$ and y_0 lie in $T(X_{\frac{\varepsilon}{2}})^-$, which means that

$$\inf_{u \in X_{\frac{\varepsilon}{2}}} \|Tu - y - y_0\| = 0 \quad \text{and} \quad \inf_{v \in X_{\frac{\varepsilon}{2}}} \|Tv - y_0\| = 0.$$

Hence (recall: $u, v \in X_{\frac{\varepsilon}{2}}$ implies $u - v \in X_\varepsilon$),

$$\begin{aligned} \inf_{x \in X_\varepsilon} \|Tx - y\| &\le \inf_{u,v \in X_{\frac{\varepsilon}{2}}} \|T(u - v) - y\| \\ &= \inf_{u,v \in X_{\frac{\varepsilon}{2}}} \|Tu - y - y_0 + y_0 - Tv\| \\ &\le \inf_{u \in X_{\frac{\varepsilon}{2}}} \|Tu - y - y_0\| + \inf_{v \in X_{\frac{\varepsilon}{2}}} \|Tv - y_0\| \,=\, 0, \end{aligned}$$

and so $y \in T(X_\varepsilon)^-$. Conclusion: $Y_\delta \subseteq T(X_\varepsilon)^-$. □

Claim 2. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $Y_\delta \subseteq T(X_\varepsilon)$ — i.e., we may even erase the closure symbol from Claim 1.

Proof. Take an arbitrary $\varepsilon > 0$, and set $\varepsilon_n = \frac{\varepsilon}{2^n}$ for every $n \in \mathbb{N}$. According to Claim 1, for each $n \in \mathbb{N}$ there exists $\delta_n \in (0, \varepsilon_n)$ such that

$$Y_{\delta_n} \subseteq T(X_{\varepsilon_n})^-.$$

If $y_n \in Y_{\delta_n} \subseteq T(X_{\varepsilon_n})^-$ for some $n \in \mathbb{N}$, then $\inf_{x \in X_{\varepsilon_n}} \|y_n - Tx\| = 0$, and so there exists $x_n \in X_n$ such that $\|y_n - Tx_n\| < \delta_{n+1}$. Set $\delta = \delta_1$ and take an arbitrary $y \in Y_\delta$. We claim that there is an $\mathcal{X}$-valued sequence $\{x_n\}$ such that

$$x_n \in X_{\varepsilon_n} \quad \text{and} \quad y - \sum_{i=1}^{n} Tx_i \in Y_{\delta_{n+1}}$$

for every $n \in \mathbb{N}$. Indeed, $y \in Y_{\delta_1}$ implies that there exists $x_1 \in X_{\varepsilon_1}$ such that $\|y - Tx_1\| < \delta_2$, and hence the above properties hold for x_1 (i.e., they hold for $i = n = 1$). Now suppose they hold for each $i = 1, \ldots, n$ up to some $n \ge 1$, so that there exists $x_i \in X_{\varepsilon_i}$ for $i = 1, \ldots, n$ and $y - \sum_{i=1}^{n} Tx_i \in Y_{\delta_{n+1}}$. This implies that there is an $x_{n+1} \in X_{\varepsilon_{n+1}}$ such that $\|y - \sum_{i=1}^{n} Tx_i - Tx_{n+1}\| < \delta_{n+2}$, and so there are $x_k \in X_{\varepsilon_k}$ for $k = 1, \ldots, n+1$ such that $y - \sum_{k=1}^{n+1} Tx_k \in Y_{\delta_{n+2}}$. Therefore, assuming that the above properties hold for some $n \ge 1$, we conclude that they hold for $n + 1$, which completes the induction argument. Now we shall use this sequence $\{x_n\}$ to show that

$$y = Tx \quad \text{for some} \quad x \in X_\varepsilon.$$

In fact, since $\|x_n\| < \frac{\varepsilon}{2^n}$ for each $n \in \mathbb{N}$, we get $\sum_{k=1}^{\infty} \|x_k\| < \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = \varepsilon$ (because $\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$), and so $\{x_n\}$ is absolutely summable. But since $\mathcal{X}$ is a Banach space, this implies that $\{x_n\}$ is summable (Proposition 4.4), which means that $\{\sum_{i=1}^{n} x_i\}$ converges in $\mathcal{X}$. That is, there exists $x \in \mathcal{X}$ such that

$$\sum_{i=1}^{n} x_i \to x \text{ in } \mathcal{X} \quad \text{as} \quad n \to \infty.$$

Moreover, $\|\sum_{i=1}^{n} x_i\| \le \sum_{i=1}^{n} \|x_i\| < \varepsilon$ for all n, so that

$$\|x\| = \Big\| \lim_n \sum_{i=1}^{n} x_i \Big\| = \lim_n \Big\| \sum_{i=1}^{n} x_i \Big\| \le \sup_n \Big\| \sum_{i=1}^{n} x_i \Big\| < \varepsilon$$

(recall: every norm is continuous), and hence $x \in X_\varepsilon$. Since T is continuous and linear, and since

$$\sum_{i=1}^{n} Tx_i \to y \text{ in } \mathcal{Y} \quad \text{as} \quad n \to \infty$$

(because $\|y - \sum_{i=1}^{n} Tx_i\| < \delta_{n+1} < \varepsilon_{n+1} = \frac{\varepsilon}{2^{n+1}}$ for every n), it follows that

$$Tx = T \lim_n \sum_{k=1}^{n} x_k = \lim_n T \sum_{k=1}^{n} x_k = \lim_n \sum_{k=1}^{n} Tx_k = y,$$

and so $y \in T(X_\varepsilon)$. That is, an arbitrary point of Y_δ lies in $T(X_\varepsilon)$. Therefore $Y_\delta \subseteq T(X_\varepsilon)$. □

Let U be any open subset of $\mathcal{X}$ and take an arbitrary $z \in T(U)$. Thus there exists $u \in U$ such that $z = Tu$. But u is an interior point of U (since U is open in $\mathcal{X}$), which means that there is a nonempty open ball included in U with center at u, say $B_\varepsilon(u) \subseteq U$. Translate this ball to 0 in $\mathcal{X}$ and note that the translated ball coincides with the open ball of radius ε about the origin of $\mathcal{X}$:

$$X_\varepsilon = B_\varepsilon(u) - u = \big\{x \in \mathcal{X}\colon\ x = v - u \text{ for some } v \in B_\varepsilon(u)\big\}.$$

According to Claim 2 there is a nonempty open ball Y_δ about the origin of $\mathcal{Y}$ such that $Y_\delta \subseteq T(X_\varepsilon)$. Translate this ball to the point z and get the open ball with center at z and radius δ:

$$B_\delta(z) = Y_\delta + z = \big\{w \in \mathcal{Y}\colon\ w = y + z \text{ for some } y \in Y_\delta\big\}.$$

Then, since T is linear,

$$B_\delta(z) \subseteq T(X_\varepsilon) + Tu = T(X_\varepsilon + u) = T(B_\varepsilon(u)) \subseteq T(U),$$

and hence z is an interior point of $T(U)$. Conclusion: Every point of $T(U)$ is interior, which means that $T(U)$ is open in $\mathcal{Y}$. □

A straightforward application of the Open Mapping Theorem says that *an injective and surjective bounded linear transformation between Banach spaces has a bounded inverse*.

Theorem 4.22. (The Inverse Mapping Theorem or The Banach Continuous Inverse Theorem). *If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is injective and surjective, then $T^{-1} \in \mathcal{B}[\mathcal{Y}, \mathcal{X}]$.*

Proof. Theorems 2.10, 3.20, and 4.21. □

If $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces, then an element T of $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is called *invertible* if it is an invertible mapping in the set-theoretic sense (i.e., injective and surjective) and its inverse T^{-1} lies in $\mathcal{B}[\mathcal{Y}, \mathcal{X}]$. According to Theorem 2.10, this can be briefly stated as: $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is *invertible* if it has a bounded inverse. We shall denote the set of all invertible elements of $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ by $\mathcal{G}[\mathcal{X}, \mathcal{Y}]$. The Inverse Mapping Theorem says that, if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, then both meanings of the term "invertible" coincide, and $T^{-1} \in \mathcal{G}[\mathcal{Y}, \mathcal{X}]$ whenever $T \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$ (for $(T^{-1})^{-1} = T$ — Problem 1.8). Note that $\mathcal{G}[\mathcal{X}, \mathcal{Y}]$ is not a linear manifold of $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$: addition is not a binary operation on $\mathcal{G}[\mathcal{X}, \mathcal{Y}]$ (e.g., $\operatorname{diag}(\{\frac{-k}{k+1}\}_{k=1}^{\infty}) + I = \operatorname{diag}(\{\frac{1}{k+1}\}_{k=1}^{\infty})$ — see Example 4.J). On the other hand, the composition (product) of invertible bounded linear transformations is again an invertible bounded linear transformation.

Corollary 4.23. *If $T \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$ and $S \in \mathcal{G}[\mathcal{Y}, \mathcal{Z}]$, then $ST \in \mathcal{G}[\mathcal{X}, \mathcal{Z}]$ and $(ST)^{-1} = T^{-1}S^{-1}$ whenever $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are Banach spaces.*

Proof. Problem 1.10, Proposition 4.16, and Theorem 4.22. □

Here is a useful corollary of Theorem 4.22, which is sometimes referred to as the *Bounded Inverse Theorem*.

Corollary 4.24. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and take $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$. The following assertions are pairwise equivalent.*

(a) *There exists $T^{-1} \in \mathcal{B}[\mathcal{R}(T), \mathcal{X}]$ (T has a bounded inverse on its range).*

(b) *T is bounded below.*

(c) *$\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T)^- = \mathcal{R}(T)$ (T is injective and has a closed range).*

Proof. According to Proposition 4.20, assertions (a) and (b) are equivalent and each of them implies $\mathcal{R}(T)^- = \mathcal{R}(T)$ whenever $\mathcal{X}$ is a Banach space. Moreover, (b) trivially implies $\mathcal{N}(T) = \{0\}$. Thus either of the equivalent assertions (a) or (b) implies (c). On the other hand, if $\mathcal{N}(T) = \{0\}$, then T is injective. If, in addition, the linear manifold $\mathcal{R}(T)$ is closed in the Banach space $\mathcal{Y}$, then it is itself a Banach space (Proposition 4.7), so that $T: \mathcal{X} \to \mathcal{R}(T)$ is an injective and surjective bounded linear transformation of the Banach space $\mathcal{X}$ onto the Banach space $\mathcal{R}(T)$. Therefore, its inverse T^{-1} lies in $\mathcal{B}[\mathcal{R}(T), \mathcal{X}]$ by the Inverse Mapping Theorem. That is, (c) implies (a). □

The role played by the Inverse Mapping Theorem (and so by the Open Mapping Theorem) in the proof of Corollary 4.24 is restricted to show that (c) implies (a); the rest of the proof does not need the Open Mapping Theorem.

Let $\mathcal{X} \neq \{0\}$ be a normed space. Consider the algebra $\mathcal{L}[\mathcal{X}]$ (of all linear transformations on $\mathcal{X}$), and also the normed algebra $\mathcal{B}[\mathcal{X}]$ (of all operators on $\mathcal{X}$), which is a subalgebra (in the purely algebraic sense) of $\mathcal{L}[\mathcal{X}]$. Let $\mathcal{G}[\mathcal{X}]$ denote the set of all invertible operators in $\mathcal{B}[\mathcal{X}]$ (i.e., set $\mathcal{G}[\mathcal{X}] = \mathcal{G}[\mathcal{X}, \mathcal{X}]$). Suppose $\mathcal{X}$ is a Banach space, and let $T \in \mathcal{B}[\mathcal{X}]$ be an invertible element of the algebra $\mathcal{L}[\mathcal{X}]$ (i.e., there exists $T^{-1} \in \mathcal{L}[\mathcal{X}]$ such that $T^{-1}T = TT^{-1} = I$, where I stands for the identity of $\mathcal{L}[\mathcal{X}]$). Thus, as we had already noticed, the Inverse Mapping Theorem ensures that the concept of an invertible element of the Banach algebra $\mathcal{B}[\mathcal{X}]$ is unambiguously defined (since $T^{-1} \in \mathcal{B}[\mathcal{X}]$). In other words, the set-theoretic inverse of an operator on a Banach space is again an operator on the same Banach space. Moreover (since the inverse of an invertible operator is itself invertible), the set $\mathcal{G}[\mathcal{X}]$ of all invertible operators in $\mathcal{B}[\mathcal{X}]$ forms a *group* under multiplication (every operator in $\mathcal{G}[\mathcal{X}] \subset \mathcal{B}[\mathcal{X}]$ has an inverse in $\mathcal{G}[\mathcal{X}]$) whenever $\mathcal{X}$ is a Banach space.

We close this section with another important application of the Open Mapping Theorem. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces (over the same scalar field) and let $\mathcal{X} \oplus \mathcal{Y}$ be their direct sum, which is a linear space whose underlying set is the Cartesian product $\mathcal{X} \times \mathcal{Y}$ of the underlying sets of the linear spaces $\mathcal{X}$ and $\mathcal{Y}$. Recall that the graph of a transformation $T\colon \mathcal{X} \to \mathcal{Y}$ is the set

$$G_T \;=\; \big\{(x, y) \in \mathcal{X} \times \mathcal{Y}\colon\; y = Tx\big\} \;=\; \big\{(x, Tx) \in \mathcal{X} \oplus \mathcal{Y}\colon\; x \in \mathcal{X}\big\}$$

(cf. Section 1.2). If T is linear, then G_T is a linear manifold of the linear space $\mathcal{X} \oplus \mathcal{Y}$ (since $\alpha(u, Tu) \oplus \beta(v, Tv) = (\alpha u + \beta v,\, T(\alpha u + \beta v))$ for every u, v in $\mathcal{X}$ and every pair of scalars α, β). Equip the linear space $\mathcal{X} \oplus \mathcal{Y}$ with any of the norms of Example 4.E and consider the normed space $\mathcal{X} \oplus \mathcal{Y}$.

Theorem 4.25. (The Closed Graph Theorem). *If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$, then T is continuous* (i.e., $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$) *if and only if G_T is closed in $\mathcal{X} \oplus \mathcal{Y}$.*

Proof. Let $P_X\colon \mathcal{X} \oplus \mathcal{Y} \to \mathcal{X}$ and $P_Y\colon \mathcal{X} \oplus \mathcal{Y} \to \mathcal{Y}$ be defined by

$$P_X(x, y) = x \quad \text{and} \quad P_Y(x, y) = y$$

for every $(x, y) \in \mathcal{X} \oplus \mathcal{Y}$. Consider the restriction $P_X|_{G_T}\colon G_T \to \mathcal{X}$ of P_X to the linear manifold G_T of the normed space $\mathcal{X} \oplus \mathcal{Y}$. Observe that P_X and P_Y are the natural projections of $\mathcal{X} \oplus \mathcal{Y}$ onto $\mathcal{X}$ and $\mathcal{Y}$, respectively, which are both linear and bounded (see Example 4.I). Thus P_Y lies in $\mathcal{B}[\mathcal{X} \oplus \mathcal{Y}, \mathcal{Y}]$ and $P_X|_{G_T}$ lies in $\mathcal{B}[G_T, \mathcal{X}]$ (Problems 2.14 and 3.30). Moreover, $P_X|_{G_T}$ is clearly surjective. Furthermore, $P_X|_{G_T}$ is also injective (if $P_X(x, Tx) = 0$, then $x = 0$ and hence $(x, Tx) = (0, 0) \in G_T$; that is, $\mathcal{N}(P_X|_{G_T}) = \{0\}$).

(a) Recall that $\mathcal{X} \oplus \mathcal{Y}$ is a Banach space whenever $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces (Example 4.E). If G_T is closed in $\mathcal{X} \oplus \mathcal{Y}$, then G_T is itself a Banach space (Proposition 4.7), and the Inverse Mapping Theorem (Theorem 4.22) ensures that the inverse $(P_X|_{G_T})^{-1}$ of $P_X|_{G_T}$ lies in $\mathcal{B}[\mathcal{X}, G_T]$. Since $T = P_Y(P_X|_{G_T})^{-1}$ (because $TP_X|_{G_T} = P_Y|_{G_T}$), it follows by Proposition 4.16 that T is bounded.

(b) Conversely, take an arbitrary sequence $\{(x_n, Tx_n)\}$ in G_T that converges in $\mathcal{X} \oplus \mathcal{Y}$ to, say, $(x, y) \in \mathcal{X} \oplus \mathcal{Y}$. Since P_X and P_Y are continuous, it follows by Corollary 3.8 that

$$\lim x_n = \lim P_X(x_n, Tx_n) = P_X \lim(x_n, Tx_n) = P_X(x, y) = x,$$

$$\lim Tx_n = \lim P_Y(x_n, Tx_n) = P_Y \lim(x_n, Tx_n) = P_Y(x, y) = y.$$

If T is continuous, then (by Corollary 3.8 again)

$$y = \lim Tx_n = T \lim x_n = Tx,$$

and hence $(x, y) = (x, Tx) \in G_T$. Therefore G_T is closed in $\mathcal{X} \oplus \mathcal{Y}$ by the Closed Set Theorem (Theorem 3.30). □

4.6 Equivalence and Finite-Dimensional Spaces

Recall that two sets are said to be *equivalent* if there exists a one-to-one correspondence (i.e., an injective and surjective mapping or, equivalently, an invertible mapping) between them (Chapter 1). Two linear spaces are *isomorphic* if there exists an isomorphism (i.e., an invertible linear transformation) of one of them onto the other (recall: the inverse of an invertible linear transformation is again a linear transformation). An isomorphism (or a linear-space isomorphism) is then a one-to-one correspondence that preserves the linear operations between the linear spaces, and hence preserves the algebraic structure (Chapter 2). Two topological spaces are *homeomorphic* if there exists a homeomorphism (i.e., an invertible continuous mapping whose inverse also is continuous) of one of them onto the other. A homeomorphism provides a one-to-one correspondence between the topologies on the respective spaces, thus preserving the topological structure. In particular, a homeomorphism preserves convergence. A uniform homeomorphism between metric spaces is a homeomorphism where continuity (in both senses) is strengthened to uniform continuity, so that a uniform homeomorphism also preserves Cauchy sequences. Two metric spaces are *uniformly homeomorphic* if there exists a uniform homeomorphism of one of them onto the other (Chapter 3).

Now, as one would expect, we shall be interested in preserving both algebraic and topological structures between two normed spaces. A mapping of a normed space $\mathcal{X}$ onto a normed space $\mathcal{Y}$ that is (simultaneously) a homeomorphism and an isomorphism is called a *topological isomorphism* (or an *equivalence*), and $\mathcal{X}$ and $\mathcal{Y}$ are said to be *topologically isomorphic* (or *equivalent*) if there exists a topological isomorphism between them. Clearly, continuity refers to the norm topologies: a topological isomorphism is a mapping of a normed space $\mathcal{X}$ onto a normed space $\mathcal{Y}$, which is a homeomorphism when $\mathcal{X}$ and $\mathcal{Y}$ are viewed as metric spaces (equipped with the metrics generated by their respective norms), and also is an isomorphism between the linear spaces

$\mathcal{X}$ and $\mathcal{Y}$. Since an isomorphism is just an injective and surjective linear transformation between linear spaces, it follows that *a topological isomorphism is simply a linear homeomorphism between normed spaces*. Thus the topological isomorphisms between $\mathcal{X}$ and $\mathcal{Y}$ are precisely the elements of $\mathcal{G}[\mathcal{X},\mathcal{Y}]$: if $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces, then $W\colon \mathcal{X} \to \mathcal{Y}$ is a topological isomorphism if and only if W is an invertible element of $\mathcal{B}[\mathcal{X},\mathcal{Y}]$. Therefore, *$\mathcal{X}$ and $\mathcal{Y}$ are topologically isomorphic if and only if there exists a linear homeomorphism between them* or, equivalently, *if and only if* $\mathcal{G}[\mathcal{X},\mathcal{Y}] \neq \varnothing$. Conversely, the elements of $\mathcal{G}[\mathcal{X},\mathcal{Y}]$ are also characterized as those linear-space isomorphisms that are bounded above and below (Theorem 4.14 and Proposition 4.20). Thus *$\mathcal{X}$ and $\mathcal{Y}$ are topologically isomorphic if and only if there exists an isomorphism W in $\mathcal{L}[\mathcal{X},\mathcal{Y}]$ and a pair of positive constants α and β such that*

$$\alpha\|x\| \leq \|Wx\| \leq \beta\|x\| \quad \text{for every} \quad x \in \mathcal{X}.$$

The Inverse Mapping Theorem says that an injective and surjective bounded linear transformation of a Banach space onto a Banach space is a homeomorphism, and hence a topological isomorphism: *if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, then they are topologically isomorphic if and only if there exists an isomorphism in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$.*

Two norms on the same linear space are said to be *equivalent* if the metrics that they generate are equivalent. In other words (see Section 3.4), $\|\ \|_1$ and $\|\ \|_2$ are equivalent norms on a linear space $\mathcal{X}$ if and only if they induce the same norm topology on $\mathcal{X}$.

Proposition 4.26. *Let $\|\ \|_1$ and $\|\ \|_2$ be two norms on the same linear space $\mathcal{X}$. The following assertions are pairwise equivalent.*

(a) $\|\ \|_1$ and $\|\ \|_2$ *are equivalent norms.*

(b) *The identity map between the normed spaces $(\mathcal{X},\|\ \|_1)$ and $(\mathcal{X},\|\ \|_2)$ is a topological isomorphism.*

(c) *There exist real constants $\alpha > 0$ and $\beta > 0$ such that*

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1 \quad \textit{for every} \quad x \in \mathcal{X}.$$

Proof. Recall that the identity obviously is an isomorphism of a linear space onto itself, and hence it is a topological isomorphism between the normed spaces $(\mathcal{X},\|\ \|_1)$ and $(\mathcal{X},\|\ \|_2)$ if and only if it is a homeomorphism. Thus assertions (a) and (b) are equivalent by Corollary 3.19. Assertion (c) simply says that the identity $I\colon (\mathcal{X},\|\ \|_1) \to (\mathcal{X},\|\ \|_2)$ is bounded above and below or, equivalently, that it is continuous and its inverse $I^{-1}\colon (\mathcal{X},\|\ \|_2) \to (\mathcal{X},\|\ \|_1)$ also is continuous (Theorem 4.14 and Proposition 4.20), which means that it is a homeomorphism. That is, assertions (b) and (c) are equivalent. □

It is worth noticing that metrics generated by norms are equivalent if and only if they are uniformly equivalent. Indeed, continuity coincides with uni-

form continuity for linear transformations between normed spaces (Theorem 4.14), and hence the identity $I\colon (\mathcal{X}, \|\ \|_1) \to (\mathcal{X}, \|\ \|_2)$ is a homeomorphism if and only if it is a uniform homeomorphism. Observe that, according to Problem 3.33, all norms of Example 4.E are equivalent. In particular, all norms of Example 4.A are equivalent.

Theorem 4.27. *If $\mathcal{X}$ is a finite-dimensional linear space, then any two norms on $\mathcal{X}$ are equivalent.*

Proof. Let $B = \{e_i\}_{i=1}^n$ be a Hamel basis for an n-dimensional linear space $\mathcal{X}$ over $\mathbb{F}$ so that every vector in $\mathcal{X}$ has a unique expansion on B,

$$x = \sum_{i=1}^n \xi_i e_i,$$

where $\{\xi_i\}_{i=1}^n$ is a family of scalars: the coordinates of x with respect to B. It is easy to show that the function $\|\ \|_0\colon \mathcal{X} \to \mathbb{R}$, defined by

$$\|x\|_0 = \max_{1\le i\le n} |\xi_i|$$

for every $x \in \mathcal{X}$, is a norm on $\mathcal{X}$. Now take an arbitrary norm $\|\ \|$ on $\mathcal{X}$. We shall verify that $\|\ \|_0$ and $\|\ \|$ are equivalent. First observe that

$$\|x\| = \Big\| \sum_{i=1}^n \xi_i e_i \Big\| \le \sum_{i=1}^n |\xi_i| \|e_i\| \le \Big(\sum_{i=1}^n \|e_i\| \Big) \|x\|_0$$

for every $x \in \mathcal{X}$. To show the other inequality, consider the linear space $\mathbb{F}^n$ equipped with the norm $\|\ \|_\infty$ of Example 4.A, that is,

$$\|\boldsymbol{a}\|_\infty = \max_{1\le i\le n} |\alpha_i|$$

for every $\boldsymbol{a} = (\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n$, and let $L\colon (\mathbb{F}^n, \|\ \|_\infty) \to (\mathcal{X}, \|\ \|)$ be a transformation defined by

$$L\boldsymbol{a} = \sum_{i=1}^n \alpha_i e_i$$

for every $\boldsymbol{a} = (\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n$. It is readily verified that L is linear. Moreover,

$$\|L\boldsymbol{a}\| = \Big\| \sum_{i=1}^n \alpha_i e_i \Big\| \le \Big(\sum_{i=1}^n \|e_i\| \Big) \|\boldsymbol{a}\|_\infty,$$

so that L is bounded, that is, L is continuous (Theorem 4.14), and so is the real-valued function $\varphi\colon (\mathbb{F}^n, \|\ \|_\infty) \to \mathbb{R}$ such that

$$\varphi(\boldsymbol{a}) = \|L\boldsymbol{a}\|$$

for every $\boldsymbol{a} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$ (because $\| \ \| : \mathcal{X} \to \mathbb{R}$ is continuous, and composition of continuous functions is a continuous function). The unit sphere $S = \partial B_1[0] = \{\boldsymbol{a} \in \mathbb{F}^n \colon \|\boldsymbol{a}\|_\infty = 1\}$ is compact in $(\mathbb{F}^n, \| \ \|_\infty)$ by Theorem 3.83, and so the continuous function φ assumes a minimum value on S (Theorem 3.86). Since $\{e_i\}_{i=1}^n$ is linearly independent, it follows that the linear transformation L is injective ($L(\boldsymbol{a}) = 0$ if and only if $\boldsymbol{a} = 0$), so that this minimum value of φ on S is positive (since $\varphi(\boldsymbol{a}) = \|L(\boldsymbol{a})\| > 0$ for every $\boldsymbol{a} \in S$). Summing up: There exists $\boldsymbol{a}^* \in S$ such that

$$0 \ < \ \varphi(\boldsymbol{a}^*) \ \leq \ \varphi(\boldsymbol{a})$$

for all $\boldsymbol{a} \in S$. Let $\boldsymbol{x} = (\xi_1, \ldots, \xi_n) \in \mathbb{F}^n$ be the n-tuple whose entries are the coordinates $\{\xi_i\}_{i=1}^n$ of an arbitrary $x = \sum_{i=1}^n \xi_i e_i$ in $\mathcal{X}$ with respect to the basis B. Note that

$$\|x\|_0 \ = \ \|\boldsymbol{x}\|_\infty \ \neq \ 0$$

whenever $x \neq 0$, and hence

$$\begin{aligned} \varphi(\boldsymbol{a}^*)\|x\|_0 \ \leq \ \varphi\big(\tfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|_\infty}\big)\|\boldsymbol{x}\|_\infty \ &= \ \big\|L\big(\tfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|_\infty}\big)\big\|\,\|\boldsymbol{x}\|_\infty \\ &= \ \|L\boldsymbol{x}\| \ = \ \Big\|\sum_{i=1}^n \xi_i e_i\Big\| \ = \ \|x\| \end{aligned}$$

for every nonzero x in $\mathcal{X}$. Therefore, by setting $\alpha = \varphi(\boldsymbol{a}^*) > 0$ and $\beta = \sum_{i=1}^n \|e_i\| > 0$ (which do not depend on x), it follows that

$$\alpha\|x\|_0 \ \leq \ \|x\| \ \leq \ \beta\|x\|_0$$

for every $x \in \mathcal{X}$. Conclusion: Every norm on $\mathcal{X}$ is equivalent to the norm $\| \ \|_0$, and hence every two norms on $\mathcal{X}$ are equivalent. □

Corollary 4.28. *Every finite-dimensional normed space is a Banach space.*

Proof. Let $\mathcal{X}$ be an n-dimensional linear space over $\mathbb{F}$ and let $B = \{e_i\}_{i=1}^n$ be a Hamel basis for $\mathcal{X}$. Take an arbitrary $\mathcal{X}$-valued sequence $\{x_k\}_{k\geq 1}$ so that

$$x_k \ = \ \sum_{i=1}^n \xi_{k(i)}\, e_i,$$

where $\{\xi_{k(i)}\}_{i=1}^n$ is a family of scalars (the coordinates of x_k with respect to the Hamel basis B) for each integer $k \geq 1$. Equip $\mathcal{X}$ with a norm $\| \ \|$ and suppose $\{x_k\}_{k\geq 1}$ is a Cauchy sequence in $(\mathcal{X}, \| \ \|)$. Consequently, for every $\varepsilon > 0$ there exists an integer $k_\varepsilon \geq 1$ such that $\|x_j - x_k\| < \varepsilon$ whenever $j, k \geq k_\varepsilon$. Now consider the equivalent norm $\| \ \|_0$ on $\mathcal{X}$ (which was defined in the proof of Theorem 4.27) so that, for every $j, k \geq k_\varepsilon$,

$$\max_{1\leq i\leq n} |\xi_{j(i)} - \xi_{k(i)}| \ = \ \|x_j - x_k\|_0 \ \leq \ \tfrac{1}{\alpha}\,\|x_j - x_k\| \ < \ \tfrac{\varepsilon}{\alpha}$$

for some positive constant α. Hence $\{\xi_{k(i)}\}_{k\geq 1}$ is a Cauchy sequence in $(\mathbb{F}, |\ |)$ for each $i = 1, \ldots, n$, where $|\ |$ stands for the usual norm on $\mathbb{F}$. As $(\mathbb{F}, |\ |)$ is a Banach space (Example 4.A), each sequence $\{\xi_{k(i)}\}_{k\geq 1}$ converges in $(\mathbb{F}, |\ |)$ to, say, $\xi_{(i)} \in \mathbb{F}$. That is, for every $\varepsilon > 0$ there exists an integer $k_{i,\varepsilon} \geq 1$ for each $i = 1, \ldots, n$ such that $|\xi_{k(i)} - \xi_{(i)}| < \varepsilon$ whenever $k \geq k_{i,\varepsilon}$. By setting $x = \sum_{i=1}^{n} \xi_{(i)} e_i$ in $\mathcal{X}$ we get

$$\|x_k - x\| \leq \sum_{i=1}^{n} |\xi_{k(i)} - \xi_{(i)}| \|e_i\| \leq \Big(\sum_{i=1}^{n} \|e_i\| \Big) \varepsilon$$

whenever $k \geq k_\varepsilon = \max\{k_{i,\varepsilon}\}_{i=1}^{n}$, and hence $x_k \to x$ in $(\mathcal{X}, \|\ \|)$ as $k \to \infty$. Conclusion: If $\mathcal{X}$ is a finite-dimensional linear space and $\|\ \|$ is any norm on $\mathcal{X}$, then every Cauchy sequence in $(\mathcal{X}, \|\ \|)$ converges in $(\mathcal{X}, \|\ \|)$, and therefore $(\mathcal{X}, \|\ \|)$ is a Banach space. □

Corollary 4.29. *Every finite-dimensional linear manifold of any normed space $\mathcal{X}$ is a subspace of $\mathcal{X}$.*

Proof. Corollary 4.28 and Theorem 3.40(a). □

Corollary 4.30. *Let $\mathcal{X}$ and $\mathcal{Y} \neq \{0\}$ be normed spaces. Every linear transformation of $\mathcal{X}$ into $\mathcal{Y}$ is continuous if and only if $\mathcal{X}$ is finite dimensional.*

Proof. If $\mathcal{Y} = \{0\}$, then $\{0\}^{\mathcal{X}} = \mathcal{L}[\mathcal{X}, \{0\}] = \mathcal{B}[\mathcal{X}, \{0\}] = \{O\}$ for every normed space $\mathcal{X}$ regardless of its dimension. Therefore, the right statement must exclude the case of $\mathcal{Y} = \{0\}$, and it can be rewritten as follows. If $\mathcal{Y} \neq \{0\}$, then

$$\dim \mathcal{X} < \infty \quad \text{if and only if} \quad \mathcal{L}[\mathcal{X}, \mathcal{Y}] = \mathcal{B}[\mathcal{X}, \mathcal{Y}].$$

(a) If $\dim \mathcal{X} = 0$, then $\mathcal{X} = \{0\}$ and so $\mathcal{L}[\mathcal{X}, \mathcal{Y}] = \mathcal{B}[\mathcal{X}, \mathcal{Y}] = \{O\}$. Thus suppose $\dim \mathcal{X} = n$ for some integer $n \geq 1$ and let $B = \{e_i\}_{i=1}^{n}$ be a Hamel basis for $\mathcal{X}$. Take an arbitrary $x \in \mathcal{X}$ and consider its (unique) expansion on B,

$$x = \sum_{i=1}^{n} \xi_i e_i,$$

where $\{\xi_i\}_{i=1}^{n}$ is a family of scalars consisting of the coordinates of x with respect to the basis B. Let $\|\ \|_X$ and $\|\ \|_Y$ denote the norms on $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$, then

$$\|Tx\|_Y = \Big\| \sum_{i=1}^{n} \xi_i T e_i \Big\|_Y \leq \sum_{i=1}^{n} |\xi_i| \|Te_i\|_Y \leq \Big(\sum_{i=1}^{n} \|Te_i\|_Y \Big) \max_{1 \leq i \leq n} |\xi_i|.$$

The norm $\|\ \|_0$ on $\mathcal{X}$ that was defined in the proof of Theorem 4.27 is equivalent to $\|\ \|_X$. Hence there exists $\alpha > 0$ such that

$$\|Tx\|_Y \leq \Big(\sum_{i=1}^{n} \|Te_i\|_Y \Big) \|x\|_0 \leq \Big(\tfrac{1}{\alpha} \sum_{i=1}^{n} \|Te_i\|_Y \Big) \|x\|_X,$$

and so $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$. Thus $\mathcal{L}[\mathcal{X}, \mathcal{Y}] \subseteq \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ (i.e., $\mathcal{L}[\mathcal{X}, \mathcal{Y}] = \mathcal{B}[\mathcal{X}, \mathcal{Y}]$).

(b) Conversely, suppose $\mathcal{X}$ is infinite dimensional and let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an indexed Hamel basis for $\mathcal{X}$. Since $\{e_\gamma\}_{\gamma\in\Gamma}$ is an infinite set, it has a countably infinite subset, say $\{e_k\}_{k\in\mathbb{N}}$. Set $\mathcal{M} = \operatorname{span}\{e_k\}_{k\in\mathbb{N}}$, a linear manifold of $\mathcal{X}$ for which $\{e_k\}_{k\in\mathbb{N}}$ is a Hamel basis. Thus every vector x in $\mathcal{M} \subseteq \mathcal{X}$ has a unique representation as a (finite) linear combination of vectors in $\{e_k\}_{k\in\mathbb{N}}$. This means that there exists a unique (similarly indexed) family of scalars $\{\xi_k\}_{k\in\mathbb{N}}$ such that $\xi_k = 0$ for all but a finite set of indices k, and

$$x = \sum_{k\in\mathbb{N}} \xi_k e_k$$

(see Section 2.4). Take an arbitrary nonzero vector y in $\mathcal{Y}$ and consider the mapping $L_M\colon \mathcal{M} \to \mathcal{Y}$ defined by

$$L_M x = \sum_{k\in\mathbb{N}} k\xi_k \|e_k\| y$$

for every $x \in \mathcal{M}$. It is readily verified that L_M is linear (recall: the above sums are finite and unique). Extend L_M to $\mathcal{X}$, according to Theorem 2.9, and get a linear transformation $L\colon \mathcal{X} \to \mathcal{Y}$ such that $L|_{\mathcal{M}} = L_M$. Since $Le_k = L_M e_k = k\|e_k\|y$ and $e_k \neq 0$ for each $k \in \mathbb{N}$, it follows that

$$k\|y\| = \tfrac{\|Le_k\|}{\|e_k\|} \leq \sup_{x\neq 0} \tfrac{\|Lx\|}{\|x\|}$$

for every $k \in \mathbb{N}$. Outcome: There is no constant $\beta \geq 0$ for which $\|Lx\| \leq \beta\|x\|$ for all $x \in \mathcal{X}$. Thus the linear transformation $L\colon \mathcal{X} \to \mathcal{Y}$ is not bounded; i.e., $L \in \mathcal{L}[\mathcal{X},\mathcal{Y}]\backslash\mathcal{B}[\mathcal{X},\mathcal{Y}]$. Conclusion: If $\mathcal{L}[\mathcal{X},\mathcal{Y}] = \mathcal{B}[\mathcal{X},\mathcal{Y}]$, then $\dim\mathcal{X} < \infty$. □

Corollary 4.30 holds, in particular, if $\mathcal{X}$ is a Banach space. In this case, the above construction (proof of Corollary 4.30, part (b)) exhibits (via Theorem 2.9 — and therefore via Zorn's Lemma) our first example of an unbounded linear transformation L whose domain $\mathcal{X}$ is an infinite-dimensional Banach space and whose codomain $\mathcal{Y}$ is any nonzero normed space.

Corollary 4.31. *Two finite-dimensional normed spaces are topologically isomorphic if and only if they have the same dimension.*

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional normed spaces. Corollaries 4.28 and 4.30 ensure that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $\mathcal{B}[\mathcal{X},\mathcal{Y}] = \mathcal{L}[\mathcal{X},\mathcal{Y}]$. Thus $\mathcal{X}$ and $\mathcal{Y}$ are topologically isomorphic if and only if they are isomorphic linear spaces (Theorem 4.22), which means that $\dim\mathcal{X} = \dim\mathcal{Y}$ (Theorem 2.12). □

Corollary 3.82 says that every compact subset of any metric space is closed and bounded. The Heine–Borel Theorem (Theorem 3.83) asserts that every closed and bounded subset of $\mathbb{F}^n$ is compact. This equivalence (i.e., *in* $\mathbb{F}^n$ *a set is compact if and only if it is closed and bounded*) actually is a property that characterizes the finite-dimensional normed spaces.

Corollary 4.32. *If $\mathcal{X}$ is a finite-dimensional normed space, then every closed and bounded subset of $\mathcal{X}$ is compact.*

Proof. Let $\mathcal{X}$ be a finite-dimensional normed space over $\mathbb{F}$ and let A be a subset of $\mathcal{X}$ that is closed and bounded (in the norm topology of $\mathcal{X}$). According to Corollary 4.31, $\mathcal{X}$ is topologically isomorphic to $\mathbb{F}^n$, where $n = \dim \mathcal{X}$ and $\mathbb{F}^n$ is equipped with any norm (which are all equivalent by Theorem 4.27). Take a topological isomorphism $W \in \mathcal{G}[\mathcal{X}, \mathbb{F}^n]$ and note that $W(A)$ is closed (because W is a closed map — Theorem 3.24) and bounded (since $\sup_{a\in A}\|Wa\| \leq \|W\| \sup_{a\in A}\|a\|$ — see Problem 4.5) in $\mathbb{F}^n$. Thus $W(A)$ is compact in $\mathbb{F}^n$ (Theorem 3.83). Therefore, since $W^{-1} \in \mathcal{G}[\mathbb{F}^n, \mathcal{X}]$ is a homeomorphism, and since compactness is a topological invariant (Theorem 3.64), it follows that $A = W^{-1}(W(A))$ is compact in $\mathcal{X}$. □

Since compact sets are always closed and bounded, Corollary 4.32 extends the Heine–Borel Theorem to any finite-dimensional normed space: *a subset of a finite-dimensional normed space is compact if and only if it is closed and bounded*. To establish the converse of Corollary 4.32, we need the next result.

Lemma 4.33. (Riesz). *Let $\mathcal{M}$ be a proper subspace of a normed space $\mathcal{X}$. For each real $\alpha \in (0, 1)$ there exists a vector x_α in $\mathcal{X}$ such that $\|x_\alpha\| = 1$ and*

$$\alpha \leq d(x_\alpha, \mathcal{M}) \leq 1.$$

Proof. The second inequality holds trivially because $\mathcal{M}$ is a linear manifold of $\mathcal{X}$, and hence it contains the origin of $\mathcal{X}$ (i.e., $d(x, \mathcal{M}) = \inf_{u\in\mathcal{M}}\|x - u\| \leq \|x - 0\| = \|x\|$ for every $x \in \mathcal{X}$). Since $\mathcal{M}$ is a proper closed subset of $\mathcal{X}$, it follows that $\mathcal{X}\backslash\mathcal{M}$ is nonempty and open in $\mathcal{X}$. Take any vector $z \in \mathcal{X}\backslash\mathcal{M}$ such that $0 < d(z, \mathcal{M})$ (e.g., take the center of an arbitrary nonempty open ball included in the open set $\mathcal{X}\backslash\mathcal{M}$). Thus for each $\varepsilon > 0$ there exists a vector $v_\varepsilon \in \mathcal{M}$ such that

$$0 < d(z, \mathcal{M}) = \inf_{u\in\mathcal{M}} \|z - u\| \leq \|z - v_\varepsilon\| \leq (1 + \varepsilon)\, d(z, \mathcal{M}).$$

Set $x_\varepsilon = \|z - v_\varepsilon\|^{-1}(z - v_\varepsilon)$ in $\mathcal{X}$ so that $\|x_\varepsilon\| = 1$ and

$$\begin{aligned} \|z - v_\varepsilon\| d(x_\varepsilon, \mathcal{M}) &= \inf_{u\in\mathcal{M}} \big\| \|z - v_\varepsilon\| x_\varepsilon - \|z - v_\varepsilon\| u \big\| \\ &= \inf_{v\in\mathcal{M}} \|z - v_\varepsilon - v\| = d(z, \mathcal{M}). \end{aligned}$$

Hence

$$\tfrac{1}{1+\varepsilon} \leq d(x_\varepsilon, \mathcal{M}),$$

which concludes the proof by setting $\alpha = \frac{1}{1+\varepsilon}$. □

This result is sometimes referred to as the *Almost Orthogonality Lemma*. (Why? — Draw a picture.) It is worth remarking that the lemma does not ensure the existence of a vector x_1 in $\mathcal{X}$ with $\|x_1\| = 1$ and $d(x_1, \mathcal{M}) = 1$.

Such a vector may not exist in a Banach space, but it certainly exists in a Hilbert space (next chapter).

Corollary 4.34. *Let $\mathcal{X}$ be a normed space. If the closed unit ball $B_1[0] = \{x \in \mathcal{X}\colon \|x\| \le 1\}$ is compact, then $\mathcal{X}$ is finite dimensional.*

Proof. If $B_1[0]$ is compact, then it is totally bounded (Corollary 3.81), so that there exists a finite $\frac{1}{2}$-net for $B_1[0]$ (Definition 3.68), say $\{u_i\}_{i=1}^n \subset B_1[0]$. Set $\mathcal{M} = \operatorname{span}\{u_i\}_{i=1}^n$, which is a finite-dimensional subspace of $\mathcal{X}$ (reason: $\operatorname{span}\{u_i\}_{i=1}^n$ is a finite-dimensional linear manifold of $\mathcal{X}$ because there exists a Hamel basis for it included in $\{u_i\}_{i=1}^n$, by Theorem 2.6, and hence it is a subspace of $\mathcal{X}$ according to Corollary 4.29). If $\mathcal{X}$ is infinite dimensional, then $\mathcal{M} \neq \mathcal{X}$ and, in this case, Lemma 4.33 ensures the existence of a vector x in $B_1[0]$ such that $\frac{1}{2} < d(x, \mathcal{M})$. Thus $\frac{1}{2} < \|x - u\|$ for every $u \in \mathcal{M}$ and, in particular, $\frac{1}{2} < \|x - u_i\|$ for every $i = 1, \dots, n$, which contradicts the fact that $\{u_i\}_{i=1}^n$ is a $\frac{1}{2}$-net for $B_1[0]$. Conclusion: If $B_1[0]$ is compact in $\mathcal{X}$, then $\mathcal{X}$ is finite dimensional. □

Corollary 4.34 says that if $\mathcal{X}$ is infinite dimensional, then there is a closed and bounded subset of $\mathcal{X}$ that is not compact. Equivalently, *if every closed and bounded subset of a normed space $\mathcal{X}$ is compact, then $\mathcal{X}$ is finite dimensional*, which is the converse of Corollary 4.32. Thus Corollaries 4.32 and 4.34 lead to the following characterization. *A normed space is finite dimensional if and only if compact sets are precisely the closed and bounded ones.*

4.7 Continuous Linear Extension and Completion

We shall now transfer the extension results of Section 3.8 on uniformly continuous mappings between metric spaces to bounded linear transformations between normed spaces. The normed-space versions of Theorem 3.45 and Corollary 3.46 read as follows.

Theorem 4.35. *Let $\mathcal{M}$ be a dense linear manifold of a normed space $\mathcal{X}$ and let $\mathcal{Y}$ be a Banach space. Every T in $\mathcal{B}[\mathcal{M}, \mathcal{Y}]$ has a unique extension $\widehat{T}$ in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$. Moreover, $\|\widehat{T}\| = \|T\|$.*

Proof. Let $\mathcal{M}$ and $\mathcal{Y}$ be normed spaces. Recall that every T in $\mathcal{B}[\mathcal{M}, \mathcal{Y}]$ is uniformly continuous (Theorem 4.14). If $\mathcal{M}$ is a dense linear manifold of a normed space $\mathcal{X}$, and $\mathcal{Y}$ is a Banach space (i.e., a complete normed space), then Theorem 3.45 ensures that there exists a unique continuous extension $\widehat{T}\colon \mathcal{X} \to \mathcal{Y}$ of T over $\mathcal{X}$. It remains to verify that

$$\widehat{T} \text{ is linear and } \|\widehat{T}\| = \|T\|.$$

Let u and v be arbitrary vectors in $\mathcal{X}$. Since $\mathcal{M}^- = \mathcal{X}$, it follows by Proposition 3.32 that there exist $\mathcal{M}$-valued sequences $\{u_n\}$ and $\{v_n\}$ converging in $\mathcal{X}$ to u and v, respectively. Then, by Corollary 3.8,

$$\widehat{T}(u+v) \,=\, \widehat{T}(\lim u_n + \lim v_n) \,=\, \widehat{T}(\lim(u_n+v_n)) \,=\, \lim \widehat{T}(u_n+v_n)$$

because addition is a continuous operation (Problem 4.1) and $\widehat{T}$ is continuous. But $T = \widehat{T}|_{\mathcal{M}}$, $\mathcal{M}$ is a linear space, and T is a linear transformation. Hence

$$\widehat{T}(u_n+v_n) \,=\, T(u_n+v_n) \,=\, Tu_n + Tv_n \,=\, \widehat{T}u_n + \widehat{T}v_n$$

for every n. The same continuity argument (Problem 4.1 and Corollary 3.8) makes the backward path:

$$\lim \widehat{T}(u_n+v_n) \,=\, \lim(\widehat{T}u_n + \widehat{T}v_n) \,=\, \lim \widehat{T}u_n + \lim \widehat{T}v_n \,=\, \widehat{T}u + \widehat{T}v.$$

This shows that $\widehat{T}$ is additive. Similarly, for every scalar α,

$$\begin{aligned}\widehat{T}(\alpha u) \,&=\, \widehat{T}(\alpha \lim u_n) \,=\, \widehat{T}(\lim \alpha u_n) \,=\, \lim \widehat{T}(\alpha u_n)\\ &=\, \lim T\alpha u_n \,=\, \lim \alpha T u_n \,=\, \lim \alpha \widehat{T} u_n \,=\, \alpha \lim \widehat{T} u_n \,=\, \alpha \widehat{T} u,\end{aligned}$$

so that $\widehat{T}$ is homogeneous. That is, $\widehat{T}$ is linear. Thus $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$. Moreover,

$$\|\widehat{T}u_n\| \,=\, \|Tu_n\| \,\le\, \|T\|\,\|u_n\|$$

for every n. Applying once again the same continuity argument, and recalling that the norm is continuous, it follows that $\lim \|u_n\| = \|u\|$ and

$$\|\widehat{T}u\| \,=\, \|\widehat{T} \lim u_n\| \,=\, \lim \|\widehat{T}u_n\| \,\le\, \|T\| \lim \|u_n\| \,=\, \|T\|\,\|u\|,$$

and so $\|\widehat{T}\| = \sup_{\|u\|=1} \|\widehat{T}u\| \le \|T\|$. On the other hand,

$$\|T\| \,=\, \sup_{0\ne w\in\mathcal{M}} \tfrac{\|Tw\|}{\|w\|} \,=\, \sup_{0\ne w\in\mathcal{M}} \tfrac{\|\widehat{T}w\|}{\|w\|} \,\le\, \sup_{0\ne x\in\mathcal{X}} \tfrac{\|\widehat{T}x\|}{\|x\|} \,=\, \|\widehat{T}\|. \qquad \square$$

Corollary 4.36. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Let $\mathcal{M}$ and $\mathcal{N}$ be dense linear manifolds of $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $W \in \mathcal{G}[\mathcal{M},\mathcal{N}]$, then there exists a unique $\widehat{W} \in \mathcal{G}[\mathcal{X},\mathcal{Y}]$ that extends W over $\mathcal{X}$.*

Proof. Every $W \in \mathcal{G}[\mathcal{M},\mathcal{N}]$ is a linear homeomorphism, and hence a uniform homeomorphism (Theorem 4.14). Thus Corollary 3.46 ensures that there is a unique homeomorphism $\widehat{W}\colon \mathcal{X} \to \mathcal{Y}$ that extends W over $\mathcal{X}$. On the other hand, Theorem 4.35 says that there exists a unique continuous linear extension of $W\colon \mathcal{M} \to \mathcal{Y}$ over $\mathcal{X}$, say $\widehat{W}' \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$. Thus $\widehat{W}$ and $\widehat{W}'$ are both continuous mappings of $\mathcal{X}$ into $\mathcal{Y}$ that coincide on a dense subset $\mathcal{M}$ of $\mathcal{X}$: $\widehat{W}|_{\mathcal{M}} = W = \widehat{W}'|_{\mathcal{M}}$. Thus $\widehat{W} = \widehat{W}'$ by Corollary 3.33, and so the homeomorphism $\widehat{W}$ of $\mathcal{X}$ onto $\mathcal{Y}$ lies in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$. Conclusion: $\widehat{W}$ is a linear homeomorphism, which means a topological isomorphism; that is, $\widehat{W} \in \mathcal{G}[\mathcal{X},\mathcal{Y}]$. $\square$

An even stronger form of homeomorphism is that of a surjective isometry (a surjective mapping that preserves distance — recall that every isometry is

an injective contraction, and the inverse of a surjective isometry is again a surjective isometry). If there exists a surjective isometry between two metric spaces, then they are said to be *isometric* or *isometrically equivalent* (Chapter 3). Thus a linear isometry of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ (i.e., a linear transformation of $\mathcal{X}$ into $\mathcal{Y}$ that is also an isometry) is necessarily an element of $\mathcal{B}[\mathcal{X},\mathcal{Y}]$, since an isometry is continuous. A surjective isometry in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ (i.e., a linear surjective isometry or, equivalently, a linear-space isomorphism that is also an isometry) is called an *isometric isomorphism*. Two normed spaces are *isometrically isomorphic* if there exists an isometric isomorphism between them. The next proposition places linear isometries in a normed-space setting.

Proposition 4.37. *Take $V \in \mathcal{L}[\mathcal{X},\mathcal{Y}]$, where $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces. The following assertions are equivalent.*

(a) *V is an isometry* (i.e., $\|Vu - Vv\| = \|u - v\|$ *for every* $u, v \in \mathcal{X}$).

(b) $\|Vx\| = \|x\|$ *for every* $x \in \mathcal{X}$.

If $\mathcal{Y} = \mathcal{X}$, then each of the above assertions also is equivalent to

(c) $\|V^n x\| = \|x\|$ *for every* $x \in \mathcal{X}$ *and every integer* $n \geq 1$.

Proof. Assertions (a) and (b) are clearly equivalent because V is linear. Indeed, set $v = 0$ in (a) to get (b) and, conversely, set $x = u - v$ in (b) to get (a). Moreover, take $V \in \mathcal{L}[\mathcal{X}]$ and suppose (b) holds true. Then (c) holds trivially for $n = 1$. If (c) holds for some $n \geq 1$, then $\|V^{n+1}x\| = \|V^n Vx\| = \|Vx\| = \|x\|$ for every $x \in \mathcal{X}$. Conclusion: (b) implies (c) by induction. Finally, note that (c) trivially implies (b). □

It is plain that every isometric isomorphism is a topological isomorphism, and that the identity I of $\mathcal{B}[\mathcal{X}]$ is an isometric isomorphism. It is also clear that the inverse of a topological (isometric) isomorphism is itself a topological (isometric) isomorphism, and that the composition (product) of two topological (isometric) isomorphisms is again a topological (isometric) isomorphism. Hence these concepts (topological isomorphism and isometric isomorphism, that is) have the defining properties of an equivalence relation (viz., reflexivity, symmetry, and transitivity).

Corollary 4.38. *Let $\mathcal{M}$ and $\mathcal{N}$ be dense linear manifolds of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $U\colon \mathcal{M} \to \mathcal{N}$ is an isometric isomorphism of $\mathcal{M}$ onto $\mathcal{N}$, then there exists a unique isometric isomorphism $\widehat{U}\colon \mathcal{X} \to \mathcal{Y}$ of $\mathcal{X}$ onto $\mathcal{Y}$ that extends U over $\mathcal{X}$.*

Proof. According to Corollary 4.36, there exists a unique topological isomorphism $\widehat{U}\colon \mathcal{X} \to \mathcal{Y}$ that extends $U\colon \mathcal{M} \to \mathcal{N}$ over $\mathcal{X}$. Note that the mappings $\|\ \|\colon \mathcal{X} \to \mathbb{R}$ and $\|\widehat{U}(\ )\|\colon \mathcal{X} \to \mathbb{R}$ are continuous (composition of continuous functions is again a continuous function), and also that they coincide on a

dense subset $\mathcal{M}$ of $\mathcal{X}$ ($\|\widehat{U}u\| = \|Uu\| = \|u\|$ for every $u \in \mathcal{M}$ since $\widehat{U}|_{\mathcal{M}} = U$ and U is an isometry — Proposition 4.37). Then, according to Corollary 3.33, they coincide on the whole space $\mathcal{X}$; that is, $\|\widehat{U}x\| = \|x\|$ for every $x \in \mathcal{X}$. Thus (Proposition 4.37 again) the topological isomorphism $\widehat{U}$ is an isometry, and hence an isometric isomorphism. □

Since every normed space is a metric space, every normed space has a completion in the sense of Definition 3.48. The question that naturally arises is whether a normed space has a completion that is a Banach space. In other words, is a completion (in the norm topology, of course) of a normed space a complete *normed space*? We shall first redefine the concept of completion in a normed-space setting.

Definition 4.39. If the image of a *linear* isometry on a normed space $\mathcal{X}$ is a dense linear manifold of a Banach space $\widehat{\mathcal{X}}$, then $\widehat{\mathcal{X}}$ is a *completion* of $\mathcal{X}$. Equivalently, if a normed space $\mathcal{X}$ is isometrically isomorphic to a dense linear manifold of a Banach space $\widehat{\mathcal{X}}$, then $\widehat{\mathcal{X}}$ is a completion of $\mathcal{X}$. (In particular, any Banach space is a completion of every dense linear manifold of it, since the inclusion map is a linear isometry.)

Theorem 4.40. *Every normed space has a completion.*

Proof. Let $(\mathcal{X}, \|\ \|_X)$ be a normed space. Consider the linear space $\mathcal{X}^{\mathbb{N}}$ of all $\mathcal{X}$-valued sequences. It is readily verified that the collection $CS(\mathcal{X})$ of all Cauchy sequences in $(\mathcal{X}, \|\ \|_X)$ is a linear manifold of $\mathcal{X}^{\mathbb{N}}$, and so a linear space (over the same field of the linear space $\mathcal{X}$). The program is to rewrite the proof of Theorem 3.49 in terms of the metric d_X generated by the norm $\|\ \|_X$. Thus

$$\|\boldsymbol{x}\| \;=\; \lim \|x_n\|_X$$

defines a seminorm on the linear space $CS(\mathcal{X})$, where $\boldsymbol{x} = \{x_n\}$ is an arbitrary sequence in $CS(\mathcal{X})$. Set

$$\mathcal{N} \;=\; \big\{\boldsymbol{x} \in CS(\mathcal{X})\colon\ \|\boldsymbol{x}\| = 0\big\}.$$

Proposition 4.5 ensures that $\mathcal{N}$ is a linear manifold of $CS(\mathcal{X})$, and also that

$$\|[\boldsymbol{x}]\|_{\widehat{X}} \;=\; \|\boldsymbol{x}\|$$

defines a norm on the quotient space $\widehat{\mathcal{X}} = CS(\mathcal{X})/\mathcal{N}$, where $\boldsymbol{x}$ is an arbitrary element of an arbitrary coset $[\boldsymbol{x}] = \boldsymbol{x} + \mathcal{N}$ in $\widehat{\mathcal{X}}$, so that

$$(\widehat{\mathcal{X}}, \|\ \|_{\widehat{X}}) \text{ is a normed space.}$$

Now consider the mapping $K\colon \mathcal{X} \to \widehat{\mathcal{X}}$ that assigns to each $x \in \mathcal{X}$ the coset $[\boldsymbol{x}] \in \widehat{\mathcal{X}}$ containing the constant sequence $\boldsymbol{x} = \{x_n\} \in CS(\mathcal{X})$ such that $x_n = x$ for all n. It is easy to show that

$$K\colon \mathcal{X} \to \widehat{\mathcal{X}} \text{ is a linear isometry.}$$

Moreover, by Claims 1 and 2 in the proof of Theorem 3.49, the range of K (which is clearly a linear manifold of $\widehat{\mathcal{X}}$) is dense in $(\widehat{\mathcal{X}}, \| \ \|_{\widehat{X}})$, and $(\widehat{\mathcal{X}}, \| \ \|_{\widehat{X}})$ is a complete normed space. That is,

$$K(\mathcal{X})^- = \widehat{\mathcal{X}} \quad \text{and} \quad \widehat{\mathcal{X}} \text{ is a Banach space.}$$

Summing up: $\mathcal{X}$ is isometrically isomorphic to $K(\mathcal{X})$, which is a dense linear manifold of $\widehat{\mathcal{X}}$, and $\widehat{\mathcal{X}}$ is a Banach space. Thus $\widehat{\mathcal{X}}$ is a completion of $\mathcal{X}$. □

The completion of a normed space is essentially unique; that is, the completion of a normed space is unique up to an isometric isomorphism (i.e., up to a linear surjective isometry).

Theorem 4.41. *Any two completions of a normed space are isometrically isomorphic.*

Proof. Replace "metric space", "surjective isometry", and "subspace", respectively, with "normed space", "isometric isomorphism", and "linear manifold" in the proof of Theorem 3.50. □

According to Definition 4.39, a Banach space $\widehat{\mathcal{X}}$ is a completion of a normed space $\mathcal{X}$ if there exists a dense linear manifold of $\widehat{\mathcal{X}}$, say $\widetilde{\mathcal{X}}$, which is isometrically isomorphic to $\mathcal{X}$; in other words, if there exists a surjective isometry U_X in $\mathcal{G}[\widetilde{\mathcal{X}}, \mathcal{X}]$ for some dense linear manifold $\widetilde{\mathcal{X}}$ of $\widehat{\mathcal{X}}$. Let $\widehat{\mathcal{Y}}$ be a completion of a normed space $\mathcal{Y}$, so that $\mathcal{Y}$ is isometrically isomorphic to a dense linear manifold $\widetilde{\mathcal{Y}}$ of the Banach space $\widehat{\mathcal{Y}}$, and let $U_Y \in \mathcal{G}[\mathcal{Y}, \widetilde{\mathcal{Y}}]$ be the surjective isometry between $\widetilde{\mathcal{Y}}$ and $\mathcal{Y}$. Take $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ and set $\widetilde{T} = U_Y T U_X \in \mathcal{B}[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}]$ as in the following commutative diagram.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ T\ } & \mathcal{Y} \\ {\scriptstyle U_X}\uparrow & & \downarrow{\scriptstyle U_Y} \\ \widetilde{\mathcal{X}} & \xrightarrow{\ \widetilde{T}\ } & \widetilde{\mathcal{Y}} \end{array}$$

Define $\widehat{T} \in \mathcal{B}[\widehat{\mathcal{X}}, \widehat{\mathcal{Y}}]$ as the extension of $\widetilde{T} \in \mathcal{B}[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}]$ over $\widehat{\mathcal{X}}$ (Theorem 4.35), so that $\|\widehat{T}\| = \|U_Y T U_X\| = \|T\|$ (see Problem 4.41). It is again usual to refer to $\widehat{T}$ as the *extension of* T *over the completion* $\widehat{\mathcal{X}}$ *of* $\mathcal{X}$ *into the completion* $\widehat{\mathcal{Y}}$ *of* $\mathcal{Y}$. Moreover, since any pair of completions of $\mathcal{X}$ and $\mathcal{Y}$, say $\{\widehat{\mathcal{X}}, \widehat{\mathcal{X}}'\}$ and $\{\widehat{\mathcal{Y}}, \widehat{\mathcal{Y}}'\}$, are isometrically isomorphic (i.e., since there exist surjective isometries $\widehat{U}_X$ in $\mathcal{G}[\widehat{\mathcal{X}}', \widehat{\mathcal{X}}]$ and $\widehat{U}_Y$ in $\mathcal{G}[\widehat{\mathcal{Y}}, \widehat{\mathcal{Y}}']$), it follows (see proof of Theorem 3.51) that any two extensions $\widehat{T} \in \mathcal{B}[\widehat{\mathcal{X}}, \widehat{\mathcal{Y}}]$ and $\widehat{T}' \in \mathcal{B}[\widehat{\mathcal{X}}', \widehat{\mathcal{Y}}']$ of $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ over the completions $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{X}}'$ into the completions $\widehat{\mathcal{Y}}$ and $\widehat{\mathcal{Y}}'$, respectively, are unique up to isometric isomorphisms in the sense that $\widehat{T}' = \widehat{U}_Y \widehat{T} \widehat{U}_X$ (and hence $\|\widehat{T}'\| = \|\widehat{T}\| = \|T\|$). The commutative diagrams

$$\begin{array}{ccccccc}
\mathcal{Y} & \xrightarrow{U_Y} & \widetilde{\mathcal{Y}} & \subset & \widehat{\mathcal{Y}} & \xrightarrow{\widehat{U}_Y} & \widehat{\mathcal{Y}}' \\
{\scriptstyle T}\uparrow & & \uparrow{\scriptstyle \widetilde{T}} & & {\scriptstyle \widehat{T}}\uparrow & & \uparrow{\scriptstyle \widehat{T}'} \\
\mathcal{X} & \xleftarrow{U_X} & \widetilde{\mathcal{X}} & \subset & \widehat{\mathcal{X}} & \xleftarrow{\widehat{U}_X} & \widehat{\mathcal{X}}'
\end{array}$$

illustrate such an extension program, which is finally stated as follows.

Theorem 4.42. *Let the Banach spaces $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ be completions of the normed spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Every $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ has an extension $\widehat{T} \in \mathcal{B}[\widehat{\mathcal{X}}, \widehat{\mathcal{Y}}]$ over the completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ into the completion $\widehat{\mathcal{Y}}$ of $\mathcal{Y}$. Moreover, $\widehat{T}$ is unique up to isometric isomorphisms and $\|\widehat{T}\| = \|T\|$.*

4.8 The Banach–Steinhaus Theorem and Operator Convergence

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and let Θ be a subset of $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$. We shall say that Θ is *strongly bounded* (or *pointwise bounded*) if for each $x \in \mathcal{X}$ the set $\Theta_x = \{Tx \in \mathcal{Y}\colon\ T \in \Theta\}$ is bounded in $\mathcal{Y}$ (with respect to the norm topology of $\mathcal{Y}$), and *uniformly bounded* if Θ is itself bounded in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ (with respect to the uniform norm topology of $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$). It is clear that uniform boundedness implies strong boundedness:

$$\sup_{T\in\Theta} \|T\| < \infty \ \text{ implies } \ \sup_{T\in\Theta} \|Tx\| < \infty \ \text{ for every } \ x \in \mathcal{X},$$

since $\|Tx\| \le \|T\|\|x\|$ for every $x \in \mathcal{X}$. The Banach–Steinhaus Theorem ensures the converse whenever $\mathcal{X}$ is a Banach space:

$$\sup_{T\in\Theta} \|Tx\| < \infty \text{ for every } x \text{ in a Banach space } \mathcal{X} \text{ implies } \sup_{T\in\Theta} \|T\| < \infty.$$

That is, *a collection Θ of bounded linear transformations of a Banach space $\mathcal{X}$ into a normed space $\mathcal{Y}$ is uniformly bounded if and only if it is strongly bounded*. This will be our second important application of the Baire Category Theorem (the first was the Open Mapping Theorem).

Theorem 4.43. (The Banach–Steinhaus Theorem or The Uniform Boundedness Principle). *Let $\{T_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary indexed family of bounded linear transformations of a Banach space $\mathcal{X}$ into a normed space $\mathcal{Y}$. If $\sup_{\gamma\in\Gamma}\|T_\gamma x\| < \infty$ for every $x \in \mathcal{X}$, then $\sup_{\gamma\in\Gamma}\|T_\gamma\| < \infty$.*

Proof. For each $n \in \mathbb{N}$ set $A_n = \big\{x \in \mathcal{X}\colon\ \|T_\gamma x\| \le n \text{ for all } \gamma \in \Gamma\big\}$.

Claim 1. A_n is closed in $\mathcal{X}$.

Proof. Let $\{x_k\}$ be an A_n-valued sequence that converges in $\mathcal{X}$ to, say, $x \in \mathcal{X}$. Take an arbitrary $\gamma \in \Gamma$ and note that

$$\begin{aligned}\|T_\gamma x\| &= \|T_\gamma(x - x_k) + T_\gamma x_k\| \\ &\le \|T_\gamma(x - x_k)\| + \|T_\gamma x_k\| \le \|T_\gamma\|\,\|x - x_k\| + n\end{aligned}$$

for every integer k. Since $\|x - x_k\| \to 0$ as $k \to \infty$, it follows that $\|T_\gamma x\| \le n$. Thus, as γ is arbitrary in Γ, $\|T_\gamma x\| \le n$ for all $\gamma \in \Gamma$, and hence $x \in A_n$. Conclusion: A_n is closed in $\mathcal{X}$ by the Closed Set Theorem (Theorem 3.30). □

Claim 2. If $\sup_{\gamma\in\Gamma}\|T_\gamma x\| < \infty$ for every $x \in \mathcal{X}$, then $\{A_n\}$ covers $\mathcal{X}$.

Proof. Take an arbitrary $x \in \mathcal{X}$. If $\sup_{\gamma\in\Gamma}\|T_\gamma x\| < \infty$, then there exists an integer $n_x \in \mathbb{N}$ such that $\sup_{\gamma\in\Gamma}\|T_\gamma x\| \le n_x$, and hence $x \in A_{n_x}$. Therefore, $\mathcal{X} \subseteq \bigcup_{n\in\mathbb{N}} A_n$ (i.e., $\mathcal{X} = \bigcup_{n\in\mathbb{N}} A_n$). □

Then, by the Baire Category Theorem (Theorem 3.58), at least one of the sets A_n, say A_{n_0}, has a nonempty interior. This means that there is a vector x_0 in $\mathcal{X}$ and a radius $\rho > 0$ such that the open ball $B_{2\rho}(x_0)$ is included in A_{n_0} (and so $\|T_\gamma x_0\| < n_0$ for all $\gamma \in \Gamma$). Take any nonzero vector x in $\mathcal{X}$ and set $x' = \rho\|x\|^{-1}x \in \mathcal{X}$. Since $x' \in B_{2\rho}(0)$, it follows that $x' + x_0 \in B_{2\rho}(x_0) \subset A_{n_0}$. Thus, for every $\gamma \in \Gamma$, $\|T_\gamma(x' + x_0)\| < n_0$ and

$$\begin{aligned}\rho\,\tfrac{\|T_\gamma x\|}{\|x\|} = \|T_\gamma x'\| &= \|T_\gamma(x' + x_0) - T_\gamma x_0\| \\ &\le \|T_\gamma(x' + x_0)\| + \|T_\gamma x_0\| < 2n_0.\end{aligned}$$

Therefore, $\|T_\gamma\| = \sup_{x\ne 0}\frac{\|T_\gamma x\|}{\|x\|} \le \frac{2n_0}{\rho}$ for all $\gamma \in \Gamma$. □

Let $\{L_n\}$ be a sequence of linear transformations of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ (i.e., an $\mathcal{L}[\mathcal{X},\mathcal{Y}]$-valued sequence). $\{L_n\}$ is *pointwise convergent* if for every $x \in \mathcal{X}$ the $\mathcal{Y}$-valued sequence $\{L_n x\}$ converges in $\mathcal{Y}$; that is, if for each x in $\mathcal{X}$ there exists a y_x in $\mathcal{Y}$ such that $L_n x \to y_x$ in $\mathcal{Y}$ as $n \to \infty$. As the limit is unique, this actually defines a mapping $L\colon \mathcal{X} \to \mathcal{Y}$ given by $L(x) = y_x$ for every $x \in \mathcal{X}$. Moreover, it is readily verified that L is linear (cf. Problem 4.1). Thus *an $\mathcal{L}[\mathcal{X},\mathcal{Y}]$-valued sequence $\{L_n\}$ is pointwise convergent if and only if there is an $L \in \mathcal{L}[\mathcal{X},\mathcal{Y}]$ such that $\|(L_n - L)x\| \to 0$ as $n \to \infty$ for every $x \in \mathcal{X}$*. Now consider a sequence $\{T_n\}$ of bounded linear transformations of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ (i.e., a $\mathcal{B}[\mathcal{X},\mathcal{Y}]$-valued sequence).

Proposition 4.44. *If $\mathcal{X}$ is a Banach space and $\mathcal{Y}$ is a normed space, then a $\mathcal{B}[\mathcal{X},\mathcal{Y}]$-valued sequence $\{T_n\}$ is pointwise convergent if and only if there is a $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ such that $\|(T_n - T)x\| \to 0$ for every $x \in \mathcal{X}$.*

Proof. According to the above italicized result (about pointwise convergence of $\mathcal{L}[\mathcal{X},\mathcal{Y}]$-valued sequences), all that remains to be shown is that T is bounded whenever $\|(T_n - T)x\| \to 0$ for every $x \in \mathcal{X}$. Since the sequence $\{T_n x\}$ converges in $\mathcal{Y}$, it is bounded in $\mathcal{Y}$ (Proposition 3.39). That is, $\sup_n\|T_n x\| < \infty$ for every $x \in \mathcal{X}$ (Problem 4.5). Then $\sup_n\|T_n\| < \infty$ by the Banach–Steinhaus Theorem (Theorem 4.43) because $\mathcal{X}$ is a Banach space. Therefore,

$$\|Tx\| \;=\; \lim_n \|T_n x\| \;\le\; \big(\limsup_n \|T_n\|\big)\|x\| \;\le\; \big(\sup_n \|T_n\|\big)\|x\|$$

for every $x \in \mathcal{X}$, so that $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ and $\|T\| \le \limsup_n \|T_n\|$. □

Definition 4.45. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, and let $\{T_n\}$ be a $\mathcal{B}[\mathcal{X},\mathcal{Y}]$-valued sequence. If $\{T_n\}$ converges in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$, that is, if there exists T in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ such that

$$\|T_n - T\| \to 0,$$

then we say that $\{T_n\}$ *converges uniformly* (or *converges in the uniform topology*, or *in the operator norm topology*) to T, which is called the *uniform limit* of $\{T_n\}$. Notation: $T_n \xrightarrow{u} T$ (or $T_n - T \xrightarrow{u} O$). If $\{T_n\}$ does not converge uniformly to T, then we write $T_n \overset{u}{\not\longrightarrow} T$. If there exists T in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ such that

$$\|(T_n - T)x\| \to 0 \quad \text{for every} \quad x \in \mathcal{X},$$

then we say that $\{T_n\}$ *converges strongly* (or *converges in the strong* (*operator*) *topology*) to T, which is called the *strong limit* of $\{T_n\}$. Notation: $T_n \xrightarrow{s} T$ (or $T_n - T \xrightarrow{s} O$). Again, if $\{T_n\}$ does not converge strongly to T, then we write $T_n \overset{s}{\not\longrightarrow} T$.

Uniform convergence implies strong convergence (to the same limit). In fact, for each n, $0 \le \|(T_n - T)x\| \le \|T_n - T\|\,\|x\|$ for every $x \in \mathcal{X}$, and hence

$$T_n \xrightarrow{u} T \quad \Longrightarrow \quad T_n \xrightarrow{s} T.$$

The next proposition says that on a finite-dimensional normed space the concepts of strong and uniform convergence coincide.

Proposition 4.46. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. Take a $\mathcal{B}[\mathcal{X},\mathcal{Y}]$-valued sequence $\{T_k\}$ and a transformation T in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$. If $\mathcal{X}$ is finite dimensional, then $T_k \xrightarrow{s} T$ if and only if $T_k \xrightarrow{u} T$.*

Proof. Suppose $(\mathcal{X}, \|\ \|_X)$ is a finite-dimensional normed space and let $B = \{e_i\}_{i=1}^n$ be a Hamel basis for the linear space $\mathcal{X}$. Take an arbitrary x in $\mathcal{X}$ and consider its unique expansion on B,

$$x \;=\; \sum_{i=1}^n \xi_i e_i,$$

where the scalars $\{\xi_i\}_{i=1}^n$ are the coordinates of x with respect to B. Set

$$\|x\|_0 \;=\; \max_{1\le i\le n} |\xi_i|,$$

which defines a norm on $\mathcal{X}$, as we saw in the proof of Theorem 4.27. If $\{T_k\}$ converges strongly to $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$, then $\|(T_k - T)e_i\|_Y \to 0$ as $k \to \infty$ for each $e_i \in B$. Thus for each $i = 1, \dots, n$ and for every $\varepsilon > 0$ there exists a positive integer $k_{i,\varepsilon}$ such that $\|(T_k - T)e_i\|_Y < \varepsilon$ whenever $k \ge k_{i,\varepsilon}$. Therefore,

$$\|(T_k - T)x\|_Y = \Big\| \sum_{i=1}^{n} \xi_i (T_k - T)e_i \Big\|_Y \le \sum_{i=1}^{n} |\xi_i| \, \|(T_k - T)e_i\|_Y$$
$$\le n \max_{1 \le i \le n} |\xi_i| \max_{1 \le i \le n} \|(T_k - T)e_i\|_Y < n \|x\|_0 \varepsilon$$

whenever $k \ge k_\varepsilon = \max\{k_{i,\varepsilon}\}_{i=1}^n$. However, $\| \ \|_0$ and $\| \ \|_X$ are equivalent norms on $\mathcal{X}$ (according to Theorem 4.27) so that there exists a constant $\alpha > 0$ such that $\alpha \|x\|_0 \le \|x\|_X$. Since n and α do not depend on x, it follows that

$$\|(T_k - T)x\|_Y < \tfrac{n}{\alpha} \varepsilon \|x\|_X$$

for every $x \in \mathcal{X}$ whenever $k \ge k_\varepsilon$. Hence, for every $\varepsilon > 0$ there exists a positive integer k_ε such that

$$k \ge k_\varepsilon \quad \text{implies} \quad \|(T_k - T)\| = \sup_{\|x\| \le 1} \|(T_k - T)x\|_Y < \tfrac{n}{\alpha} \varepsilon,$$

and so $\{T_k\}$ converges uniformly to T. Summing up: $T_k \xrightarrow{s} T$ implies $T_k \xrightarrow{u} T$ whenever $\mathcal{X}$ is a finite-dimensional normed space. This concludes the proof once uniform convergence always implies strong convergence. □

Proposition 4.47. *If a Cauchy sequence in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is strongly convergent, then it is uniformly convergent.*

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. If T lies in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ and if $\{T_n\}$ is a $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$-valued sequence, then, for each pair of indices m and n,

$$\|(T_n - T)x\| = \|(T_n - T_m)x + (T_m - T)x\| \le \|T_n - T_m\| \, \|x\| + \|(T_m - T)x\|$$

for every $x \in \mathcal{X}$. Take an arbitrary $\varepsilon > 0$. If $\{T_n\}$ is a Cauchy sequence in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$, then there is a positive integer n_ε such that $\|T_n - T_m\| < \frac{\varepsilon}{2}$, and so

$$\|(T_n - T)x\| < \tfrac{\varepsilon}{2} \|x\| + \|(T_m - T)x\|$$

for every $x \in \mathcal{X}$, whenever $m, n \ge n_\varepsilon$. If $\{T_n\}$ converges strongly to T, then for each $x \in \mathcal{X}$ there exists an integer $m_{\varepsilon,x} \ge n_\varepsilon$ such that $\|(T_m - T)x\| < \frac{\varepsilon}{2}$ whenever $m \ge m_{\varepsilon,x}$. Hence

$$\|(T_n - T)x\| < \tfrac{\varepsilon}{2} \|x\| + \tfrac{\varepsilon}{2}$$

for every $x \in \mathcal{X}$, and therefore

$$\|T_n - T\| = \sup_{\|x\| \le 1} \|(T_n - T)x\| < \varepsilon,$$

if $n \ge n_\varepsilon$. Conclusion: For every $\varepsilon > 0$ there exists an integer $n_\varepsilon \ge 1$ such that

$$n \ge n_\varepsilon \quad \text{implies} \quad \|T_n - T\| < \varepsilon,$$

which means that $\{T_n\}$ converges in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ to T. □

Outcome: If $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces and $\{T_n\}$ is a $\mathcal{B}[\mathcal{X},\mathcal{Y}]$-valued sequence, then $\{T_n\}$ *is uniformly convergent if and only if it is a strongly convergent Cauchy sequence in* $\mathcal{B}[\mathcal{X},\mathcal{Y}]$.

Remark: If $\{T_n\}$ is strongly convergent, then it is strongly bounded (Proposition 3.39). Now, if $\mathcal{X}$ is a Banach space, then pointwise convergence coincides with strong convergence (Proposition 4.44), and strong boundedness coincides with uniform boundedness (Theorem 4.43). Hence

$$T_n \xrightarrow{s} T \quad \Longrightarrow \quad \sup_n \|T_n\| < \infty \text{ whenever } \mathcal{X} \text{ is Banach.}$$

Take $T \in \mathcal{B}[\mathcal{X}]$ and consider the power sequence $\{T^n\}$ in the normed algebra $\mathcal{B}[\mathcal{X}]$. The operator T is called *uniformly stable* if the power sequence $\{T^n\}$ converges uniformly to the null operator; that is, if $T^n \xrightarrow{u} O$ (equivalently, $\|T^n\| \to 0$). T is *strongly stable* if the power sequence $\{T^n\}$ converges strongly to the null operator; that is, if $T^n \xrightarrow{s} O$ (equivalently, $\|T^n x\| \to 0$ for every $x \in \mathcal{X}$). T is called *power bounded* if $\{T^n\}$ is a bounded sequence in $\mathcal{B}[\mathcal{X}]$; that is, if $\sup_n \|T^n\| < \infty$. If $\mathcal{X}$ is a Banach space, then the Banach–Steinhaus Theorem ensures that T is power bounded if and only if $\sup_n \|T^n x\| < \infty$ for every $x \in \mathcal{X}$. Clearly, uniform stability implies strong stability,

$$T^n \xrightarrow{u} O \quad \Longrightarrow \quad T^n \xrightarrow{s} O,$$

which in turn implies power boundedness whenever $\mathcal{X}$ is a Banach space:

$$T^n \xrightarrow{s} O \quad \Longrightarrow \quad \sup_n \|T^n\| < \infty \text{ if } \mathcal{X} \text{ is Banach.}$$

However, the converses fail.

Example 4.K. Consider the diagonal operator $D_a \in \mathcal{B}[\ell^p_+]$ of Example 4.H (for some $p \geq 1$, where $a = \{\alpha_k\}_{k=0}^{\infty}$ lies in ℓ^∞_+), and recall that $\|D_a\| = \sup_k |\alpha_k| = \|a\|_\infty$. It is readily verified by induction that the nth power of D_a, D_a^n, is again a diagonal operator in $\mathcal{B}[\ell^p_+]$. Indeed,

$$D_a^n x = \{\alpha_k^n \xi_k\}_{k=0}^{\infty} \quad \text{for every} \quad x = \{\xi_k\}_{k=0}^{\infty} \in \ell^p_+$$

so that $\|D_a^n\| = \sup_k |\alpha_k|^n = \|a\|_\infty^n$ for every $n \geq 0$. Hence $D_a \in \mathcal{B}[\ell^p_+]$ is uniformly stable if and only if $\|a\|_\infty < 1$; that is,

$$D_a^n \xrightarrow{u} O \quad \text{if and only if} \quad \sup_k |\alpha_k| < 1.$$

Next we shall investigate strong stability for D_a. If $\|D_a^n x\| \to 0$ for every $x \in \ell^p_+$, then (Example 4.H) $\|D_a^n e_j\| = |\alpha_j|^n \to 0$ as $n \to \infty$ and so $|\alpha_j| < 1$ for every $j \geq 0$. On the other hand, take an arbitrary $x = \{\xi_k\}_{k=0}^{\infty}$ in ℓ^p_+. Note that

$$\begin{aligned} 0 \leq \|D_a^n x\|^p &= \sum_{k=0}^{\infty} |\alpha_k|^{np} |\xi_k|^p = \sum_{k=0}^{m} |\alpha_k|^{np} |\xi_k|^p + \sum_{k=m+1}^{\infty} |\alpha_k|^{np} |\xi_k|^p \\ &\leq \max_{0 \leq k \leq m} |\alpha_k|^{np} \sum_{k=0}^{m} |\xi_k|^p + \sup_{k>m} |\alpha_k|^{np} \sum_{k=m+1}^{\infty} |\xi_k|^p \end{aligned}$$

for every pair of integers $m, n \geq 0$. If $|\alpha_k| < 1$ for every integer $k \geq 0$, then $\sup_k |\alpha_k|^{np} \leq 1$ for all $n \geq 0$. Thus, as $\sum_{k=0}^{m} |\xi_k|^p \leq \|x\|^p$ for all $m \geq 0$,

$$0 \leq \|D_a^n x\|^p \leq \max_{0 \leq k \leq m} |\alpha_k|^{np} \|x\|^p + \sum_{k=m+1}^{\infty} |\xi_k|^p$$

for every $m, n \geq 0$. Now take an arbitrary $\varepsilon > 0$. Since $\sum_{k=m+1}^{\infty} |\xi_k|^p \to 0$ as $m \to \infty$ (because $\sum_{k=0}^{\infty} |\xi_k|^p = \|x\|^p < \infty$ — Problem 3.11), it follows that there exists a positive integer m_ε such that

$$m \geq m_\varepsilon \quad \text{implies} \quad \sum_{k=m+1}^{\infty} |\xi_k|^p < \tfrac{\varepsilon^p}{2}.$$

Moreover, if $|\alpha_k| < 1$ for every integer $k \geq 0$, then $\max\{|\alpha_k|^p\}_{k=0}^{m} < 1$ so that $\lim_n \max_{0 \leq k \leq m} |\alpha_k|^{np} = \lim_n (\max_{0 \leq k \leq m} |\alpha_k|^p)^n = 0$ for every nonnegative integer m. In particular, $\lim_n \max_{0 \leq k \leq m_\varepsilon} |\alpha_k|^{np} = 0$. Then there exists a positive integer n_ε such that

$$n \geq n_\varepsilon \quad \text{implies} \quad \max_{0 \leq k \leq m_\varepsilon} |\alpha_k|^{np} < \tfrac{\varepsilon^p}{2}.$$

But $0 \leq \|D_a^n x\|^p \leq \max_{0 \leq k \leq m_\varepsilon} |\alpha_k|^{np} \|x\|^p + \sum_{k=m_\varepsilon+1}^{\infty} |\xi_k|^p$, and hence

$$n \geq n_\varepsilon \quad \text{implies} \quad 0 \leq \|D_a^n x\| < \varepsilon.$$

Therefore, if $|\alpha_k| < 1$ for every $k \geq 0$, then $\|D_a^n x\| \to 0$ for every $x \in \ell_+^p$. Hence $D_a \in \mathcal{B}[\ell_+^p]$ is strongly stable if and only if $|\alpha_k| < 1$ for every $k \geq 0$; that is,

$$D_a^n \xrightarrow{s} O \quad \text{if and only if} \quad |\alpha_k| < 1 \text{ for every } k \geq 0.$$

For instance, the diagonal operator $D_a = \operatorname{diag}\big(\{\frac{k+1}{k+2}\}_{k=0}^{\infty}\big) \in \mathcal{B}[\ell_+^p]$ of Example 4.H is strongly stable but not uniformly stable.

Example 4.L. Consider the mapping $S_-: \ell_+^p \to \ell_+^p$ defined by

$$S_- x = \{\xi_{k+1}\}_{k=0}^{\infty} \quad \text{for every} \quad x = \{\xi_k\}_{k=0}^{\infty} \in \ell_+^p$$

(i.e., $S_-(\xi_0, \xi_1, \xi_2, \ldots) = (\xi_1, \xi_2, \xi_3, \ldots)$), which is also represented as an infinite matrix as follows.

$$S_- = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots \end{pmatrix},$$

where every entry immediately above the main diagonal is equal to 1 and the remaining entries are all zero. This is the *backward unilateral shift* on ℓ_+^p. It is readily verified that S_- is linear and bounded so that $S_- \in \mathcal{B}[\ell_+^p]$. Actually,

$\|S_-x\|^p = \sum_{k=1}^{\infty}|\xi_k|^p \le \sum_{k=0}^{\infty}|\xi_k|^p = \|x\|^p$ for every $x = \{\xi_k\}_{k=0}^{\infty} \in \ell_+^p$, so that S_- is, in fact, a contraction (i.e., $\|S_-\| \le 1$). Consider the power sequence $\{S_-^n\}$ in $\mathcal{B}[\ell_+^p]$. A trivial induction shows that, for each $n \in \mathbb{N}_0$,

$$S_-^n x \,=\, \{\xi_{k+n}\}_{k=0}^{\infty} \quad \text{for every} \quad x = \{\xi_k\}_{k=0}^{\infty} \in \ell_+^p.$$

Hence $\|S_-^n x\|^p = \sum_{k=n}^{\infty}|\xi_k|^p \to 0$ as $n \to \infty$ for every $x = \{\xi_k\}_{k=0}^{\infty} \in \ell_+^p$ (cf. Problem 3.11) so that S_- is strongly stable; that is,

$$S_-^n \xrightarrow{\;s\;} O.$$

However, S_- is not uniformly stable. Indeed, $\|S_-^n\| \le \|S_-\|^n \le 1$ for every integer $n \ge 0$ (see Problem 4.47(a)). On the other hand, consider the ℓ_+^p-valued sequence $\{e_j\}_{j=0}^{\infty}$ where, for each integer $j \ge 0$, e_j is a scalar-valued sequence with just one nonzero entry (equal to 1) at the jth position (i.e., $e_j = \{\delta_{jk}\}_{k=0}^{\infty} \in \ell_+^p$ for every $j \ge 0$). Observe that $\|e_j\| = 1$ for every $j \ge 0$ and $S_-^n e_{n+1} = e_1$ for every $n \ge 0$. Thus $\|S_-^n\| = \sup_{\|x\|=1}\|S_-^n x\| \ge \|S_-^n e_{n+1}\| = \|e_1\| = 1$ for every nonnegative integer n. Therefore,

$$\|S_-^n\| = 1 \quad \text{for every} \quad n \ge 0,$$

so that $S_-^n \xrightarrow{\;u\;} O$. Conclusion: The power sequence

$$\{S_-^n\} \text{ does not converge uniformly.}$$

Reason: Since $S_-^n \xrightarrow{\;s\;} O$, and since uniform convergence implies strong convergence to the same limit, it follows that either $S_-^n \xrightarrow{\;u\;} O$ or $\{S_-^n\}$ does not converge uniformly.

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and let Θ be a subset of $\mathcal{B}[\mathcal{X},\mathcal{Y}]$. According to the Closed Set Theorem, Θ is closed in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ if and only if every Θ-valued sequence that converges in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ has its limit in Θ. Equivalently, every uniformly convergent sequence $\{T_n\}$ of bounded linear transformations in $\Theta \subseteq \mathcal{B}[\mathcal{X},\mathcal{Y}]$ has its (uniform) limit T in Θ. In this case the set $\Theta \subseteq \mathcal{B}[\mathcal{X},\mathcal{Y}]$ is also called *uniformly closed* (or *closed in the uniform topology of* $\mathcal{B}[\mathcal{X},\mathcal{Y}]$). We say that a subset Θ of $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ is *strongly closed* in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ if every Θ-valued strongly convergent sequence $\{T_n\}$ has its (strong) limit T in Θ.

Proposition 4.48. *If $\Theta \subseteq \mathcal{B}[\mathcal{X},\mathcal{Y}]$ is strongly closed in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$, then it is (uniformly) closed in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$.*

Proof. Take an arbitrary Θ-valued uniformly convergent sequence, say $\{T_n\}$, and let $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ be its (uniform) limit. Since uniform convergence implies strong convergence to the same limit, it follows that $\{T_n\}$ converges strongly to T. If every Θ-valued strongly convergent sequence has its (strong) limit in Θ, then $T \in \Theta$. Conclusion: Every Θ-valued uniformly convergent sequence has its (uniform) limit in Θ. □

Remark: If $\mathcal{X}$ is finite dimensional, then strong convergence coincides with uniform convergence (Proposition 4.46). Thus the concepts of strongly and uniformly closed in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ coincide if $\mathcal{X}$ is a finite-dimensional normed space.

Example 4.M. Take an arbitrary $p \geq 1$ and let $\Delta \subset \mathcal{B}[\ell^p_+]$ be the collection of all diagonal operators in $\mathcal{B}[\ell^p_+]$. That is (see Example 4.H),

$$\Delta \,=\, \big\{D_a \in \mathcal{L}[\ell^p_+]\colon\ D_a = \mathrm{diag}(\{\alpha_k\}_{k=0}^{\infty})\ \text{ and }\ a = \{\alpha_k\}_{k=0}^{\infty} \in \ell^\infty_+\big\}.$$

Set

$$\Delta_\infty \,=\, \big\{D_a \in \Delta\colon\ a \in \ell^{c_o}_+\big\},$$

the collection of all diagonal operators $D_a = \mathrm{diag}(\{\alpha_k\}_{k=0}^{\infty})$ in $\mathcal{B}[\ell^p_+]$ such that the scalar-valued sequence $a = \{\alpha_k\}_{k=0}^{\infty}$ converges to zero. As a matter of fact, both Δ and Δ_∞ are subalgebras of the Banach algebra $\mathcal{B}[\ell^p_+]$. Let $\{D_{a_n}\}_{n\geq 1}$ be an arbitrary Δ_∞-valued uniformly convergent sequence. Hence $D_{a_n} \xrightarrow{u} D$ for some $D \in \mathcal{B}[\ell^p_+]$, with each $D_{a_n} = \mathrm{diag}(\{\alpha_{k(n)}\}_{k=0}^{\infty})$ in Δ_∞, so that each $a_n = \{\alpha_{k(n)}\}_{k=0}^{\infty}$ lies in $\ell^{c_o}_+$. This implies that $D_{a_n} \xrightarrow{s} D$ and, according to Problem 4.51, D is a diagonal operator; that is, $D = D_a = \mathrm{diag}(\{\alpha_k\}_{k=0}^{\infty})$ in Δ for some $a = \{\alpha_k\}_{k=0}^{\infty}$ in ℓ^∞_+. Thus (see Example 4.H),

$$\|D_{a_n} - D_a\| \,=\, \|a_n - a\|_\infty \,=\, \sup_k |\alpha_{k(n)} - \alpha_k|.$$

Now take an arbitrary $\varepsilon > 0$. Since $\sup_k|\alpha_{k(n)} - \alpha_k| \to 0$ as $n \to \infty$ (because $\|D_{a_n} - D_a\| \to 0$), there is a positive integer n_ε such that

$$\sup_k |\alpha_{k(n_\varepsilon)} - \alpha_k| \,<\, \varepsilon.$$

Since $\alpha_{k(n_\varepsilon)} \to 0$ as $k \to \infty$ ($a_{n_\varepsilon} \in \ell^{c_o}_+$), there is a positive integer k_ε such that

$$|\alpha_{k(n_\varepsilon)}| < \varepsilon \quad \text{ whenever } \quad k \geq k_\varepsilon.$$

Recall that $\big||\alpha_k| - |\alpha_{k(n_\varepsilon)}|\big| \leq |\alpha_k - \alpha_{k(n_\varepsilon)}|$, and hence

$$|\alpha_k| \,\leq\, \sup_k |\alpha_k - \alpha_{k(n_\varepsilon)}| + |\alpha_{k(n_\varepsilon)}|,$$

for every k. Therefore,

$$k \geq k_\varepsilon \quad \text{ implies } \quad |\alpha_k| < 2\varepsilon$$

so that $\alpha_k \to 0$. Thus $a = \{\alpha_k\}_{k=0}^{\infty} \in \ell^{c_o}_+$ and so $D_a \in \Delta_\infty$. Conclusion:

$$\Delta_\infty \ \text{ is closed in } \ \mathcal{B}[\ell^p_+].$$

Next consider a Δ_∞-valued sequence $\{D_n\}_{n\geq 1}$, where each D_n is a diagonal operator whose only nonzero entries are the first n entries in the main diagonal, which are all equal to 1:

$$D_n \,=\, \mathrm{diag}(1,\ldots,1,0,0,0,\ldots) \,\in\, \Delta_\infty.$$

Note that $\|D_n x - x\|^p = \sum_{k=n+1}^{\infty} |\xi_k|^p \to 0$ as $n \to \infty$ for every $x = \{\xi_k\}_{k=0}^{\infty}$ (because $\sum_{k=0}^{\infty} |\xi_k|^p = \|x\| < \infty$ — Problem 3.11). Hence

$$D_n \xrightarrow{s} I.$$

But the identity operator in $\mathcal{B}[\ell_+^p]$ is a diagonal operator that does not lie in Δ_∞ (i.e., $I \in \Delta \backslash \Delta_\infty$). Outcome:

$$\Delta_\infty \text{ is not strongly closed in } \mathcal{B}[\ell_+^p].$$

4.9 Compact Operators

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. A linear transformation $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is *compact* (or *completely continuous*) if it maps bounded subsets of $\mathcal{X}$ into relatively compact subsets of $\mathcal{Y}$; that is, if $T(A)^-$ is compact in $\mathcal{Y}$ whenever A is bounded in $\mathcal{X}$. Equivalently, $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is compact if $T(A)$ lies in a compact subset of $\mathcal{Y}$ whenever A is bounded in $\mathcal{X}$ (see Theorem 3.62).

Theorem 4.49. *If $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is compact, then $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$.*

Proof. Take an arbitrary bounded subset A of $\mathcal{X}$. If $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is compact, then $T(A)^-$ is a compact subset of $\mathcal{Y}$. Thus $T(A)^-$ is totally bounded in $\mathcal{Y}$ (Corollary 3.81), and so $T(A)^-$ is bounded in $\mathcal{Y}$ (Corollary 3.71). Hence $T(A)$ is a bounded subset of $\mathcal{Y}$. Thus T is bounded by Proposition 4.12. □

In other words, *every compact linear transformation is continuous*. The converse is clearly false, for the identity I of an infinite-dimensional normed space $\mathcal{X}$ into itself is not compact by Corollary 4.34. Recall that a transformation $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is of *finite rank* (or T is a *finite-dimensional linear transformation*) if it has a finite-dimensional range (see Problem 2.18).

Proposition 4.50. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. If $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is of finite rank, then it is compact.*

Proof. Take an arbitrary bounded subset A of $\mathcal{X}$. If $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$, then $T(A)$ is a bounded subset of $\mathcal{Y}$ (Proposition 4.12). Thus $T(A)^-$ is closed and bounded in $\mathcal{Y}$. If T is of finite rank, then the range $\mathcal{R}(T)$ of T is a finite-dimensional subspace of $\mathcal{Y}$ (Corollary 4.29), and hence $\mathcal{R}(T)$ is a closed subset of $\mathcal{Y}$. Thus $T(A)^-$ is closed and bounded in $\mathcal{R}(T)$, according to Problem 3.38(d), and so it is a compact subset of $\mathcal{R}(T)$ by Corollary 4.32. Since the metric space $\mathcal{R}(T)$ is a subspace of the metric space $\mathcal{Y}$, it follows that $T(A)^-$ is compact in $\mathcal{Y}$. □

Remark: The assumption "T is bounded" cannot be removed from the statement of Proposition 4.50. Actually, we have exhibited (see proof of Corollary 4.30, part (b)) an unbounded linear transformation L of an arbitrary infinite-dimensional normed space $\mathcal{X}$ into an arbitrary nonzero normed space $\mathcal{Y}$. If $\dim \mathcal{Y} = 1$, then $L: \mathcal{X} \to \mathcal{Y}$ is a linear transformation of rank 1 ($\dim \mathcal{R}(L) = 1$) that is not even bounded, and so L is not compact by Theorem 4.49.

Corollary 4.51. *If $\mathcal{X}$ is a finite-dimensional normed space and $\mathcal{Y}$ is any normed space, then every $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is of finite rank and compact $\mathcal{Y}$.*

Proof. If $\mathcal{X}$ is finite dimensional and $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$, then $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ (Corollary 4.30) and $\dim \mathcal{R}(T) < \infty$ (Problems 2.17 and 2.18). Thus T is bounded and of finite rank, and hence compact by Proposition 4.50. □

Theorem 4.52. *Let T be a linear transformation of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$. The following four assertions are pairwise equivalent.*

(a) *T is compact* (i.e., *T maps bounded sets into relatively compact sets*).

(b) *T maps the unit ball $B_1[0]$ into a relatively compact set.*

(c) *T maps every $B_1[0]$-valued sequence into a sequence that has a convergent subsequence.*

(d) *T maps bounded sequences into sequences that have a convergent subsequence.*

Moreover, each of the above equivalent assertions implies that

(e) *T maps bounded sets into totally bounded sets*,

which in turn implies that

(f) *T maps the unit ball $B_1[0]$ into a totally bounded set.*

Furthermore, if $\mathcal{Y}$ is a Banach space, then these six assertions are all pairwise equivalent.

Proof. Note that (a)⇒(b) trivially. Hence, in order to verify that (a), (b), (c), and (d) are pairwise equivalent, it is enough to show that (b)⇒(c)⇒(d)⇒(a).

Proof of (b)⇒(c). Take an arbitrary $B_1[0]$-valued sequence $\{x_n\}$. If (b) holds, then $\{Tx_n\}$ lies in a compact subset of $\mathcal{Y}$ or, equivalently, in a sequentially compact subset of $\mathcal{Y}$ (Corollary 3.81). Thus, according to Definition 3.76, the $\mathcal{Y}$-valued sequence $\{Tx_n\}$ has a convergent subsequence. Then (b) implies (c).

Proof of (c)⇒(d). Take an arbitrary $\mathcal{X}$-valued bounded sequence $\{x_n\}$ so that there is a real number $\beta \geq \sup_n \|x_x\|$ for which $\{\beta^{-1}x_n\}$ is a $B_1[0]$-valued sequence. If (c) holds, then $\{T(\beta^{-1}x_n)\} = \beta^{-1}\{Tx_n\}$ has a convergent subsequence, and so $\{Tx_n\}$ has a convergent subsequence. Thus (c) implies (d).

Proof of (d)⇒(a). Take an arbitrary bounded subset A of $\mathcal{X}$. If (d) holds, then every $T(A)$-valued sequence has a convergent subsequence, and hence every $T(A)^-$-valued sequence has a subsequence that converges in $T(A)^-$ by Theorem 3.30, which means that $T(A)^-$ is sequentially compact (Definition 3.76). Therefore (d) implies (a) by Corollary 3.81.

Moreover, (a) implies (e). Indeed, if $T(A)^-$ is a compact subset of $\mathcal{Y}$ whenever A is a bounded subset of $\mathcal{X}$, then $T(A)^-$ (and so $T(A)$) is totally bounded in $\mathcal{Y}$ whenever A is bounded in $\mathcal{X}$, by Corollary 3.81. Also, (e) trivially implies (f).

Conversely, suppose (f) holds true so that $T(B_1[0])$ is a totally bounded subset of $\mathcal{Y}$. If $\mathcal{Y}$ is a Banach space, then Corollary 3.84(b) ensures that $T(B_1[0])$ is relatively compact in $\mathcal{Y}$. Thus (f) implies (b) if $\mathcal{Y}$ is a Banach space. □

Remark: On a finite-dimensional normed space every operator is compact (cf. Corollary 4.51). So, by Theorem 4.52 and Corollary 4.34, *the identity operator on a normed space $\mathcal{X}$ is compact if and only if $\mathcal{X}$ is finite dimensional*.

Let $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ denote the collection of all compact linear transformations between normed spaces $\mathcal{X}$ and $\mathcal{Y}$. Thus $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}] \subseteq \mathcal{B}[\mathcal{X},\mathcal{Y}]$ by Theorem 4.49, and $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}] = \mathcal{B}[\mathcal{X},\mathcal{Y}]$ if $\dim \mathcal{X} < \infty$ by Corollary 4.51. Write $\mathcal{B}_\infty[\mathcal{X}]$ for $\mathcal{B}_\infty[\mathcal{X},\mathcal{X}]$: the collection of all compact operators on a normed space $\mathcal{X}$.

Theorem 4.53. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces.*

(a) *$\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ is a linear manifold of $\mathcal{B}[\mathcal{X},\mathcal{Y}]$.*

(b) *If $\mathcal{Y}$ is a Banach space, then $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ is a subspace of $\mathcal{B}[\mathcal{X},\mathcal{Y}]$.*

Proof. It is trivially verified that $\alpha T \in \mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ for every scalar α whenever $T \in \mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ (Theorem 4.52(d)). In order to verify that $S+T \in \mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ for every $S,T \in \mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$, proceed as follows. Take S,T in $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}] \subseteq \mathcal{B}[\mathcal{X},\mathcal{Y}]$ and let $\{x_n\}_{n\geq 1}$ be an arbitrary $\mathcal{X}$-valued bounded sequence. Theorem 4.52(d) ensures that there exists a subsequence of $\{Tx_n\}_{n\geq 1}$, say $\{Tx_{n_k}\}_{k\geq 1}$, that converges in $\mathcal{Y}$. Now consider the subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$, which is clearly bounded. Then (Theorem 4.52(d) again) the sequence $\{Sx_{n_k}\}_{k\geq 1}$ has a subsequence, say $\{Sx_{n_{k_j}}\}_{j\geq 1}$, that converges in $\mathcal{Y}$. Since $\{Tx_{n_{k_j}}\}_{j\geq 1}$ is a subsequence of the convergent sequence $\{Tx_{n_k}\}_{k\geq 1}$, it follows that $\{Tx_{n_{k_j}}\}_{j\geq 1}$ also converges in $\mathcal{Y}$ (Proposition 3.5). Thus $\{(S+T)x_{n_{k_j}}\}_{j\geq 1} = \{Sx_{n_{k_j}}\}_{j\geq 1} + \{Tx_{n_{k_j}}\}_{j\geq 1}$ is a convergent subsequence of $\{(S+T)x_n\}_{n\geq 1}$, and therefore $S+T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ is compact by Theorem 4.52(d). Hence $S+T \in \mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$. Conclusion: $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ is a linear manifold of $\mathcal{B}[\mathcal{X},\mathcal{Y}]$.

Claim. If $\mathcal{Y}$ is a Banach space, then $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ is closed in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$.

Proof. Take an arbitrary $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$-valued sequence $\{T_n\}$ that converges (uniformly) in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ to, say, $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$. Thus, for each $\varepsilon > 0$ there exists a positive integer n_ε such that

$$\|(T - T_{n_\varepsilon})x\| \leq \|T - T_{n_\varepsilon}\|\,\|x\| < \tfrac{\varepsilon}{2}\|x\|$$

for every $x \in \mathcal{X}$. Since T_{n_ε} is compact, it follows by Theorem 4.52(f) that the image $T_{n_\varepsilon}(B_1[0])$ of the unit ball $B_1[0]$ is totally bounded in $\mathcal{Y}$, and hence $T_{n_\varepsilon}(B_1[0])$ has a finite $\frac{\varepsilon}{2}$-net, say Y_ε (Definition 3.68). Therefore, for each $x \in B_1[0]$ there exists $y \in Y_\varepsilon$ such that $\|T_{n_\varepsilon}x - y\| < \frac{\varepsilon}{2}$, and so

$$\begin{aligned}\|Tx - y\| &= \|Tx - T_{n_\varepsilon}x + T_{n_\varepsilon}x - y\| \\ &\leq \|(T - T_{n_\varepsilon})x\| + \|T_{n_\varepsilon}x - y\| < \tfrac{\varepsilon}{2}\|x\| + \tfrac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

That is, Y_ε is a finite $\frac{\varepsilon}{2}$-net for $T(B_1[0])$, which means that $T(B_1[0])$ is totally bounded in $\mathcal{Y}$. Thus, if $\mathcal{Y}$ is a Banach space, then T is compact by Theorem 4.52. Conclusion: Every $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$-valued sequence that converges in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ has its limit in $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$. Hence $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ is closed in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ by the Closed Set Theorem (Theorem 3.30). □

Outcome: If $\mathcal{Y}$ is a Banach space, then $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ is a subspace of $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ (which are Banach spaces by Propositions 4.15 and 4.7). □

Recall that a *two-sided ideal* $\mathcal{I}$ of an algebra $\mathcal{A}$ is a subalgebra of $\mathcal{A}$ such that the product (both left and right products) of every element of $\mathcal{I}$ with any element of $\mathcal{A}$ is again an element of $\mathcal{I}$ (see Problem 2.30).

Proposition 4.54. *If $\mathcal{X}$ is a normed space, then $\mathcal{B}_\infty[\mathcal{X}]$ is a two-sided ideal of the normed algebra $\mathcal{B}[\mathcal{X}]$.*

Proof. $\mathcal{B}_\infty[\mathcal{X}]$ is a linear manifold of $\mathcal{B}[\mathcal{X}]$ by Theorem 4.53. Take $S \in \mathcal{B}_\infty[\mathcal{X}]$ and $T \in \mathcal{B}[\mathcal{X}]$ arbitrary.

Claim. Both ST and TS lie in $\mathcal{B}_\infty[\mathcal{X}]$.

Proof. Let A be any bounded subset of $\mathcal{X}$ so that $T(A)$ is bounded by Proposition 4.12. Since S is compact, it follows (by definition) that $S(T(A))^-$ is compact. Thus the composition ST maps bounded sets into relatively compact sets, which means that ST is compact. Moreover, $S(A)^-$ is compact as well. Since T is continuous, it follows that $T(S(A)^-)$ is compact by Theorem 3.64 (and so $T(S(A)^-)$ is closed — Theorem 3.62), and also that $T(S(A))^- = T(S(A)^-)^- = T(S(A)^-)$ according to Problem 3.46. Therefore $T(S(A))^-$ is compact. Thus the composition TS maps bounded sets into relatively compact sets, which means that TS is compact. □

Conclusion: $\mathcal{B}_\infty[\mathcal{X}]$ is a two-sided ideal of $\mathcal{B}[\mathcal{X}]$. □

Let $\mathcal{B}_0[\mathcal{X},\mathcal{Y}]$ denote the collection of all finite-rank bounded linear transformations of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$. Proposition 4.50 says that $\mathcal{B}_0[\mathcal{X},\mathcal{Y}] \subseteq \mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$, and $\mathcal{B}_0[\mathcal{X},\mathcal{Y}] = \mathcal{B}_\infty[\mathcal{X},\mathcal{Y}] = \mathcal{B}[\mathcal{X},\mathcal{Y}] = \mathcal{L}[\mathcal{X},\mathcal{Y}]$ if $\mathcal{X}$ is finite dimensional by Corollaries 4.30 and 4.51. We shall write $\mathcal{B}_0[\mathcal{X}]$ for $\mathcal{B}_0[\mathcal{X},\mathcal{X}]$: the collection of all finite-rank operators on $\mathcal{X}$. It is readily verified that both ST and TS lie in $\mathcal{B}_0[\mathcal{X}]$ for every $S \in \mathcal{B}_0[\mathcal{X}]$ and every $T \in \mathcal{B}[\mathcal{X}]$. Indeed, it is clear that ST is of finite rank (because the range of ST is trivially included in the range of S). Moreover, TS is of finite rank since $TS = T|_{\mathcal{R}(S)}S$ and the domain of $T|_{\mathcal{R}(S)}$ is the finite-dimensional range of S (and so its own range is finite dimensional as well — see Problem 2.17). Therefore $\mathcal{B}_0[\mathcal{X}]$ *is also a two-sided ideal of* $\mathcal{B}[\mathcal{X}]$.

Corollary 4.55. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. If $\mathcal{Y}$ is a Banach space, then every $\mathcal{B}_0[\mathcal{X},\mathcal{Y}]$-valued sequence that converges (uniformly) in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ has its limit in $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$.*

Proof. This is a straightforward application of Theorem 4.53(b) because $\mathcal{B}_0[\mathcal{X},\mathcal{Y}] \subseteq \mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$. Indeed, since $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ is closed in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$, the Closed Set Theorem says that every $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$-valued sequence (and so every $\mathcal{B}_0[\mathcal{X},\mathcal{Y}]$-valued sequence) that converges in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ has its limit in $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$. □

Example 4.N. Consider the diagonal operator $D_a \in \mathcal{B}[\ell^p_+]$ of Example 4.K (for some $p \geq 1$, where $a = \{\alpha_k\}_{k=0}^\infty$ lies in ℓ^∞_+). We shall show that

$$D_a \text{ is compact if and only if } \alpha_k \to 0 \text{ as } k \to \infty.$$

Consider the ℓ^p_+-valued sequence $\{e_j\}_{j=0}^\infty$ with $e_j = \{\delta_{jk}\}_{k=0}^\infty \in \ell^p_+$ for each $j \geq 0$ (just one nonzero entry equal to 1 at the jth position) of Example 4.H.

(a) For each nonnegative integer n set $D_{a_n} = \operatorname{diag}(\alpha_0, \ldots, \alpha_n, 0, 0, 0, \ldots)$ in $\mathcal{B}[\ell^p_+]$. It is readily verified that each D_{a_n} is of finite rank. (Indeed, y is in $\mathcal{R}(D_{a_n})$ if and only if $y = \sum_{i=0}^n \alpha_i \xi_i e_i$ for some $x = \{\xi_j\}_{j=0}^\infty$ in ℓ^p_+, so that $\mathcal{R}(D_{a_n}) \subseteq \operatorname{span}\{e_i\}_{i=0}^n$, and hence $\dim \mathcal{R}(D_{a_n}) \leq n$ by Theorem 2.6.) Moreover (Example 4.H again), $\|D_{a_n} - D_a\| = \sup_{k \geq n+1} |\alpha_k|$ for every $n \geq 0$. If $\lim_n |\alpha_n| = 0$, then $\limsup_n |\alpha_n| = \lim_n \sup_{k \geq n} |\alpha_k| = 0$ (Problem 3.13), and so $\lim_n \|D_{a_n} - D_a\| = 0$. Thus $\alpha_n \to 0$ implies $D_{a_n} \xrightarrow{u} D_a$, which in turn implies that D_a is compact by Corollary 4.55.

(b) Conversely, if $\alpha_j \not\to 0$, then $\{\alpha_j\}_{j=0}^\infty$ has a subsequence, say $\{\alpha_{j_n}\}_{n=0}^\infty$, such that $\inf_n |\alpha_{j_n}| > 0$. Set $\varepsilon = \inf_n |\alpha_{j_n}| > 0$, and note that $\|D_a e_{j_m} - D_a e_{j_n}\|^p = \|\alpha_{j_m} e_{j_m} - \alpha_{j_n} e_{j_n}\|^p = |\alpha_{j_m}|^p + |\alpha_{j_n}|^p \geq 2\varepsilon^p$ whenever $m \neq n$. Then every subsequence of $\{D_a e_{j_n}\}_{n=0}^\infty$ is not a Cauchy sequence, and so does not converge in ℓ^p_+. Therefore, the bounded sequence $\{e_{j_n}\}_{n=0}^\infty$ is such that its image under D_a, $\{D_a e_{j_n}\}_{n=0}^\infty$, has no convergent subsequence. Hence D_a is not compact by Theorem 4.52(c). Conclusion: If D_a is compact, then $\alpha_i \to 0$ as $i \to \infty$.

Example 4.O. According to Theorem 4.53 and Corollary 4.55, it follows that

$$\mathcal{B}_0[\mathcal{X},\mathcal{Y}]^- \text{ lies in } \mathcal{B}_\infty[\mathcal{X},\mathcal{Y}],$$

and

$$\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}] \text{ is closed in } \mathcal{B}[\mathcal{X},\mathcal{Y}],$$

whenever $\mathcal{Y}$ is a Banach space. Now consider the setup of Examples 4.M and 4.N, and note that $\Delta_\infty = \{D_a \in \Delta\colon a \in \ell^{c_o}_+\}$ is precisely the collection of all compact diagonal operators on the Banach space ℓ^p_+. It was shown in Example 4.M that Δ_∞ is not strongly closed in $\mathcal{B}[\ell^p_+]$ by exhibiting a sequence $\{D_n\}$ of finite-rank diagonal operators (and hence a sequence of compact diagonal operators) that converges strongly to the identity I on ℓ^p_+, which is not even compact. Moreover, strong convergence to compact diagonals is shown in Problem 4.53. Conclusion:

$$\mathcal{B}_0[\ell^p_+] \text{ is not strongly closed in } \mathcal{B}_\infty[\ell^p_+],$$

and

$$\mathcal{B}_0[\ell^p_+] \text{ and } \mathcal{B}_\infty[\ell^p_+] \text{ are not strongly closed in } \mathcal{B}[\ell^p_+].$$

It may be tempting to think that the converse of Corollary 4.55 holds. It in fact does hold whenever the Banach space $\mathcal{Y}$ has a Schauder basis (Problem 4.11), and it also holds if $\mathcal{Y}$ is a Hilbert space (next chapter). In these cases, *every T in $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ is the uniform limit of a $\mathcal{B}_0[\mathcal{X},\mathcal{Y}]$-valued sequence*. But such an italicized result fails in general (see Problem 4.58). However, every compact linear transformation comes close to having a finite-dimensional range in the following sense.

Proposition 4.56. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and take T in $\mathcal{L}[\mathcal{X},\mathcal{Y}]$. If T is compact, then for each $\varepsilon > 0$ there exists a finite-dimensional subspace $\mathcal{R}_\varepsilon$ of the range $\mathcal{R}(T)$ of T such that*

$$d(Tx, \mathcal{R}_\varepsilon) \le \varepsilon \|x\| \quad \text{for every} \quad x \in \mathcal{X}.$$

Proof. Take an arbitrary $\varepsilon > 0$. If $T \in \mathcal{L}[\mathcal{X},\mathcal{Y}]$ is compact, then the image $T(B_1[0])$ of the closed unit ball $B_1[0]$ is totally bounded in the normed space $\mathcal{R}(T)$ by Theorem 4.52(f). Thus there exists a finite ε-net for $T(B_1[0])$, say $\{v_i\}_{i=1}^{n_\varepsilon} \subset \mathcal{R}(T)$. That is, for every $y \in T(B_1[0])$ there exists $v_y \in \{v_i\}_{i=1}^{n_\varepsilon}$ such that $\|y - v_y\| < \varepsilon$. Set $\mathcal{R}_\varepsilon = \operatorname{span}\{v_i\}_{i=1}^{n_\varepsilon} \subseteq \mathcal{R}(T)$. $\mathcal{R}_\varepsilon$ is a finite-dimensional subspace of $\mathcal{R}(T)$ (Theorem 2.6, Proposition 4.7, and Corollary 4.28), and

$$\begin{aligned} \tfrac{1}{\|x\|} d(Tx, \mathcal{R}_\varepsilon) &= \tfrac{1}{\|x\|} \inf_{u \in \mathcal{R}_\varepsilon} \|Tx - u\| = \inf_{u \in \mathcal{R}_\varepsilon} \left\| \tfrac{Tx}{\|x\|} - \tfrac{u}{\|x\|} \right\| \\ &= \inf_{v \in \mathcal{R}_\varepsilon} \left\| T\big(\tfrac{x}{\|x\|}\big) - v \right\| \le \inf_{v \in \{v_i\}_{i=1}^{n_\varepsilon}} \left\| T\big(\tfrac{x}{\|x\|}\big) - v \right\| < \varepsilon \end{aligned}$$

for every nonzero x in $\mathcal{X}$, which concludes the proof (since $0 \in \mathcal{R}_\varepsilon$). □

Proposition 4.57. *The range of a compact linear transformation is separable.*

Proof. Let $\mathcal{X}$ be a normed space. Consider the collection $\{B_n(0)\}_{n\in\mathbb{N}}$ of all open balls with center at the origin of $\mathcal{X}$ and radius n. This covers $\mathcal{X}$, that is,

$$\mathcal{X} = \bigcup_n B_n(0).$$

Let $\mathcal{Y}$ be a normed space and let $T\colon \mathcal{X} \to \mathcal{Y}$ be any mapping of $\mathcal{X}$ into $\mathcal{Y}$ so that (Problem 1.2(e))

$$\mathcal{R}(T) = T(\mathcal{X}) = T\Big(\bigcup_n B_n(0)\Big) = \bigcup_n T(B_n(0)).$$

If $T \in \mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$, then each $T(B_n(0))$ is separable (Theorem 4.52 and Proposition 3.72), and hence each $T(B_n(0))$ has a countable dense subset. That is, for each $n \in \mathbb{N}$ there exists a countable set $A_n \subseteq T(B_n(0))$ such that $A_n^- = T(B_n(0))^-$ (Problem 3.38(g)). Therefore,

$$\bigcup_n A_n \subseteq \bigcup_n T\big(B_n(0)\big) \subseteq \bigcup_n A_n^- \subseteq \Big(\bigcup_n A_n\Big)^-$$

(Section 3.5). Thus $\big(\bigcup_n A_n\big)^- = \mathcal{R}(T)^-$ so that $\bigcup_n A_n$ is dense in $\mathcal{R}(T)$ (Problem 3.38(g)). Moreover, since each A_n is countable, it follows by Corollary 1.11 that the countable union $\bigcup_n A_n$ is countable as well. Outcome: $\bigcup_n A_n$ is a countable dense subset of $\mathcal{R}(T)$, which means that $\mathcal{R}(T)$ is separable. □

If $T\colon \mathcal{X} \to \mathcal{Y}$ is a compact linear transformation of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$, then the restriction $T|_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{Y}$ of T to a linear manifold $\mathcal{M}$ of $\mathcal{X}$ is a linear transformation (Problem 2.14). Moreover, it is clear by the very definition of compact linear transformations that $T|_{\mathcal{M}}$ is compact as well: *the restriction of a compact linear transformation $T\colon \mathcal{X} \to \mathcal{Y}$ to a linear manifold of $\mathcal{X}$ is again a compact linear transformation* (i.e., $T|_{\mathcal{M}}$ lies in $\mathcal{B}_\infty[\mathcal{M}, \mathcal{Y}]$ whenever T lies in $\mathcal{B}_\infty[\mathcal{X}, \mathcal{Y}]$). On the other hand, if $\mathcal{M}$ is a linear manifold of an infinite-dimensional Banach space $\mathcal{X}$, and $T\colon \mathcal{M} \to \mathcal{Y}$ is a compact linear transformation of $\mathcal{M}$ into a Banach space $\mathcal{Y}$, then it is easy to show that an arbitrary bounded linear extension of T over $\mathcal{X}$ may not be compact (e.g., see Problem 4.60). However, if $\mathcal{M}$ is dense in $\mathcal{X}$, then the extension of T over $\mathcal{X}$ must be compact. In fact, *the extension of a compact linear transformation $T\colon \mathcal{X} \to \mathcal{Y}$ over a completion of $\mathcal{X}$ into a completion of $\mathcal{Y}$ is again a compact linear transformation*. Recall that every bounded linear transformation T of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ has a unique (up to an isometric isomorphism) bounded linear extension $\widehat{T}$ over the (essentially unique) completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ into the (essentially unique) completion $\widehat{\mathcal{Y}}$ of $\mathcal{Y}$ (Theorems 4.40 through 4.42). The next theorem says that $\widehat{T} \in \mathcal{B}[\widehat{\mathcal{X}}, \widehat{\mathcal{Y}}]$ is compact whenever$T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is compact.

Theorem 4.58. *Let the Banach spaces $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ be completions of the normed spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. If T lies in $\mathcal{B}_\infty[\mathcal{X}, \mathcal{Y}]$, then its bounded linear extension $\widehat{T}\colon \widehat{\mathcal{X}} \to \widehat{\mathcal{Y}}$ lies in $\mathcal{B}_\infty[\widehat{\mathcal{X}}, \widehat{\mathcal{Y}}]$.*

Proof. Let $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ be completions of $\mathcal{X}$ and $\mathcal{Y}$. Thus there exist dense linear manifolds $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$ of $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ that are isometrically isomorphic to $\mathcal{X}$ and $\mathcal{Y}$, respectively (cf. Definition 4.39 and Theorem 4.40). Let $U_X \in \mathcal{G}[\widetilde{\mathcal{X}}, \mathcal{X}]$ and $U_Y \in \mathcal{G}[\mathcal{Y}, \widetilde{\mathcal{Y}}]$ denote such isometric isomorphisms. Take $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ and set $\widetilde{T} = U_Y T U_X \in \mathcal{B}[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}]$ so that the diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{U_Y} & \widetilde{\mathcal{Y}} \\ {\scriptstyle T}\uparrow & & \uparrow{\scriptstyle \widetilde{T}} \\ \mathcal{X} & \xleftarrow{U_X} & \widetilde{\mathcal{X}} \end{array}$$

commutes. Now take an arbitrary bounded $\widehat{\mathcal{X}}$-valued sequence $\{\widehat{x}_n\}$. Since $\widetilde{\mathcal{X}}^- = \widehat{\mathcal{X}}$ (i.e., since $\inf\{\|\widehat{x} - \widetilde{x}\|\colon \widetilde{x} \in \widetilde{\mathcal{X}}\} = 0$ for every $\widehat{x} \in \widehat{\mathcal{X}}$ — see Proposition 3.32), it follows that there exists an $\widetilde{\mathcal{X}}$-valued sequence $\{\widetilde{x}_n\}$ equiconvergent with $\{\widehat{x}_n\}$ (i.e., such that $\|\widehat{x}_n - \widetilde{x}_n\| \to 0$; for instance, for each integer

n take $\widetilde{x}_n$ in $\widetilde{\mathcal{X}}$ such that $\|\widehat{x}_n - \widetilde{x}_n\| < \frac{1}{n+1}$), which is bounded (since $\|\widetilde{x}_n\| \leq \|\widehat{x}_n - \widetilde{x}_n\| + \|\widehat{x}_n\|$ for every n). Consider the $\mathcal{X}$-valued sequence $\{x_n\}$ such that $x_n = U_X \widetilde{x}_n$ for each n, which is bounded too: $\|x_n\| = \|U_X \widetilde{x}_n\| = \|\widetilde{x}_n\|$ for every n. If T is compact, then the $\mathcal{Y}$-valued sequence $\{Tx_n\}$ has a convergent subsequence (Theorem 4.52), say, $\{Tx_{n_k}\}$. Thus $\{U_Y T x_{n_k}\}$ converges in $\widetilde{\mathcal{Y}}$ (because U_Y is a homeomorphism). Therefore $\{\widetilde{T}\widetilde{x}_{n_k}\}$ converges in $\widetilde{\mathcal{Y}}$ (since $\widetilde{T}\widetilde{x}_{n_k} = U_Y T U_X \widetilde{x}_{n_k} = U_Y T x_{n_k}$ for each k) to, say, $\widetilde{y} \in \widetilde{\mathcal{Y}} \subseteq \widehat{\mathcal{Y}}$. Hence $\{\widehat{T}\widehat{x}_{n_k}\}$ converges in $\widehat{\mathcal{Y}}$. Indeed,

$$\begin{aligned} \|\widehat{T}\widehat{x}_{n_k} - \widetilde{y}\| &= \|\widehat{T}(\widehat{x}_{n_k} - \widetilde{x}_{n_k}) + \widehat{T}\widetilde{x}_{n_k} - \widetilde{y}\| \\ &\leq \|\widehat{T}\|\,\|\widehat{x}_{n_k} - \widetilde{x}_{n_k}\| + \|\widetilde{T}\widetilde{x}_{n_k} - \widetilde{y}\| \end{aligned}$$

for every k because $\widetilde{T} = \widehat{T}|_{\widetilde{\mathcal{X}}}$, so that $\widehat{T}\widehat{x}_{n_k} \to \widetilde{y}$ (reason: $\|\widetilde{T}\widetilde{x}_{n_k} - \widetilde{y}\| \to 0$ and $\|\widehat{x}_{n_k} - \widetilde{x}_{n_k}\| \to 0$, see Proposition 3.5) as $k \to \infty$. Conclusion: $\widehat{T}$ maps bounded sequences into sequences that have a convergent subsequence; that is, $\widehat{T}$ is compact (Theorem 4.52). □

4.10 The Hahn–Banach Theorem and Dual Spaces

Three extremely important results on continuous (i.e., bounded) linear transformations, which yield a solid foundation for a large portion of modern analysis, are the *Open Mapping Theorem*, the *Banach–Steinhaus Theorem*, and the *Hahn–Banach Theorem*. The Hahn–Banach Theorem is concerned with the existence of bounded linear extensions for bounded linear functionals (i.e., for scalar-valued bounded linear transformations), and it is the basis for several existence results that are often applied in functional analysis. In particular, the Hahn–Banach Theorem ensures the existence of a large supply of continuous linear functionals on a normed space $\mathcal{X}$, and hence it is of fundamental importance for introducing the *dual space* of $\mathcal{X}$ (the collection of all continuous linear functionals on $\mathcal{X}$).

Let $\mathcal{M}$ be any linear manifold of a linear space $\mathcal{X}$ and consider a linear transformation $L: \mathcal{M} \to \mathcal{Y}$ of $\mathcal{M}$ into a linear space $\mathcal{Y}$. From a purely algebraic point of view, a plain linear extension $\widehat{L}: \mathcal{X} \to \mathcal{Y}$ of L over $\mathcal{X}$ has already been investigated in Theorem 2.9. On the other hand, if $\mathcal{M}$ is a dense linear manifold of the normed space $\mathcal{X}$ and $T: \mathcal{M} \to \mathcal{Y}$ is a bounded linear transformation of $\mathcal{M}$ into a Banach space $\mathcal{Y}$, then T has a unique bounded linear extension $\widehat{T}: \mathcal{X} \to \mathcal{Y}$ over $\mathcal{X}$ (Theorem 4.35). In particular, every bounded linear functional on a dense linear manifold $\mathcal{M}$ of a normed space $\mathcal{X}$ has a bounded linear extension over $\mathcal{X}$. The results of Section 3.8 (and also of Section 4.7), which ensure the existence of a uniformly continuous extension over a metric space X of a uniformly continuous mapping on a dense subset of X,

are called *extension by continuity*. What the Hahn–Banach Theorem does is to ensure the existence of a bounded linear extension $\widehat{f}\colon \mathcal{X} \to \mathbb{F}$ over $\mathcal{X}$ for every bounded linear functional $f\colon \mathcal{M} \to \mathbb{F}$ on *any* linear manifold $\mathcal{M}$ of the normed space $\mathcal{X}$. (Here $\mathcal{M}$ is not necessarily dense in $\mathcal{X}$ so that extension by continuity collapses.)

We shall approach the Hahn–Banach Theorem step by step. The first steps are purely algebraic and, as such, could have been introduced in Chapter 2. To begin with, we consider the following lemma on linear functionals, acting on a linear manifold of a real linear space, that are dominated by a sublinear functional (i.e., by a nonnegative homogeneous and subadditive functional).

Lemma 4.59. *Let $\mathcal{M}_0$ be a proper linear manifold of a real linear space $\mathcal{X}$. Take $x_1 \in \mathcal{X}\backslash\mathcal{M}_0$ and consider the linear manifold $\mathcal{M}_1$ of $\mathcal{X}$ generated by $\mathcal{M}_0$ and x_1,*

$$\mathcal{M}_1 \;=\; \mathcal{M}_0 + \operatorname{span}\{x_1\}.$$

Let $p\colon \mathcal{X} \to \mathbb{R}$ be a sublinear functional on $\mathcal{X}$. If $f_0\colon \mathcal{M}_0 \to \mathbb{R}$ is a linear functional on $\mathcal{M}_0$ such that

$$f_0(x) \;\le\; p(x) \quad \text{for every} \quad x \in \mathcal{M}_0,$$

then there exists a linear extension $f_1\colon \mathcal{M}_1 \to \mathbb{R}$ of f_0 over $\mathcal{M}_1$ such that

$$f_1(x) \;\le\; p(x) \quad \text{for every} \quad x \in \mathcal{M}_1.$$

Proof. Take an arbitrary vector x_1 in $\mathcal{X}\backslash\mathcal{M}_0$.

Claim. There exists a real number c_1 such that

$$-p(-w - x_1) - f_0(w) \;\le\; c_1 \;\le\; p(w + x_1) - f_0(w)$$

for every $w \in \mathcal{M}_0$.

Proof. Since the linear functional $f_0\colon \mathcal{M}_0 \to \mathbb{R}$ is dominated by a subadditive functional $p\colon \mathcal{X} \to \mathbb{R}$, it follows that

$$\begin{aligned} f_0(v) - f_0(u) \;=\; f_0(v-u) \;&\le\; p(v-u) \;=\; p(v + x_1 - u - x_1) \\ &\le\; p(v + x_1) + p(-u - x_1) \end{aligned}$$

for every $u, v \in \mathcal{M}_0$. Therefore,

$$-p(-u - x_1) - f_0(u) \;\le\; p(v + x_1) - f_0(v)$$

for every $u \in \mathcal{M}_0$ and every $v \in \mathcal{M}_0$. Set

$$a_1 = \sup_{u \in \mathcal{M}_0} \left(-p(-u - x_1) - f_0(u)\right) \quad \text{and} \quad b_1 = \inf_{v \in \mathcal{M}_0} \left(p(v + x_1) - f_0(v)\right).$$

The above inequality ensures that a_1 and b_1 are real numbers, and also that $a_1 \le b_1$. Thus the claimed result holds for any $c_1 \in [a_1, b_1]$. □

Recall that every x in $\mathcal{M}_1 = \mathcal{M}_0 + \operatorname{span}\{x_1\}$ can be *uniquely* written as $x = x_0 + \alpha x_1$ with x_0 in $\mathcal{M}_0$ and α in $\mathbb{R}$. Consider the functional $f_1\colon \mathcal{M}_1 \to \mathbb{R}$ defined by the formula

$$f_1(x) \;=\; f_0(x_0) + \alpha c_1$$

for every $x \in \mathcal{M}_1$, where the pair (x_0, α) in $\mathcal{M}_0 \times \mathbb{R}$ stands for the unique representation of x in $\mathcal{M}_0 + \operatorname{span}\{x_1\}$. It is readily verified that f_1 is a linear extension of f_0 over $\mathcal{M}_1$ (i.e., f_1 inherits the linearity of f_0 and $f_1|_{\mathcal{M}_0} = f_0$). We show now that p also dominates f_1. Take an arbitrary $x = x_0 + \alpha x_1$ in $\mathcal{M}_1$ and consider the three possibilities, viz., $\alpha = 0$, $\alpha > 0$, or $\alpha < 0$. If $\alpha = 0$, then $f_1(x) \le p(x)$ trivially (in this case, $f_1(x) = f_0(x_0) \le p(x_0) = p(x)$). Next recall that p is nonnegative homogeneous (i.e., $p(\gamma z) = \gamma p(z)$ for every $z \in \mathcal{X}$ and every $\gamma \ge 0$), and consider the above claimed inequalities. If $\alpha > 0$, then

$$\begin{aligned} f_1(x) \;=\; f_0(x_0) + \alpha c_1 \;&\le\; f_0(x_0) + \alpha p\big(\tfrac{x_0}{\alpha} + x_1\big) - \alpha f_0\big(\tfrac{x_0}{\alpha}\big) \\ &=\; p(x_0 + \alpha x_1) \;=\; p(x). \end{aligned}$$

On the other hand, if $\alpha < 0$, then

$$\begin{aligned} f_1(x) \;=\; f_0(x_0) + \alpha c_1 \;&=\; f_0(x_0) - |\alpha| c_1 \\ &\le\; f_0(x_0) + |\alpha| p\big(\tfrac{-x_0}{\alpha} - x_1\big) + |\alpha| f_0\big(\tfrac{-x_0}{\alpha}\big) \\ &=\; p(x_0 - |\alpha| x_1) \;=\; p(x_0 + \alpha x_1) \;=\; p(x), \end{aligned}$$

which concludes the proof. □

Theorem 4.60. (Real Hahn–Banach Theorem). *Let $\mathcal{M}$ be a linear manifold of a real linear space $\mathcal{X}$ and let $p\colon \mathcal{X} \to \mathbb{R}$ be a sublinear functional on $\mathcal{X}$. If $f\colon \mathcal{M} \to \mathbb{R}$ is a linear functional on $\mathcal{M}$ such that*

$$f(x) \;\le\; p(x) \quad \text{for every} \quad x \in \mathcal{M},$$

then there exists a linear extension $\widehat{f}\colon \mathcal{X} \to \mathbb{R}$ of f over $\mathcal{X}$ such that

$$\widehat{f}(x) \;\le\; p(x) \quad \text{for every} \quad x \in \mathcal{X}.$$

Proof. First note that, except for the dominance condition that travels from f to $\widehat{f}$, this could be viewed as a particular case of Theorem 2.9, whose kernel's proof is Zorn's Lemma. Let

$$\mathcal{K} \;=\; \big\{\varphi \in \mathcal{L}[\mathcal{N}, \mathbb{R}]\colon\; \mathcal{N} \in \mathcal{L}at(\mathcal{X}),\; \mathcal{M} \subseteq \mathcal{N} \text{ and } f = \varphi|_{\mathcal{M}}\big\}$$

be the collection of all linear functionals on linear manifolds of the real linear space $\mathcal{X}$ which extend the linear functional $f\colon \mathcal{M} \to \mathbb{R}$, and set

$$\mathcal{K}' \;=\; \big\{\varphi \in \mathcal{K}\colon\; \varphi(x) \le p(x) \text{ for every } x \in \mathcal{N} = \mathcal{D}(\varphi)\big\}.$$

Note that $\mathcal{K}'$ is not empty (for $f \in \mathcal{K}'$). Following the proof of Theorem 2.9, $\mathcal{K}$ is partially ordered (in the extension ordering, and so is its subcollection $\mathcal{K}'$),

and every chain in $\mathcal{K}$ has a supremum in $\mathcal{K}$. Then every chain $\{\varphi_\gamma\}$ in $\mathcal{K}'$ has a supremum $\bigvee_\gamma \varphi_\gamma$ in $\mathcal{K}$, which actually lies in $\mathcal{K}'$. Indeed, since each φ_γ is such that $\varphi_\gamma(x) \le p(x)$ for every x in the domain $\mathcal{D}(\varphi_\gamma)$ of φ_γ, and since $\{\varphi_\gamma\}$ is a chain, it follows that $\big(\bigvee_\gamma \varphi_\gamma\big)(x) \le p(x)$ for every x in the domain $\bigcup_\gamma \mathcal{D}(\varphi_\gamma)$ of $\bigvee_\gamma \varphi_\gamma$. (In fact, if $x \in \bigcup_\gamma \mathcal{D}(\varphi_\gamma)$, then $x \in \mathcal{D}(\varphi_\mu)$ for some $\varphi_\mu \in \{\varphi_\gamma\}$, and so $\big(\bigvee_\gamma \varphi_\gamma\big)(x) = \varphi_\mu(x) \le p(x)$ because $\big(\bigvee_\gamma \varphi_\gamma\big)|_{\mathcal{D}(\varphi_\mu)} = \varphi_\mu$.) Therefore, every chain in $\mathcal{K}'$ has a supremum (and hence an upper bound) in $\mathcal{K}'$. Thus, according to Zorn's Lemma, $\mathcal{K}'$ has a maximal element, say, $\varphi_0 : \mathcal{N}_0 \to \mathbb{R}$. Now we apply Lemma 4.59 to show that $\mathcal{N}_0 = \mathcal{X}$. Suppose $\mathcal{N}_0 \ne \mathcal{X}$. Take $x_1 \in \mathcal{X} \backslash \mathcal{N}_0$ and consider the linear manifold $\mathcal{N}_1$ of $\mathcal{X}$ generated by $\mathcal{N}_0$ and x_1,

$$\mathcal{N}_1 \;=\; \mathcal{N}_0 + \operatorname{span}\{x_1\},$$

which properly includes $\mathcal{N}_0$. Since $\varphi_0 \in \mathcal{K}'$, it follows that $\varphi_0(x) \le p(x)$ for every $x \in \mathcal{N}_0$. Thus, according to Lemma 4.59, there exists a linear extension $\varphi_1 : \mathcal{N}_1 \to \mathbb{R}$ of φ_0 over $\mathcal{N}_1$ such that $\varphi_1(x) \le p(x)$ for every $x \in \mathcal{N}_1$. Therefore $\varphi_0 \le \varphi_1 \in \mathcal{K}'$, which contradicts the fact that φ_0 is maximal in $\mathcal{K}'$ (for $\varphi_0 \ne \varphi_1$). Conclusion: $\mathcal{N}_0 = \mathcal{X}$. Outcome: φ_0 is a linear extension of f over $\mathcal{X}$ which is dominated by p. □

Theorem 4.61. (Hahn–Banach Theorem). *Let $\mathcal{M}$ be a linear manifold of a linear space $\mathcal{X}$ over $\mathbb{F}$ and let $p : \mathcal{X} \to \mathbb{R}$ be a seminorm on $\mathcal{X}$. If $f : \mathcal{M} \to \mathbb{F}$ is a linear functional on $\mathcal{M}$ such that*

$$|f(x)| \;\le\; p(x) \quad \textit{for every} \quad x \in \mathcal{M},$$

then there exists a linear extension $\widehat{f} : \mathcal{X} \to \mathbb{F}$ of f over $\mathcal{X}$ such that

$$|\widehat{f}(x)| \;\le\; p(x) \quad \textit{for every} \quad x \in \mathcal{X}.$$

Proof. As we have agreed in the introduction of this chapter, $\mathbb{F}$ denotes either the complex field $\mathbb{C}$ or the real field $\mathbb{R}$. Recall that a seminorm is a convex (i.e., an absolutely homogeneous and subadditive) nonnegative functional.

(a) If $\mathbb{F} = \mathbb{R}$, then this is an easy corollary of the previous theorem. Indeed, if $\mathbb{F} = \mathbb{R}$, then the condition $|f| \le p$ trivially implies $f \le p$ on $\mathcal{M}$. As a seminorm is a sublinear functional, Theorem 4.60 ensures the existence of a extension $\widehat{f} : \mathcal{X} \to \mathbb{R}$ of f over $\mathcal{X}$ such that $\widehat{f} \le p$ on $\mathcal{X}$. Since $\widehat{f}$ is linear and p is absolutely homogeneous, it follows that $-\widehat{f}(x) = \widehat{f}(-x) \le p(-x) = |-1|\, p(x) = p(x)$ for every $x \in \mathcal{X}$. Hence $-p \le \widehat{f}$, and so $|\widehat{f}\,| \le |p| = p$ on $\mathcal{X}$ (for p is nonnegative).

(b) Suppose $\mathbb{F} = \mathbb{C}$. Note that the complex linear space $\mathcal{X}$ can also be viewed as a real linear space (where scalar multiplication now means multiplication by *real* scalars only). Moreover, if $\mathcal{M}$ is a linear manifold of the (complex) linear space $\mathcal{X}$, then it is also a (real) linear manifold of $\mathcal{X}$ when $\mathcal{X}$ is regarded as a real linear space. Furthermore, if $f : \mathcal{M} \to \mathbb{C}$ is a complex-valued functional on $\mathcal{M}$, and if $g : \mathcal{M} \to \mathbb{R}$ and $h : \mathcal{M} \to \mathbb{R}$ are defined by $g(x) = \operatorname{Re} f(x)$ and $h(x) = \operatorname{Im} f(x)$ for every $x \in \mathcal{X}$, then

$$f \,=\, g + ih.$$

Now recall that $f\colon \mathcal{M} \to \mathbb{C}$ is linear. Thus, for an arbitrary $\alpha \in \mathbb{R}$,

$$g(\alpha x) + ih(\alpha x) \,=\, f(\alpha x) \,=\, \alpha f(x) \,=\, \alpha g(x) + i\alpha h(x),$$

and hence $g(\alpha x) = \alpha g(x)$ and $h(\alpha x) = \alpha h(x)$ for every $x \in \mathcal{M}$ (because g and h are real valued). Similarly, for every $x, y \in \mathcal{M}$,

$$g(x+y) + ih(x+y) = f(x+y) = f(x) + f(y) = g(x) + ih(x) + g(y) + ih(y),$$

and so $g(x+y) = g(x) + g(y)$ and $h(x+y) = h(x) + h(y)$.

Conclusion: $g\colon \mathcal{M} \to \mathbb{R}$ and $h\colon \mathcal{M} \to \mathbb{R}$ are *linear* functionals on $\mathcal{M}$ when $\mathcal{M}$ is regarded as a real linear space.

Observe that $f(ix) = if(x)$, and hence $g(ix) + ih(ix) \,=\, ig(x) - h(x)$, for every $x \in \mathcal{M}$. Since $g(x)$, $g(ix)$, $h(x)$, and $h(ix)$ are real numbers, it follows that $h(x) = -g(ix)$, and therefore

$$f(x) \,=\, g(x) - ig(ix),$$

for every $x \in \mathcal{M}$. If $|f| < p$ on $\mathcal{M}$, then $g = \operatorname{Re} f \le p$ on $\mathcal{M}$. Since g is a linear functional on the (real) linear manifold $\mathcal{M}$, and since p is a sublinear functional on the (real) linear space $\mathcal{X}$ (because it is a seminorm on the complex linear space $\mathcal{X}$), it follows by Theorem 4.60 that there exists a real-valued linear extension $\widehat{g}$ of g over the (real) linear space $\mathcal{X}$ such that $\widehat{g} < p$ on $\mathcal{X}$. Consider the functional $\widehat{f}\colon \mathcal{X} \to \mathbb{C}$ (on the complex linear space $\mathcal{X}$) defined by

$$\widehat{f}(x) \,=\, \widehat{g}(x) - i\,\widehat{g}(ix)$$

for every $x \in \mathcal{X}$. It is clear that $\widehat{f}$ extends f over $\mathcal{X}$ (reason: if $x \in \mathcal{M}$ — so that $ix \in \mathcal{M}$ — then $\widehat{f}(x) = g(x) - ig(ix) = f(x)$). It is also readily verified that $\widehat{f}$ is a linear functional on the complex space $\mathcal{X}$. Indeed, additivity and (real) homogeneity are trivially verified (because $\widehat{g}$ is additive and homogeneous on the real linear space $\mathcal{X}$). Thus it suffices to verify that $\widehat{f}(ix) = i\widehat{f}(x)$ for every $x \in \mathcal{X}$. In fact, $\widehat{f}(ix) \,=\, \widehat{g}(ix) - i\widehat{g}(-x) \,=\, \widehat{g}(ix) + i\widehat{g}(x) \,=\, i\widehat{f}(x)$ for every $x \in \mathcal{X}$. Therefore, $\widehat{f}$ *is a linear extension of* f *over* $\mathcal{X}$. Finally, we show that

$$|\widehat{f}(x)| \,\le\, p(x)$$

for every $x \in \mathcal{X}$. Take an arbitrary x in $\mathcal{X}$ and write the complex number $\widehat{f}(x)$ in polar form: $\widehat{f}(x) = \rho e^{i\theta}$ (if $\widehat{f}(x) = 0$, then $\rho = 0$ and θ is any number in $[0, 2\pi]$). Since $\widehat{f}$ is a linear functional on the complex space $\mathcal{X}$, it follows that $\widehat{f}(e^{-i\theta}x) = \rho = |\widehat{f}(x)|$, a real number. Then $\widehat{f}(e^{-i\theta}x) = \widehat{g}(e^{-i\theta}x)$, and hence

$$|\widehat{f}(x)| \,=\, \widehat{g}(e^{-i\theta}x) \,\le\, p(e^{-i\theta}x) \,=\, |e^{-i\theta}|p(x) \,=\, p(x),$$

since $p\colon \mathcal{X} \to \mathbb{R}$ is absolutely homogeneous on the complex space $\mathcal{X}$. □

Theorems 4.60 and 4.61 are called *Dominated Extension Theorems* (in which no topology is involved). The next one is the *Continuous Extension Theorem*.

Theorem 4.62. (Hahn–Banach Theorem in Normed Space). *Let $\mathcal{M}$ be a linear manifold of a normed space $\mathcal{X}$. Every bounded linear functional $f\colon \mathcal{M} \to \mathbb{F}$ defined on $\mathcal{M}$ has a bounded linear extension $\widehat{f}\colon \mathcal{X} \to \mathbb{F}$ over the whole space $\mathcal{X}$ such that $\|\widehat{f}\,\| = \|f\|$.*

Proof. Take an arbitrary $f \in \mathcal{B}[\mathcal{M}, \mathbb{F}]$ so that $|f(x)| \le \|f\|\,\|x\|$ for each $x \in \mathcal{M}$. Set $p(x) = \|f\|\,\|x\|$ for every $x \in \mathcal{X}$, which defines a seminorm on $\mathcal{X}$. (Indeed, $p\colon \mathcal{X} \to \mathbb{R}$ is a norm on $\mathcal{X}$ whenever $f \neq 0$ because it is a multiple of a norm $\|\ \|\colon \mathcal{X} \to \mathbb{R}$.) Since $|f(x)| \le p(x)$ for every $x \in \mathcal{M}$, it follows by the previous theorem that there exists a linear extension $\widehat{f}\colon \mathcal{X} \to \mathbb{F}$ of f over $\mathcal{X}$ such that

$$|\widehat{f}(x)| \;\le\; p(x) \;=\; \|f\|\,\|x\|$$

for every $x \in \mathcal{X}$. Thus $\widehat{f}$ is bounded (i.e., $\widehat{f} \in \mathcal{B}[\mathcal{X}, \mathbb{F}]$) and $\|\widehat{f}\,\| \le \|f\|$. On the other hand, $f(x) = \widehat{f}(x)$ for every $x \in \mathcal{M}$ (because $f = \widehat{f}|_{\mathcal{M}}$), and hence

$$\|f\| \;=\; \sup_{x\in\mathcal{M}} |\widehat{f}(x)| \;\le\; \sup_{x\in\mathcal{X}} |\widehat{f}(x)| \;=\; \|\widehat{f}\,\|.$$

Therefore $\|\widehat{f}\,\| = \|f\|$. □

Here are some useful consequences of the Hahn–Banach Theorem.

Corollary 4.63. *Let $\mathcal{M}$ be a proper subspace of a normed space $\mathcal{X}$. For each vector x in $\mathcal{X}\backslash\mathcal{M}$ there exists a bounded linear functional $f\colon \mathcal{X} \to \mathbb{F}$ such that $f(x) = 1$, $f(\mathcal{M}) = \{0\}$, and $\|f\| = d(x, \mathcal{M})^{-1}$.*

Proof. Take an arbitrary $x \in \mathcal{X}\backslash\mathcal{M}$. Since $\mathcal{M} = \mathcal{M}^-$, the distance from x to $\mathcal{M}$ is strictly positive, that is, $d(x, \mathcal{M}) \neq 0$ (Problem 3.43(b)). Thus $d(x, \mathcal{M})^{-1}$ is well defined. Consider the linear manifold $\mathcal{M}_x$ of $\mathcal{X}$ generated by $\mathcal{M}$ and x,

$$\mathcal{M}_x \;=\; \mathcal{M} + \operatorname{span}\{x\},$$

so that every y in $\mathcal{M}_x$ can be uniquely written as $y = u + \alpha x$ with u in $\mathcal{M}$ and α in $\mathbb{F}$. Let $f_x\colon \mathcal{M}_x \to \mathbb{F}$ be a functional on $\mathcal{M}_x$ defined by

$$f_x(u + \alpha x) \;=\; \alpha$$

for every $y = u + \alpha x \in \mathcal{M}_x$. It is easy to verify that f_x is linear, $f_x(\mathcal{M}) = \{0\}$, and $f_x(x) = 1$. Next we show that f_x is bounded (so that $f_x \in \mathcal{B}[\mathcal{M}_x, \mathbb{F}]$) and $\|f_x\| = d(x, \mathcal{M})^{-1}$. Consider the set

$$S \;=\; \big\{(u, \alpha) \in \mathcal{M}\times\mathbb{F}\colon\ (u, \alpha) \neq (0, 0)\big\}$$

and its partition $\{S_1, S_2\}$, where

$$S_1 \,=\, \big\{(u,\alpha)\in \mathcal{M}\times\mathbb{F}\colon\ \alpha\neq 0\big\},$$

$$S_2 \,=\, \big\{(u,\alpha)\in \mathcal{M}\times\mathbb{F}\colon\ u\neq 0 \text{ and } \alpha = 0\big\}.$$

Note that $y = u+\alpha x = 0$ (in $\mathcal{M}_x$) if and only if $u = 0$ (in $\mathcal{M}$) and $\alpha = 0$ (in $\mathbb{F}$) — reason: $\mathcal{M}$ is a linear manifold of $\mathcal{X}$ and $x\in\mathcal{X}\backslash\mathcal{M}$, so that $\operatorname{span}\{x\}\cap\mathcal{M} = \{0\}$. Hence $y = u+\alpha x\neq 0$ in $\mathcal{M}_x$ if and only if $(u,\alpha)\in S$, which implies that

$$\|f_x\| \,=\, \sup_{0\neq y\in\mathcal{M}_x} \tfrac{|f_x(y)|}{\|y\|} \,=\, \sup_{(u,\alpha)\in S} \tfrac{|\alpha|}{\|u+\alpha x\|} \,=\, \sup_{(u,\alpha)\in S_1} \tfrac{|\alpha|}{\|u+\alpha x\|},$$

since $\sup_{(u,\alpha)\in S_2} \frac{|\alpha|}{\|u+\alpha x\|} = 0$ and $S = S_1\cup S_2$. However, $\inf_{v\in\mathcal{M}}\|v+x\| = \inf_{v\in\mathcal{M}}\|x-v\| = d(x,\mathcal{M})\neq 0$, and so (Problem 4.5)

$$\sup_{(u,\alpha)\in S_1} \tfrac{|\alpha|}{\|u+\alpha x\|} \,=\, \sup_{(u,\alpha)\in S_1} \tfrac{1}{\|\alpha^{-1}u+x\|} \,=\, \sup_{v\in\mathcal{M}} \tfrac{1}{\|v+x\|} \,=\, \tfrac{1}{d(x,\mathcal{M})}.$$

Summing up: f_x is a bounded linear functional on the linear manifold $\mathcal{M}_x$ of $\mathcal{X}$ such that $f_x(x) = 1$, $f_x(\mathcal{M}) = \{0\}$, and $\|f_x\| = d(x,\mathcal{M})^{-1}$. Thus, according to Theorem 4.62, $f_x\colon\mathcal{M}_x\to\mathbb{F}$ has a bounded linear extension $f\colon\mathcal{X}\to\mathbb{F}$ over $\mathcal{X}$ such that $\|f\| = \|f_x\| = d(x,\mathcal{M})^{-1}$. Moreover, since $f|_{\mathcal{M}_x} = f_x$, we get $f(x) = f_x(x) = 1$ and $f(\mathcal{M}) = f_x(\mathcal{M}) = \{0\}$, which concludes the proof. □

Recall that $\{0\}$ is a proper subspace of any nonzero normed space $\mathcal{X}$. Thus, according to Corollary 4.63, for each $x\neq 0$ in $\mathcal{X}\neq\{0\}$ there exists a bounded linear functional $f\colon\mathcal{X}\to\mathbb{F}$ such that $f(x) = 1$ and $\|f\| = d(x,\{0\})^{-1} = \|x\|^{-1}$. Now set $f_x = \|x\|f$, which is again a bounded linear functional on $\mathcal{X}$, so that $f_x(x) = \|x\|$ and $\|f_x\| = 1$. Moreover, if $x = 0$, then take $x'\neq 0$ in $\mathcal{X}$ and a bounded linear functional $f_{x'}$ on $\mathcal{X}$ such that $f_{x'}(x') = \|x'\|$ and $\|f_{x'}\| = 1$. Since $f_{x'}$ is linear, $f_{x'}(x) = f_{x'}(0) = 0 = \|x\|$. This proves the next corollary.

Corollary 4.64. *For each vector x in a normed space $\mathcal{X}\neq\{0\}$ there exists a bounded linear functional $f\colon\mathcal{X}\to\mathbb{F}$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Consequently, there exist nonzero bounded linear functionals defined on every nonzero normed space.*

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces over the same field, and consider the normed space $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ of all bounded linear transformations of $\mathcal{X}$ into $\mathcal{Y}$. If $\mathcal{X}\neq\{0\}$, then Corollary 4.64 ensures the existence of $f\neq 0$ in $\mathcal{B}[\mathcal{X},\mathbb{F}]$. Suppose $\mathcal{Y}\neq\{0\}$, take any $y\neq 0$ in $\mathcal{Y}$, and set

$$T(x) \,=\, f(x)y$$

for every $x\in\mathcal{X}$. This defines a nonzero mapping $T\colon\mathcal{X}\to\mathcal{Y}$ which certainly is linear and bounded. Conclusion: *There exists $T\neq O$ in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ whenever $\mathcal{X}$ and $\mathcal{Y}$ are nonzero normed spaces.*

Example 4.P. Proposition 4.15 says that *$\mathcal{B}[\mathcal{X},\mathcal{Y}]$ is a Banach space whenever $\mathcal{Y}$ is a Banach space*. If $\mathcal{X} = \{0\}$, then $\mathcal{B}[\mathcal{X},\mathcal{Y}] = \{O\}$, which is a trivial

Banach space regardless of whether $\mathcal{Y}$ is Banach or not. Thus the converse of Proposition 4.15 should read as follows.

If $\mathcal{X} \neq \{0\}$ and $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is a Banach space, then $\mathcal{Y}$ is a Banach space.

Corollary 4.64 asserts that there exists $f \neq 0$ in $\mathcal{B}[\mathcal{X}, \mathbb{F}]$ whenever $\mathcal{X} \neq \{0\}$. Take an arbitrary Cauchy sequence $\{y_n\}$ in $\mathcal{Y}$ and consider the $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$-valued sequence $\{T_n\}$ such that, for each n,

$$T_n x \;=\; f(x)\, y_n$$

for every $x \in \mathcal{X}$. Each T_n in fact lies in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ because f lies in $\mathcal{B}[\mathcal{X}, \mathbb{F}]$. Indeed, for each integer n,

$$\|T_n\| \;=\; \sup_{\|x\| \leq 1} \|T_n x\| \;=\; \sup_{\|x\| \leq 1} |f(x)|\, \|y_n\| \;=\; \|f\|\, \|y_n\|$$

and, for any pair of integers m and n,

$$\|T_m - T_n\| \;=\; \sup_{\|x\| \leq 1} \|(T_m - T_n)x\| \;=\; \|f\|\, \|y_m - y_n\|.$$

Hence $\{T_n\}$ is a Cauchy sequence in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$. If $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is complete, then $\{T_n\}$ converges in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ to, say, $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$. Since $f \neq 0$, there exists $x_0 \in \mathcal{X}$ such that $f(x_0) \neq 0$. Therefore,

$$y_n = f(x_0)^{-1} T_n(x_0) \;\to\; f(x_0)^{-1} T(x_0) \quad \text{in} \quad \mathcal{Y}$$

(since uniform convergence implies strong convergence to the same limit), and so $\{y_n\}$ converges in $\mathcal{Y}$. Conclusion: *If* $\mathcal{X} \neq \{0\}$ *and* $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ *is complete, then* $\mathcal{Y}$ *is complete*.

The *dual space* (or *conjugate space*) of a normed space $\mathcal{X}$, denoted by $\mathcal{X}^*$, is the normed space of all continuous linear functionals on $\mathcal{X}$ (i.e., $\mathcal{X}^* = \mathcal{B}[\mathcal{X}, \mathbb{F}]$, where $\mathbb{F}$ stands either for the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, whether $\mathcal{X}$ is a real or complex normed space, respectively). Obviously, $\mathcal{X}^* = \{0\}$ whenever $\mathcal{X} = \{0\}$. Corollary 4.64 ensures the converse: $\mathcal{X}^* \neq \{0\}$ whenever $\mathcal{X} \neq \{0\}$. Indeed, if $f(x) = 0$ for all $f \in \mathcal{X}^*$, then $x = 0$. As a matter of fact, if $x \neq y$ in $\mathcal{X}$, then there exists $f \in \mathcal{X}^*$ such that $f(x) - f(y) = f(x - y) = \|x - y\| \neq 0$ (Corollary 4.64 again), and so $f(x) \neq f(y)$. This is usually expressed by saying that $\mathcal{X}^*$ *separates the points of* $\mathcal{X}$. Still from Corollary 4.64, for each nonzero $x \in \mathcal{X}$ there exists $f_0 \in \mathcal{X}^*$ such that $\|f_0\| = 1$ and $\|x\| = |f_0(x)|$. Therefore,

$$\|x\| \;=\; \frac{|f_0(x)|}{\|f_0\|} \;\leq\; \left\{ \begin{array}{l} \sup_{f \in \mathcal{X}^*,\, \|f\| \leq 1} |f(x)| \\ \sup_{0 \neq f \in \mathcal{X}^*} \frac{|f(x)|}{\|f\|} \end{array} \right\} \;\leq\; \|x\|$$

(recall that $|f(x)| \leq \|f\|\, \|x\|$ for every $x \in \mathcal{X}$ and every $f \in \mathcal{X}^*$), which shows a symmetry in the definitions of the norms in $\mathcal{X}$ and $\mathcal{X}^*$:

$$\|x\| \;=\; \sup_{f\in\mathcal{X}^*,\ \|f\|\le 1} |f(x)| \;=\; \sup_{0\neq f\in\mathcal{X}^*} \tfrac{|f(x)|}{\|f\|}$$

for every $x \in \mathcal{X}$. Note that, by Proposition 4.15, $\mathcal{X}^*$ *is a Banach space for every normed space* $\mathcal{X}$ (reason: $\mathcal{X}^* = \mathcal{B}[\mathcal{X}, \mathbb{F}]$ and $(\mathbb{F}, |\ |)$ is a Banach space).

Proposition 4.65. *If the dual space* $\mathcal{X}^*$ *of a normed space* $\mathcal{X}$ *is separable, then* $\mathcal{X}$ *itself is separable.*

Proof. If $\mathcal{X}^* = \{0\}$, then $\mathcal{X} = \{0\}$ and the result holds trivially. Thus suppose $\mathcal{X}^* \neq \{0\}$ is separable and consider the unit sphere about the origin of $\mathcal{X}^*$, viz., $S_1 = \{f \in \mathcal{X}^*\colon \|f\| = 1\}$. Since every subset of a separable metric space is separable (Corollary 3.36), it follows that S_1 includes a countable dense subset, say, $\{f_n\}$. For each f_n there exists $x_n \in \mathcal{X}$ such that

$$\|x_n\| = 1 \quad \text{and} \quad \tfrac{1}{2} < |f_n(x_n)|.$$

Indeed, $\sup_{\|x\|=1}|f_n(x)| = \|f_n\| = 1$ because $f_n \in S_1$. Consider the countable subset $\{x_n\}$ of $\mathcal{X}$ and set

$$\mathcal{M} \;=\; \operatorname{span}\{x_n\}.$$

If $\mathcal{M}^- \neq \mathcal{X}$, then Corollary 4.63 ensures that there exists $0 \neq f \in \mathcal{X}^*$ such that $f(\mathcal{M}) = \{0\}$. Set $f_0 = \|f\|^{-1} f$ in $\mathcal{X}^*$ so that

$$f_0 \in S_1 \quad \text{and} \quad f_0(x_n) = 0 \ \text{ for every } \ n.$$

Hence $|f_n(x_n)| \le |(f_n - f_0)(x_n)| \le \|f_n - f_0\|$, and therefore

$$\tfrac{1}{2} \;\le\; \|f_n - f_0\|$$

for every n. But this implies that the set $\{f_n\}$ is not dense in S_1 (see Proposition 3.32), which contradicts the very definition of $\{f_n\}$. Outcome: $\mathcal{M}^- = \mathcal{X}$, and so $\mathcal{X}$ is separable by Proposition 4.9(b). □

If $\mathcal{X} \neq \{0\}$, then $\mathcal{X}^* \neq \{0\}$ and so $(\mathcal{X}^*)^*$, the dual of $\mathcal{X}^*$, is again a nonzero Banach space. We shall write $\mathcal{X}^{**}$ instead of $(\mathcal{X}^*)^*$, which is called the *second dual* (or *bidual*) of $\mathcal{X}$. It is clear that $\mathcal{X}$, $\mathcal{X}^*$, and $\mathcal{X}^{**}$ are normed spaces over the same scalar field. The next result shows that $\mathcal{X}$ can be identified with a linear manifold of $\mathcal{X}^{**}$. Thus $\mathcal{X}$ is naturally embedded in its second dual $\mathcal{X}^{**}$.

Theorem 4.66. *Every normed space* $\mathcal{X}$ *is isometrically isomorphic to a linear manifold of* $\mathcal{X}^{**}$.

Proof. Suppose $\mathcal{X} \neq \{0\}$ (otherwise the result is trivial), take an arbitrary x in $\mathcal{X}$, and consider the functional $\varphi_x\colon \mathcal{X}^* \to \mathbb{F}$ defined on the dual $\mathcal{X}^*$ of $\mathcal{X}$ by

$$\varphi_x(f) \;=\; f(x)$$

for every f in $\mathcal{X}^*$. Since $\mathcal{X}^*$ is a linear space,

$$\varphi_x(\alpha f + \beta g) \,=\, (\alpha f + \beta g)(x) \,=\, \alpha f(x) + \beta g(x) \,=\, \alpha\varphi_x(f) + \beta\varphi_x(g)$$

for every $f, g \in \mathcal{X}^*$ and every $\alpha, \beta \in \mathbb{F}$, so that φ_x is linear. Moreover, since the elements of $\mathcal{X}^*$ are bounded and linear, it also follows that

$$|\varphi_x(f)| \,=\, |f(x)| \,\le\, \|f\|\,\|x\|$$

for every $f \in \mathcal{X}^*$, and so φ_x is bounded. Thus $\varphi_x \in \mathcal{X}^{**}$. Indeed,

$$\|\varphi_x\| \,=\, \|x\|,$$

since Corollary 4.64 ensures the existence of $f_0 \in \mathcal{X}^*$ such that $\|f_0\| = 1$ and $|f_0(x)| = \|x\|$, and therefore

$$\|\varphi_x\| \,=\, \sup_{\|f\|\le 1} |f(x)| \,\le\, \|x\| \,=\, |f_0(x)| \,=\, |\varphi_x(f_0)| \,\le\, \|\varphi_x\|\,\|f_0\| \,=\, \|\varphi_x\|.$$

Let $\Phi\colon \mathcal{X} \to \mathcal{X}^{**}$ be the mapping that assigns to each vector x in $\mathcal{X}$ the functional φ_x in $\mathcal{X}^{**}$; that is,

$$\Phi(x) \,=\, \varphi_x$$

for every $x \in \mathcal{X}$. It is easy to verify that Φ is linear. Since $\|\Phi(x)\| = \|x\|$ for every $x \in \mathcal{X}$, it follows that Φ is a linear isometry of $\mathcal{X}$ into $\mathcal{X}^{**}$ (Proposition 4.37). Hence $\Phi\colon \mathcal{X} \to \mathcal{R}(\Phi) \subseteq \mathcal{X}^{**}$ is an isometric isomorphism of $\mathcal{X}$ onto $\mathcal{R}(\Phi) = \Phi(\mathcal{X})$, the range of Φ. Thus the range $\Phi(\mathcal{X})$ of Φ is a linear manifold of $\mathcal{X}^{**}$ isometrically isomorphic to $\mathcal{X}$. □

Remark: If $\mathcal{X}$ is a Banach space, then the linear isometry $\Phi\colon \mathcal{X} \to \mathcal{X}^{**}$ has a closed range $\mathcal{R}(\Phi)$ — see Problem 4.41(d). Thus the previous theorem ensures that *every Banach space $\mathcal{X}$ is isometrically isomorphic to a subspace of $\mathcal{X}^{**}$.*

This linear isometry $\Phi\colon \mathcal{X} \to \mathcal{X}^{**}$ is known as the *natural embedding* of the normed space $\mathcal{X}$ into its second dual $\mathcal{X}^{**}$. If Φ is surjective (i.e., if $\Phi(\mathcal{X}) = \mathcal{X}^{**}$), then we say that $\mathcal{X}$ is *reflexive*. Equivalently, $\mathcal{X}$ is reflexive if and only if the *natural embedding* $\Phi\colon \mathcal{X} \to \mathcal{X}^{**}$ is an isometric isomorphism of $\mathcal{X}$ onto $\mathcal{X}^{**}$. Thus, if $\mathcal{X}$ is reflexive, then $\mathcal{X}$ and $\mathcal{X}^{**}$ are isometrically isomorphic (notation: $\mathcal{X} \cong \mathcal{X}^{**}$). The converse, however, fails: $\mathcal{X} \cong \mathcal{X}^{**}$ clearly implies $\Phi(\mathcal{X}) \cong \mathcal{X}^{**}$ (since $\Phi(\mathcal{X}) \cong \mathcal{X} \cong \mathcal{X}^{**}$ and composition of isometric isomorphisms is again an isometric isomorphism) but does not imply $\Phi(\mathcal{X}) = \mathcal{X}^{**}$. Since $\mathcal{X}^{**}$ (the dual of $\mathcal{X}^*$) always is a Banach space, it follows by Problem 4.37(b) that *every reflexive normed space is a Banach space*. Again, the converse fails (i.e., there exist nonreflexive Banach spaces, as we shall see in Example 4.S). Recall that separability is a topological invariant (see Problem 3.48), so that the converse of Proposition 4.65 holds for reflexive Banach spaces. Indeed, if $\mathcal{X} \cong \mathcal{X}^{**}$, then $\mathcal{X}$ is separable if and only if $\mathcal{X}^{**}$ is separable, which implies that $\mathcal{X}^*$ is separable by Proposition 4.65. Therefore, if $\mathcal{X}$ is separable and $\mathcal{X}^*$ is not separable, then $\mathcal{X} \ncong \mathcal{X}^{**}$ and so $\mathcal{X}$ is not reflexive: *a separable Banach space with a nonseparable dual is not reflexive*. This provides a necessary condition for reflexivity. Here is an equivalent condition.

Proposition 4.67. *A Banach space $\mathcal{X}$ is reflexive if and only for each $\varphi \in \mathcal{X}^{**}$ there exists $x \in \mathcal{X}$ such that*

$$\varphi(f) = f(x) \quad \textit{for every} \quad f \in \mathcal{X}^*.$$

Proof. Let $\Phi\colon \mathcal{X} \to \mathcal{X}^{**}$ be the natural embedding of $\mathcal{X}$ into $\mathcal{X}^{**}$ and take an arbitrary $\varphi \in \mathcal{X}^{**}$. There exists $x \in \mathcal{X}$ such that $\varphi(f) = f(x)$ for every $f \in \mathcal{X}^*$ if and only if $\varphi = \varphi_x$, which means that $\varphi \in \mathcal{R}(\Phi)$ (cf proof of Theorem 4.66). Therefore, for each $\varphi \in \mathcal{X}^{**}$ there exists an $x \in \mathcal{X}$ such that $\varphi(f) = f(x)$ for every $f \in \mathcal{X}^*$ if and only if Φ is surjective. □

Example 4.Q. Let $\mathcal{X}$ be a finite-dimensional normed space. In this case, $\dim \mathcal{X} = \dim \mathcal{X}^*$ by Problem 4.64. Moreover, $\dim \mathcal{X}^* = \dim \mathcal{X}^{**}$ because $\mathcal{X}^*$ is finite dimensional (Problem 4.64 again). Hence $\dim \mathcal{X} = \dim \mathcal{X}^{**}$. Take the natural embedding $\Phi\colon \mathcal{X} \to \mathcal{X}^{**}$ of $\mathcal{X}$ into $\mathcal{X}^{**}$. Since $\Phi(\mathcal{X})$ is a linear manifold of the finite-dimensional linear space $\mathcal{X}^{**}$, it follows by Problem 2.7 that $\Phi(\mathcal{X})$ also is a finite-dimensional linear space. Thus, as $\mathcal{X}$ and $\Phi(\mathcal{X})$ are topologically isomorphic finite-dimensional normed spaces, $\dim \Phi(\mathcal{X}) = \dim \mathcal{X}$ by Corollary 4.31. Then $\Phi(\mathcal{X})$ is a linear manifold of the finite-dimensional space $\mathcal{X}^{**}$ and $\dim \Phi(\mathcal{X}) = \dim \mathcal{X}^{**}$. Hence $\Phi(\mathcal{X}) = \mathcal{X}^{**}$ (Problem 2.7 again). Conclusion:

Every finite-dimensional normed space is reflexive.

Example 4.R. Take an arbitrary pair $\{p, q\}$ of Hölder conjugates (i.e., take real numbers $p > 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$), and consider the Banach spaces ℓ_+^p and ℓ_+^q of Example 4.B. It can be shown that there exists a natural isometric isomorphism $J_p\colon \ell_+^p \to (\ell_+^q)^*$ of ℓ_+^p onto the dual of ℓ_+^q. Thus, symmetrically, there exists a natural isometric isomorphism $J_q\colon \ell_+^q \to (\ell_+^p)^*$ of ℓ_+^q onto the dual of ℓ_+^p. It then follows by Problem 4.65 that there exists an isometric isomorphism $J_q^*\colon (\ell_+^q)^* \to (\ell_+^p)^{**}$ of the dual of ℓ_+^q onto the second dual of ℓ_+^p. Thus the composition $J_q^* J_p\colon \ell_+^p \to (\ell_+^p)^{**}$ is an isometric isomorphism of ℓ_+^p onto its second dual so that $\ell_+^p \cong (\ell_+^p)^{**}$. Moreover, it can also be shown that this isometric isomorphism $J_q^* J_p$ actually coincides with the natural embedding $\Phi\colon \ell_+^p \to (\ell_+^p)^{**}$. Conclusion:

ℓ_+^p is a reflexive Banach space for every $p > 1$.

In particular, the very special space ℓ_+^2, besides being reflexive, is also isometrically equivalent to its own dual (actually, as we shall see in Section 5.11, the *real* space ℓ_+^2 is isometrically isomorphic to $(\ell_+^2)^*$).

Example 4.S. There are, however, nonreflexive (also referred to as *irreflexive*) Banach spaces. For instance, consider the linear spaces ℓ_+^∞ and ℓ_+^1 equipped with their usual norms ($\|\ \|_\infty$ and $\|\ \|_1$, respectively). Since $\ell_+^{c_o}$ is a linear manifold of the linear space ℓ_+^∞, equip it with the sup-norm as well. Recall that $\ell_+^{c_o}$ and ℓ_+^1 are separable Banach spaces but the Banach space ℓ_+^∞ is not

separable (see Examples 3.P to 3.S and also Problems 3.49 and 3.59). It is not very difficult to verify that $(\ell_+^{c_o})^* \cong \ell_+^1$ and $(\ell_+^1)^* \cong \ell_+^\infty$, and so $(\ell_+^{c_o})^{**} \cong \ell_+^\infty$ (Problem 4.65). Thus ℓ_+^1 is a separable Banach space with a nonseparable dual (reason: $(\ell_+^1)^*$ is not separable because $(\ell_+^1)^* \cong \ell_+^\infty$ and separability is a topological invariant). Hence,

$$\ell_+^1 \text{ is a nonreflexive Banach space.}$$

Since $\ell_+^{c_o}$ is not even homeomorphic to ℓ_+^∞ (Problem 3.49) and $(\ell_+^{c_o})^{**} \cong \ell_+^\infty$, it follows that $\ell_+^{c_o} \ncong (\ell_+^{c_o})^{**}$. Therefore,

$$\ell_+^{c_o} \text{ is a nonreflexive Banach space.}$$

Suggested Reading

Bachman and Narici [1]
Banach [1]
Beauzamy [1]
Berberian [2]
Brown and Pearcy [2]
Conway [1]
Day [1]
Douglas [1]
Dunford and Schwartz [1]
Goffman and Pedrick [1]
Goldberg [1]
Hille and Phillips [1]
Istrăţescu [1]
Kantorovich and Akilov [1]
Kolmogorov and Fomin [1]
Kreyszig [1]
Lax [1]
Maddox [1]
Naylor and Sell [1]
Pietsch [1]
Reed and Simon [1]
Robertson and Robertson [1]
Rudin [1]
Rynne and Youngson [1]
Schwartz [1]
Simmons [1]
Taylor and Lay [1]
Yosida [1]

Problems

Problem 4.1. We shall say that a topology $\mathcal{T}$ on a linear space $\mathcal{X}$ over a field $\mathbb{F}$ is *compatible with the linear structure* of $\mathcal{X}$ if vector addition and scalar multiplication are continuous mappings of $\mathcal{X}\times\mathcal{X}$ into $\mathcal{X}$ and of $\mathbb{F}\times\mathcal{X}$ into $\mathcal{X}$, respectively. In this case $\mathcal{T}$ is said to be a *compatible topology* (or a *linear topology*) on $\mathcal{X}$. When we refer to continuity of the mappings $\mathcal{X}\times\mathcal{X} \to \mathcal{X}$ and $\mathbb{F}\times\mathcal{X} \to \mathcal{X}$ defined by $(x, y) \mapsto x + y$ and $(\alpha, x) \mapsto \alpha x$, respectively, it is understood that $\mathcal{X}\times\mathcal{X}$ and $\mathbb{F}\times\mathcal{X}$ are equipped with their product topology. If $\mathcal{X}$ is a metric space, then these are the topologies induced by any of the uniformly equivalent metrics of Problems 3.9 and 3.33. If $\mathcal{X}$ is a general topological space, then these are the product topologies (cf. remark in Problem 3.64). A *topological vector space* (or *topological linear space*) is a linear space $\mathcal{X}$ equipped with a compatible topology.

(a) Show that, for each y in a topological vector space $\mathcal{X}$ and each nonzero α in $\mathbb{F}$, the translation mapping $x \mapsto x + y$ and the scaling mapping $x \mapsto \alpha x$ are homeomorphisms of $\mathcal{X}$ onto itself.

(b) Show that every normed space is a topological vector space (a metrizable topological vector space, that is).

In other words, show that vector addition and scalar multiplication are continuous mappings of $\mathcal{X}\times\mathcal{X}$ into $\mathcal{X}$ and of $\mathbb{F}\times\mathcal{X}$ into $\mathcal{X}$, respectively, with respect to the norm topology on $\mathcal{X}$.

Problem 4.2. Consider the definitions of convex set and convex hull in a linear space (Problem 2.2). Recall that the closure of a subset of a topological space is the intersection of all closed subsets that include it. Prove the following assertions.

(a) In a topological vector space the closure of a convex set is convex.

The intersection of all closed and convex subsets that include a subset A of a topological vector space is called the *closed convex hull* of A.

(b) In a topological vector space the closed convex hull of a set A coincides with $\mathrm{co}(A)^-$ (i.e., it coincides with the closure of the convex hull of A).

A subset A of a linear space is *balanced* if $\alpha x \in A$ for every vector $x \in A$ and every scalar α such that $|\alpha| \le 1$ (i.e., if $\alpha A \subseteq A$ whenever $|\alpha| \le 1$). A subset of a linear space is *absolutely convex* if it is both convex and balanced.

(c) A subset A of a linear space is absolutely convex if and only if $\alpha x + \beta y \in A$ for every $x, y \in A$ and every scalars α, β such that $|\alpha| + |\beta| \le 1$ (hence the term "absolutely convex").

(d) The interior A° of an absolutely convex set A contains the origin whenever A° is nonempty.

(e) In a topological vector space the closure of a balanced set is balanced, and therefore the closure of an absolutely convex set is absolutely convex.

A subset A of a linear space $\mathcal{X}$ is *absorbing* (or *absorbent*) if for each vector $x \in \mathcal{X}$ there exists $\varepsilon > 0$ such that $\alpha x \in A$ for every scalar α with $|\alpha| \le \varepsilon$. Equivalently, if for each $x \in \mathcal{X}$ there exists $\lambda > 0$ such that $x \in \mu A$ for every scalar μ with $|\mu| \ge \lambda$.

(f) In a topological vector space every neighborhood of the origin is absorbing.

A subset A of a linear space $\mathcal{X}$ *absorbs* a subset B of $\mathcal{X}$ (or B is *absorbed* by A) if there is a $\beta > 0$ such that $x \in B$ implies $x \in \mu A$ for every scalar μ with $|\mu| \ge \beta$ (i.e., if $B \subseteq \mu A$ whenever $|\mu| \ge \beta$). Thus A is absorbing if and only if it absorbs every singleton $\{x\}$ in $\mathcal{X}$. A subset B of a topological vector space is said to be *bounded* if it is absorbed by every neighborhood of the origin.

Problem 4.3. Let $\mathcal{X}$ be a linear space over a field $\mathbb{F}$ (either $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$). A *quasinorm* on $\mathcal{X}$ is a real-valued positive subadditive functional $\|\ \|\colon \mathcal{X} \to \mathbb{R}$ on $\mathcal{X}$ that satisfies the axioms (i), (ii), and (iv) of Definition 4.1 but, instead of axiom (iii), it satisfies the following ones.

$$\begin{aligned} &\text{(iii}'\text{)} \quad \|\alpha x\| \le \|x\| \ \text{ whenever } \ |\alpha| \le 1, \\ &\text{(iii}''\text{)} \quad \|\alpha_n x\| \to 0 \ \text{ whenever } \ \alpha_n \to 0, \end{aligned}$$

for every $x \in \mathcal{X}$ (where α stands for a scalar in $\mathbb{F}$ and $\{\alpha_n\}$ for a scalar-valued sequence). A linear space $\mathcal{X}$ equipped with a quasinorm is called a *quasinormed space*. Consider the mapping $\mathrm{d}\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ defined by $\mathrm{d}(x, y) = \|x - y\|$ for every $x, y \in \mathcal{X}$, where $\|\ \|\colon \mathcal{X} \to \mathbb{R}$ is a quasinorm on $\mathcal{X}$.

(a) Show that d is an additively invariant metric on $\mathcal{X}$ that also satisfies

$$\mathrm{d}(\alpha x, \alpha y) \ \le \ \mathrm{d}(x, y)$$

for every $x, y \in \mathcal{X}$ and every $\alpha \in \mathbb{F}$ such that $|\alpha| \le 1$.

This is called the *metric generated* by the quasinorm $\|\ \|$.

(b) Show that a norm on $\mathcal{X}$ is a quasinorm on $\mathcal{X}$, so that every normed space is a quasinormed space.

A quasinormed space that is complete as a metric space (with respect to the metric generated by the quasinorm) is called an *F-space*.

(c) Show that every Banach space is an F-space.

Problem 4.4. A *neighborhood base* at a point x in a topological vector space is a collection $\mathcal{N}$ of neighborhoods of x with the property that every neighborhood of x includes some neighborhood in $\mathcal{N}$. A *locally convex space* (or simply a *convex space*) is a topological vector space that has a neighborhood base at the origin consisting of convex sets.

(a) Show that every normed space is locally convex.

A *barrel* is a subset of a locally convex space that is convex, balanced, absorbing, and closed. It can be shown that every locally convex space has a neighborhood base at the origin consisting of barrels. A locally convex space is called *barreled* if every barrel is a neighborhood of the origin. Barreled spaces can be thought of as a generalization of Banach spaces. Indeed, a sequence $\{x_n\}$ in a locally convex space is a *Cauchy sequence* if for each neighborhood N at the origin there exists an integer $n_{_N}$ such that $x_m - x_n \in N$ for all $m, n \ge n_{_N}$. It can be verified that every convergent sequence in a metrizable locally convex space $\mathcal{X}$ (i.e., every sequence that is eventually in every open neighborhood of a point x in $\mathcal{X}$) is a Cauchy sequence. A set A in a metrizable locally convex space is *complete* if every Cauchy sequence in A converges to a point of A. A complete metrizable locally convex space is called a *Fréchet space*. Recall that every Banach space is an F-space (see Problem 4.3).

(b) Show that every F-space is a Fréchet space.

(c) Show that every Fréchet space is a barreled space.

Hint: A Fréchet space $\mathcal{X}$ is a complete metric space. Take any barrel B in $\mathcal{X}$. Show that the countable collection $\{nB\}_{n\geq 1}$ of closed sets covers $\mathcal{X}$. Apply the Baire Category Theorem (Theorem 3.58); see Problem 4.2(d).

We shall return to barreled spaces in Problem 4.44.

Problem 4.5. Recall that subset A of a metric space is bounded if and only if $\operatorname{diam}(A) < \infty$ (see Section 3.1). Prove the following propositions.

(a) A subset A of a normed space is bounded if and only if $\sup_{x\in A}\|x\| < \infty$. (By convention, $\sup_{x\in\varnothing}\|x\| = 0$.)

Now consider the definition of a bounded subset of a topological vector space as given in Problem 4.2.

(b) A set A is bounded as a subset of a normed space $\mathcal{X}$ if and only if it is bounded as a subset of the topological vector space $\mathcal{X}$. In other words, $\sup_{x\in A}\|x\| < \infty$ if and only if A is absorbed by every neighborhood of the origin of $\mathcal{X}$. Thus the notion of a bounded subset of a normed space is unambiguously defined.

Let A be a subset of a normed space that contains the origin ($A\backslash\{0\} \neq \varnothing$).

(c) $\sup_{x\in A}\|x\| < \infty$ implies $\inf_{x\in A\backslash\{0\}}\|x\|^{-1} = \big(\sup_{x\in A}\|x\|\big)^{-1}$.

(d) $\inf_{x\in A}\|x\| \neq 0$ implies $\sup_{x\in A\backslash\{0\}}\|x\|^{-1} = \big(\inf_{x\in A}\|x\|\big)^{-1}$.

(e) $\inf_{x\in A}\|x\| \neq 0$ if and only if $\sup_{x\in A\backslash\{0\}}\|x\|^{-1} < \infty$.

Note: $\inf_{x\in A}\|x\| \neq 0$ if and only if $\inf_{x\in A}\|x\| > 0$, $\inf_{x\in A}\|x\| \leq \sup_{x\in A}\|x\|$, and $\inf_{x\in A}\|x\| < \infty$ even if A is unbounded.

A subset $A \neq \varnothing$ of a normed space is *bounded away from zero* if $\inf_{x\in A}\|x\| \neq 0$. A mapping F of a nonempty set S into a normed space $\mathcal{X}$ is bounded if and only if $\sup_{s\in S}\|F(s)\| < \infty$, and it is bounded away from zero if and only if $\inf_{s\in S}\|F(s)\| \neq 0$. Thus an $\mathcal{X}$-valued sequence $\{x_n\}$ is bounded if and only if $\sup_n\|x_n\| < \infty$, and it is bounded away from zero if and only if $\inf_n\|x_n\| \neq 0$.

Problem 4.6. This problem is entirely based on the triangle inequality. Consider the space $(\ell^1_+, \|\ \|_1)$ and $(\ell^\infty_+, \|\ \|_\infty)$ of Example 4.B. Take $x = \{\xi_k\}_{k=1}^\infty$ in ℓ^∞_+ and let $\{x_n\}_{n=1}^\infty$ be an ℓ^1_+-valued sequence (i.e., each $x_n = \{\xi_{n(k)}\}_{k=1}^\infty$ lies in ℓ^1_+). Recall: $\ell^1_+ \subset \ell^\infty_+$. Show that

(a) if $\|x_n - x\|_\infty \to 0$ and $\sup_n\|x_n\|_1 < \infty$, then $x \in \ell^1_+$.

Hint: $\sum_{k=1}^m|\xi_k| \leq m\|x_n - x\|_\infty + \sup_n\|x_n\|_1$ for each $m \geq 1$.

Now suppose $x \in \ell^1_+$ and show that

(b) if $\|x_n - x\|_\infty \to 0$ and $\|x_n\|_1 \to \|x\|_1$, then $\|x_n - x\|_1 \to 0$.

Hint: If $x = \{\xi_k\}_{k=1}^\infty$ and $z = \{\zeta_k\}_{k=1}^\infty$ are in ℓ_+^1, then, for each $m \geq 1$,

$$\|z\|_1 + \|x\|_1 \leq \|z - x\|_1 + 2m\|z\|_\infty + 2\textstyle\sum_{k=m+1}^\infty |\xi_k|.$$

Observe that $\|z\|_1 + \|x\|_1 = \sum_{k=1}^m (|\xi_k| - |\zeta_k|) + \sum_{k=m+1}^\infty (|\zeta_k| - |\xi_k|) + 2\big(\sum_{k=1}^m |\zeta_k| + \sum_{k=m+1}^\infty |\xi_k|\big)$ and $\big||\xi_k| - |\zeta_k|\big| \leq |\xi_k - \zeta_k|$. Prove the above auxiliary inequality and conclude: if $x, y \in \ell_+^1$, then

$$\|y - x\|_1 \leq \big|\, \|y\|_1 - \|x\|_1 \big| + 2m\|y - x\|_\infty + 2\textstyle\sum_{k=m+1}^\infty |\xi_k|$$

for each $m \geq 1$. Show that, under the assumptions of (b),

$$\limsup_n \|x_n - x\|_1 \leq 2\textstyle\sum_{k=m+1}^\infty |\xi_k| \quad \text{for all} \quad m \geq 1.$$

Next suppose the sequence $\{x_n\}_{n=1}^\infty$ is dominated by a vector y in ℓ_+^1. That is, $|\xi_{n(k)}| \leq |\upsilon_k|$ for each $k \geq 1$ and all $n \geq 1$, for some $y = \{\upsilon_k\}_{k=1}^\infty$ in ℓ_+^1. Equivalently, $\sum_{k=1}^\infty \sup_n |\xi_{n(k)}| < \infty$. Show that

(c) if $\|x_n - x\|_\infty \to 0$ and $\sum_{k=1}^\infty \sup_n |\xi_{n(k)}| < \infty$, then $\|x_n - x\|_1 \to 0$.

Hint: Item (a) and the dominance condition ensure that $x \in \ell_+^1$ (since $\sup_n \|x_n\|_1 \leq \|y\|_1$ for some $y \in \ell_+^1$). Moreover, for each $m \geq 1$,

$$\|x_n - x\|_1 \leq m\|x_n - x\|_\infty + \textstyle\sum_{k=m+1}^\infty |\xi_{n(k)}| + \sum_{k=m+1}^\infty |\xi_k|.$$

Extend these results to $\ell_+^1(\mathcal{X})$ and $\ell_+^\infty(\mathcal{X})$ as in Example 4.F.

Problem 4.7. Let $\{x_i\}_{i=1}^\infty$ be a sequence in a normed space $\mathcal{X}$ and consider the sequence $\{y_n\}_{n=1}^\infty$ of partial sums of $\{x_i\}_{i=1}^\infty$; that is, $y_n = \sum_{i=1}^n x_i$ for each $n \geq 1$. Prove that the following assertions are pairwise equivalent.

(a) $\{y_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{X}$.

(b) The real sequence $\{\|\sum_{i=n}^{n+k} x_i\|\}_{n=1}^\infty$ converges to zero uniformly in k, i.e.,

$$\lim_n \sup_{k \geq 0} \Big\| \sum_{i=n}^{n+k} x_i \Big\| = 0.$$

(c) For each real number $\varepsilon > 0$ there exists an integer $n_\varepsilon \geq 1$ such that

$$\Big\| \sum_{i=m}^{n} x_i \Big\| < \varepsilon \quad \text{whenever} \quad n_\varepsilon \leq m \leq n.$$

(*Hint*: Problem 3.51.) Observe that, according to item (b), $x_n \to 0$ in $\mathcal{X}$ whenever $\{\sum_{i=1}^n x_i\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{X}$ and, in particular, whenever the infinite series $\sum_{i=1}^\infty x_i$ converges in $\mathcal{X}$. That is, $x_n \to 0$ *in* $\mathcal{X}$ *for every summable sequence* $\{x_i\}_{i=1}^\infty$ *in* $\mathcal{X}$. Now suppose $\mathcal{X}$ is a Banach space. Since $\mathcal{X}$ is complete, the very definition of completeness says that each of the above equivalent assertions also is equivalent to the following one.

(d) The infinite series $\sum_{i=1}^{\infty} x_i$ converges in $\mathcal{X}$ (i.e., the sequence $\{x_i\}_{i=1}^{\infty}$ is summable).

If $\mathcal{X}$ is a Banach space, then condition (c) (or condition (b)) is referred to as the *Cauchy criterion* for convergent infinite series.

Problem 4.8. Proposition 4.4 says that every absolutely summable sequence in a Banach space is summable.

(a) Consider the ℓ_+^2-valued sequence $\{x_i\}_{i=1}^{\infty}$ with each $x_i = \frac{1}{i}e_i$, where $e_i = \{\delta_{ik}\}_{k=1}^{\infty}$ in ℓ_+^2 (just one nonzero entry equal to 1 at the ith position) for each positive integer i. Show that $\{x_i\}_{i=1}^{\infty}$ is a summable sequence in the Banach space ℓ_+^2 but is not absolutely summable.

Problem 4.7 says that $\{x_i\}_{i=1}^{\infty}$ is a summable sequence in a Banach space if and only if $\lim_n \sup_{k\geq 0}\|\sum_{i=n}^{n+k} x_i\| = 0$, which implies that $\lim_n \|\sum_{i=n}^{n+k} x_i\| = 0$ for every integer $k \geq 0$.

(b) Give an example of a nonsummable sequence (or, equivalently, an example of a nonconvergent series) in a Banach space such that $\lim_n \|\sum_{i=n}^{n+k} x_i\| = 0$ for every integer $k \geq 0$. (*Hint*: $\xi_i = \frac{1}{i}$ in $\mathbb{R}$.)

Problem 4.9. Prove the following propositions.

(a) If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are summable sequences in a normed space $\mathcal{X}$, then $\{\alpha x_n + \beta y_n\}_{n=1}^{\infty} = \alpha\{x_n\}_{n=1}^{\infty} + \beta\{y_n\}_{n=1}^{\infty}$ is again a summable sequence in $\mathcal{X}$, and

$$\sum_{j=1}^{\infty}(\alpha x_j + \beta y_j) \;=\; \alpha\sum_{j=1}^{\infty} x_j + \beta\sum_{j=1}^{\infty} y_j,$$

for every pair of scalars $\alpha, \beta \in \mathbb{F}$. This shows that the collection of all $\mathcal{X}$-valued summable sequences is a linear manifold of the linear space $\mathcal{X}^{\mathbb{N}}$ (see Example 2.F), and hence is a linear space itself.

(b) If $\{x_n\}_{n=1}^{\infty}$ is a summable sequences in a normed space $\mathcal{X}$, then

$$\sum_{j=n}^{\infty} x_j \to 0 \;\text{ in }\; \mathcal{X} \quad \text{as} \quad n \to \infty.$$

Hint: $\sum_{j=1}^{m} x_j = \sum_{j=1}^{n-1} x_j + \sum_{j=n}^{m} x_j$ for every $1 < n \leq m$. Now use item (a) to verify that the series $\sum_{j=n}^{\infty} x_j = \sum_{j=1}^{\infty} x_{n+1-j}$ converges in $\mathcal{X}$, and $\sum_{j=1}^{\infty} x_j = \sum_{j=1}^{n-1} x_j + \sum_{j=n}^{\infty} x_j$ for every $n > 1$. Thus the result follows by uniqueness of the limit.

Problem 4.10. Take a sequence $\boldsymbol{x} = \{x_i\}_{i=1}^{\infty}$ of linearly independent vectors in a normed space $\mathcal{X}$. Let $\mathcal{A}_{\boldsymbol{x}}$ be the collection of all scalar-valued sequences $a = \{\alpha_i\}_{i=1}^{\infty}$ for which the series $\sum_{j=1}^{\infty} \alpha_j x_j$ converges in $\mathcal{X}$ (i.e., such that $\{\alpha_i x_i\}_{i=1}^{\infty}$ is a summable sequence in $\mathcal{X}$). Show that

(a) $\mathcal{A}_{\boldsymbol{x}}$ is a linear manifold of the linear space $\mathbb{F}^{\mathbb{N}}$.

Recall: $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, whether $\mathcal{X}$ is a complex or real linear space, respectively. Verify that the function $\|\ \| : \mathcal{A}_{\boldsymbol{x}} \to \mathbb{R}$, defined by

$$\text{(b)} \qquad \|a\| \,=\, \sup_n \Big\| \sum_{i=1}^{n} \alpha_i x_i \Big\|_X$$

for every $a = \{\alpha_i\}_{i=1}^{\infty}$ in $\mathcal{A}_{\boldsymbol{x}}$, is a norm on $\mathcal{A}_{\boldsymbol{x}}$. Take an arbitrary integer $i \ge 1$. Show that

$$|\alpha_i| \|x_i\|_X \,\le\, 2\|a\|$$

for every $a = \{\alpha_i\}_{i=1}^{\infty}$ in $\mathcal{A}_{\boldsymbol{x}}$. (*Hint*: $\alpha_i x_i = \sum_{j=1}^{i} \alpha_j x_j - \sum_{j=1}^{i-1} \alpha_j x_j$.) Hence, since $\mathcal{A}_{\boldsymbol{x}}$ is a normed space by (a) and (b),

$$\text{(c)} \qquad |\alpha_i - \beta_i| \|x_i\|_X \,\le\, 2\|a - b\|$$

for every $a = \{\alpha_i\}_{i=1}^{\infty}$ and $b = \{\beta_i\}_{i=1}^{\infty}$ in $\mathcal{A}_{\boldsymbol{x}}$. Now let $\{a_k\}_{k=1}^{\infty}$ be a Cauchy sequence in the normed space $\mathcal{A}_{\boldsymbol{x}}$. That is, each $a_k = \{\alpha_{k(i)}\}_{i=1}^{\infty}$ lies in $\mathcal{A}_{\boldsymbol{x}}$ and the $\mathcal{A}_{\boldsymbol{x}}$-valued sequence $\{a_k\}_{k=1}^{\infty}$ is Cauchy in $(\mathcal{A}_{\boldsymbol{x}}, \|\ \|)$, where $\|\ \|$ is the norm in (b). According to (c), $|\alpha_{k(i)} - \alpha_{\ell(i)}| \,\le\, 2\|x_i\|_X^{-1} \|a_k - a_\ell\|$ for each $i \ge 1$ and every $k, \ell \ge 1$ (note that $\|x_i\|_X \ne 0$ for every $i \ge 1$ because $\{x_i\}_{i=1}^{\infty}$ is linearly independent). Thus the scalar-valued sequence $\{\alpha_{k(i)}\}_{k=1}^{\infty}$ is Cauchy in $\mathbb{F}$, and so convergent in $\mathbb{F}$, for every integer $i \ge 1$. Set

$$\widetilde{\alpha}_i \,=\, \lim_k \alpha_{k(i)}$$

in $\mathbb{F}$ for each $i \ge 1$ and consider the sequence $\widetilde{a} = \{\widetilde{\alpha}_i\}_{i=1}^{\infty}$ in $\mathbb{F}^{\mathbb{N}}$. Take an arbitrary $\varepsilon > 0$. Show that there exists an integer $k_\varepsilon \ge 1$ such that

$$\Big\| \sum_{i=m}^{n} (\alpha_{k(i)} - \alpha_{\ell(i)})\, x_i \Big\|_X \,<\, 2\varepsilon$$

for every $1 \le m \le n$, whenever $k, \ell \ge k_\varepsilon$. (*Hint*: $\sum_{i=m}^{n} (\alpha_{k(i)} - \alpha_{\ell(i)})\, x_i \,=\, \sum_{i=1}^{n} (\alpha_{k(i)} - \alpha_{\ell(i)})\, x_i - \sum_{i=1}^{m-1} (\alpha_{k(i)} - \alpha_{\ell(i)})\, x_i$ and $\{a_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{A}_{\boldsymbol{x}}$.) Hence

$$\text{(d)} \qquad \Big\| \sum_{i=m}^{n} (\alpha_{k(i)} - \widetilde{\alpha}_i)\, x_i \Big\|_X \,<\, 2\varepsilon$$

for every pair of nonnegative integers $m \le n$, whenever $k \ge k_\varepsilon$. (*Hint*: Note that $\|\sum_{i=m}^{n} (\alpha_{k(i)} - \lim_\ell \alpha_{\ell(i)})\, x_i\|_X = \lim_\ell \|\sum_{i=m}^{n} (\alpha_{k(i)} - \alpha_{\ell(i)})\, x_i\|_X$. Why?) Next prove the following claims.

(e) If $\mathcal{X}$ is a Banach space, then $\widetilde{a} \in \mathcal{A}_{\boldsymbol{x}}$.

(*Hint*: (d) implies that the infinite series $\sum_{i=1}^{\infty} (\alpha_{k(i)} - \widetilde{\alpha}_i)\, x_i$ converges in the Banach space $\mathcal{X}$ for every $k \ge k_\varepsilon$ — see Problem 4.7: Cauchy criterion — and hence $a_k - \widetilde{a} \in \mathcal{A}_{\boldsymbol{x}}$ for each $k \ge k_\varepsilon$.)

(f) $$\text{If } \widetilde{a} \in \mathcal{A}_{\boldsymbol{x}}, \text{ then } a_k \to \widetilde{a} \text{ in } \mathcal{A}_{\boldsymbol{x}}.$$

(*Hint*: $\widetilde{a} \in \mathcal{A}_{\boldsymbol{x}}$ implies $\|a_k - \widetilde{a}\| < 2\varepsilon$ whenever $k \geq k_\varepsilon$ by setting $m = 1$ in (d).) Finally conclude from (e) and (f):

If $(\mathcal{X}, \| \|_X)$ is a Banach space, then $(\mathcal{A}_{\boldsymbol{x}}, \| \|)$ is a Banach space.

Problem 4.11. Let $\mathcal{X}$ be a normed space. A sequence $\{x_k\}$ of vectors in $\mathcal{X}$ (indexed either by $\mathbb{N}$ or by $\mathbb{N}_0$) is a *Schauder basis* for $\mathcal{X}$ if for each x in $\mathcal{X}$ there exists a *unique* (similarly indexed) sequence of scalars $\{\alpha_k\}$ such that

$$x = \sum_k \alpha_k x_k$$

(i.e., $x = \lim_n \sum_{k=1}^n \alpha_k x_k$). The entries of the scalar-valued sequence $\{\alpha_k\}$ are called the *coefficients* of x with respect to the Schauder basis $\{x_k\}$, and the (convergent) series $\sum_{k=1}^\infty \alpha_k x_k$ is the *expansion* of x with respect to $\{x_k\}$. Prove the following assertions.

(a) Every Schauder basis for $\mathcal{X}$ is a sequence of linearly independent vectors in $\mathcal{X}$ that spans $\mathcal{X}$. That is, if $\{x_k\}$ is a Schauder basis for $\mathcal{X}$, then

(i) $\{x_k\}$ is linearly independent, and
(ii) $\bigvee\{x_k\} = \mathcal{X}$.

(b) If a normed space has a Schauder basis, then it is separable.

Hint: Proposition 4.9(b).

Remark: An infinite sequence of linearly independent vectors exists only in an infinite-dimensional linear space. It is readily verified that finite-dimensional normed spaces are separable Banach spaces (see Problem 4.37) but, in this case, the purely algebraic notion of a (finite) Hamel basis is enough. Therefore, when the concept of a Schauder basis is under discussion, only infinite-dimensional spaces are considered. Does every separable Banach space have a Schauder basis? This is a famous question, raised by Banach himself in the early 1930s, that remained open for a long period. Each separable Banach space that ever came up in analysis during that period (and this includes all classical examples) had a Schauder basis. The surprising negative answer to that question was given by Enflo in 1973, who constructed a separable Banach space that has no Schauder basis. See also the remark in Problem 4.58.

Problem 4.12. For each integer $k \geq 1$ let e_k be a scalar-valued sequence with just one nonzero entry (equal to 1) at the kth position (i.e., $e_k = \{\delta_{jk}\}_{j=1}^\infty$ for each $k \geq 1$). Consider the Banach spaces ℓ_+^∞ and ℓ_+^p for any $p \geq 1$ as in Example 4.B. Show that the sequence $\{e_k\}$ is a Schauder basis for every ℓ_+^p, and verify that ℓ_+^∞ has no Schauder basis. (*Hint*: Example 3.Q.)

Problem 4.13. Let $\mathcal{M}$ be a subspace of a normed space $\mathcal{X}$. If $\mathcal{X}$ is a Banach space, then $\mathcal{M}$ and $\mathcal{X}/\mathcal{M}$ are both Banach spaces (Propositions 4.7 and 4.10). Conversely, *if $\mathcal{M}$ and $\mathcal{X}/\mathcal{M}$ are both Banach spaces, then $\mathcal{X}$ is a Banach space*. Indeed, suppose $\mathcal{M}$ and $\mathcal{X}/\mathcal{M}$ are Banach spaces, and let $\{x_n\}$ be an arbitrary Cauchy sequence in $\mathcal{X}$ (indexed, for instance, by $\mathbb{N}$).

(a) Show that $\{[x_n]\}$ is a Cauchy sequence in $\mathcal{X}/\mathcal{M}$ and conclude that $\{[x_n]\}$ converges in $\mathcal{X}/\mathcal{M}$ to, say, $[x] \in \mathcal{X}/\mathcal{M}$.

(b) Take any x in $[x]$. For each n there exists $\widetilde{x}_n$ in $[x_n - x]$ such that

$$0 \le \|\widetilde{x}_n\|_X \le \|[x_n - x]\| + \tfrac{1}{n} = \|[x_n] - [x]\| + \tfrac{1}{n}.$$

Prove the above assertion and conclude: $\widetilde{x}_n \to 0$ in $\mathcal{X}$.

Observe that $\widetilde{x}_n - x_n + x$ lies in $\mathcal{M}$ for each n. In fact, $[\widetilde{x}_n] = [x_n - x]$ (since $\widetilde{x}_n \in [x_n - x]$), and so $[\widetilde{x}_n - x_n + x] = [\widetilde{x}_n] - [x_n - x] = [0] = \mathcal{M}$.

(c) Set $u_n = \widetilde{x}_n - x_n + x$ in $\mathcal{M}$ and show that $\{u_n\}$ is a Cauchy sequence in $\mathcal{M}$. Thus conclude that $\{u_n\}$ converges in $\mathcal{M}$ (and so in $\mathcal{X}$) to, say, $u \in \mathcal{M}$.

Since $x_n = \widetilde{x}_n + x - u_n$ for each n, it follows that the sequence $\{x_n\}$ converges in $\mathcal{X}$ to $x - u \in \mathcal{X}$ (recall that vector addition and scalar multiplication are continuous mappings — see Problem 4.1). Outcome: Every Cauchy sequence in $\mathcal{X}$ converges in $\mathcal{X}$, which means that $\mathcal{X}$ is a Banach space.

Problem 4.14. Consider the linear space $\ell_+^{c_o}$ and let $\{a_k\}_{k=1}^{\infty}$ be an $\ell_+^{c_o}$-valued sequence. That is, each $a_k = \{\alpha_{k(n)}\}_{n=1}^{\infty}$ is a scalar-valued sequence that converges to zero:

$$\lim_n |\alpha_k(n)| = 0 \quad \text{for every} \quad k \ge 1.$$

Suppose further that there exists a real number $\alpha \ge 0$ such that $|\alpha_{k(n)}| \le \alpha$ for all $k, n \ge 1$ or, equivalently, suppose

$$\sup_k \sup_n |\alpha_{k(n)}| < \infty.$$

Take an arbitrary $x = \{\xi_k\}_{k=1}^{\infty}$ in ℓ_+^1 so that

$$\sum_{k=1}^{\infty} |\xi_k| < \infty.$$

Consider the above assumptions and prove the following proposition.

(a) $$\lim_n \sup_k |\alpha_{k(n)}||\xi_k| = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \sup_n |\alpha_{k(n)}||\xi_k| < \infty.$$

Hint: $\sup_k |\alpha_{k(n)}||\xi_k| \le \max\left\{\max_{1\le k\le m} |\alpha_{k(n)}||\xi_k|,\ \alpha \sum_{k=m+1}^{\infty} |\xi_k|\right\}$ for every $m, n \ge 1$, which implies

$$\limsup_n \sup_k |\alpha_{k(n)}||\xi_k| \le \alpha \sum_{k=m+1}^{\infty} |\xi_k| \quad \text{for every} \quad m \ge 1.$$

Next use the dominated convergence of Problem 4.6(c) to show that

(b) $$\lim_n \sum_{k=1}^{\infty} |\alpha_{k(n)}||\xi_k| = 0.$$

Then conclude: For each $n \ge 1$ the infinite series $\sum_{k=1}^{\infty} \alpha_{k(n)} \xi_k$ converges (in the Banach space $\mathbb{F}$) and the scalar-valued sequence $\{\sum_{k=1}^{\infty} \alpha_{k(n)} \xi_k\}_{n=1}^{\infty}$ is an element of $\ell_+^{c_o}$. Therefore, every infinite matrix $A = [\alpha_{k(n)}]_{k,n \ge 1}$ whose rows a_k satisfy the above assumptions represents a mapping of ℓ_+^1 into $\ell_+^{c_o}$. Equip ℓ_+^1 and $\ell_+^{c_o}$ with their usual norms ($\| \ \|_1$ on ℓ_+^1 and $\| \ \|_\infty$ on $\ell_+^{c_o}$). Observe that the assumptions on A simply say that $\{a_k\}_{k=1}^{\infty}$ is a bounded (i.e., $\sup_k \|a_k\|_\infty < \infty$) $\ell_+^{c_o}$-valued sequence. Show that, under these assumptions, such a mapping in fact is a bounded linear transformation of ℓ_+^1 into $\ell_+^{c_o}$:

$$A \in \mathcal{B}[\ell_+^1, \ell_+^{c_o}] \quad \text{and} \quad \|A\| = \sup_k \|a_k\|_\infty.$$

Problem 4.15. Let $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ be a continuous linear transformation of a normed $\mathcal{X}$ into a normed space $\mathcal{Y}$. (a) If $\{x_k\}_{k=0}^{\infty}$ is a summable sequence in $\mathcal{X}$, then show that $\{Tx_k\}_{k=0}^{\infty}$ is a summable sequence in $\mathcal{Y}$ and

$$\sum_{k=0}^{\infty} Tx_k = T\Big(\sum_{k=0}^{\infty} x_k\Big).$$

(b) If A is any set of vectors in $\mathcal{X}$, then show that

$$\bigvee TA = \Big(T \bigvee A\Big)^{-}.$$

Problem 4.16. Let $(\mathcal{X}_1, \| \ \|_1)$ and $(\mathcal{X}_2, \| \ \|_2)$ be normed spaces and consider the normed space $\mathcal{X}_1 \oplus \mathcal{X}_2$ equipped with any of the norms of Example 4.E, which we shall denote by $\| \ \|$. That is, for every (x_1, x_2) in $\mathcal{X}_1 \oplus \mathcal{X}_2$, either $\|(x_1, x_2)\|^p = \|x_1\|_1^p + \|x_2\|_2^p$ for some $p \ge 1$ or $\|(x_1, x_2)\| = \max\{\|x_1\|_1, \|x_2\|_2\}$. Let T_1 and T_2 be operators on $\mathcal{X}_1$ and $\mathcal{X}_2$ (i.e., $T_1 \in \mathcal{B}[\mathcal{X}_1]$ and $T_2 \in \mathcal{B}[\mathcal{X}_2]$). Consider the direct sum $T \in \mathcal{L}[\mathcal{X}_1 \oplus \mathcal{X}_2]$ of T_1 and T_2 (defined in Section 2.9):

$$T = T_1 \oplus T_2 = \begin{pmatrix} T_1 & O \\ O & T_2 \end{pmatrix},$$

where $T_1 = T|_{\mathcal{X}_1}$ and $T_2 = T|_{\mathcal{X}_2}$ are the direct summands of T.

(a) Show that $T \in \mathcal{B}[\mathcal{X}_1 \oplus \mathcal{X}_2]$ and $\|T\| = \max\big\{\|T_1\|, \|T_2\|\big\}$.

Hint: For any norm $\| \ \|$ that equips $\mathcal{X}_1 \oplus \mathcal{X}_2$ (among those of Example 4.E), $\|T(x_1, x_2)\| \le \max\big\{\|T_1\|, \|T_2\|\big\}\|(x_1, x_2)\|$ for every $(x_1, x_2) \in \mathcal{X}_1 \oplus \mathcal{X}_2$.

Generalize to a countable direct sum. That is, let $\{\mathcal{X}_k\}$ be a countable family of normed spaces and consider the normed space $\bigoplus_k \mathcal{X}_k$ of Examples 4.E or 4.F (equipped with any of those norms, so that either $\bigoplus_k \mathcal{X}_k = \big[\bigoplus_k \mathcal{X}_k\big]_p$ or $\bigoplus_k \mathcal{X}_k = \big[\bigoplus_k \mathcal{X}_k\big]_\infty$ in case of a countably infinite family as in Example 4.F). Let $\{T_k\}$ be a similarly indexed countable family of operators on $\mathcal{X}_k$ (each T_k lying in $\mathcal{B}[\mathcal{X}_k]$) such that $\sup_k \|T_k\| < \infty$. Set

$$T\{x_k\} \;=\; \{T_k x_k\} \quad \text{for each} \quad \{x_k\} \in \textstyle\bigoplus_k \mathcal{X}_k.$$

(b) Show that this actually defines a bounded linear transformation T of $\bigoplus_k \mathcal{X}_k$ into itself. Such an operator is usually denoted by

$$T = \bigoplus_k T_k \quad \text{in} \quad \mathcal{B}\big[\textstyle\bigoplus_k \mathcal{X}_k\big],$$

and referred to as the *direct sum* of $\{T_k\}$. Moreover, verify that $T_k = T|_{\mathcal{X}_k}$ (in the sense of Section 2.9) for each k. These are the *direct summands* of T. Finally, show that

$$\|T\| \;=\; \sup_k \|T_k\|.$$

If $\mathcal{X}_k = \mathcal{X}$ for a single normed space $\mathcal{X}$, then $T = \bigoplus_k T_k$ is sometimes referred to as a *block diagonal operator* acting on $\ell^p_+(\mathcal{X})$ or on $\ell^\infty_+(\mathcal{X})$.

Problem 4.17. Let $\mathcal{X}_i$ and $\mathcal{Y}_i$ be normed spaces for $i = 1, 2$ and consider the normed spaces consisting of the direct sums $\mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y}_1 \oplus \mathcal{Y}_2$ equipped with any of the norms of Example 4.E.

(a) Take T_{ij} in $\mathcal{B}[\mathcal{X}_j, \mathcal{Y}_i]$ for $i, j = 1, 2$ so that $T_{11}x_1 + T_{12}x_2$ lies in $\mathcal{Y}_1$ and $T_{21}x_1 + T_{22}x_2$ lies in $\mathcal{Y}_2$ for every (x_1, x_2) in $\mathcal{X}_1 \oplus \mathcal{X}_2$. Set

$$T(x_1, x_2) = (T_{11}x_1 + T_{12}x_2,\, T_{21}x_2 + T_{22}x_2) \quad \text{in} \quad \mathcal{Y}_1 \oplus \mathcal{Y}_2.$$

Show that this defines a mapping $T\colon \mathcal{X}_1 \oplus \mathcal{X}_2 \to \mathcal{Y}_1 \oplus \mathcal{Y}_2$, which in fact lies in $\mathcal{B}[\mathcal{X}_1 \oplus \mathcal{X}_2,\, \mathcal{Y}_1 \oplus \mathcal{Y}_2]$, and

$$\max_{\substack{i=1,2\\ j=1,2}} \big\{\|T_{ij}\|\big\} \;\le\; \|T\| \;\le\; 4 \max_{\substack{i=1,2\\ j=1,2}} \big\{\|T_{ij}\|\big\}.$$

(b) Conversely, suppose $T \in \mathcal{B}[\mathcal{X}_1 \oplus \mathcal{X}_2, \mathcal{Y}_1 \oplus \mathcal{Y}_2]$. If x_1 is any vector in $\mathcal{X}_1$, then

$$T(x_1, 0) \;=\; (T_{11}x_1,\, T_{21}x_1)$$

in $\mathcal{Y}_1 \oplus \mathcal{Y}_2$, where T_{11} is a mapping of $\mathcal{X}_1$ into $\mathcal{Y}_1$ and T_{21} is a mapping of $\mathcal{X}_1$ into $\mathcal{Y}_2$. Similarly, if x_2 is any vector in $\mathcal{X}_2$, then

$$T(0, x_2) \;=\; (T_{12}x_2,\, T_{22}x_2)$$

in $\mathcal{Y}_1 \oplus \mathcal{Y}_2$, where T_{12} is a mapping of $\mathcal{X}_2$ into $\mathcal{Y}_1$ and T_{22} is a mapping of $\mathcal{X}_2$ into $\mathcal{Y}_2$. Show that $T_{ij} \in \mathcal{B}[\mathcal{X}_j, \mathcal{Y}_i]$ and $\|T_{ij}\| \le \|T\|$ for every $i, j = 1, 2$.

Consider the bounded linear transformation $T \in \mathcal{B}[\mathcal{X}_1 \oplus \mathcal{X}_2, \mathcal{Y}_1 \oplus \mathcal{Y}_2]$ in (b). Since $T(x_1, x_2) = T(x_1, 0) \oplus T(0, x_2)$ in $\mathcal{Y}_1 \oplus \mathcal{Y}_2$, it follows that $T(x_1, x_2) = (T_{11}x_1 + T_{12}x_2,\ T_{21}x_1 + T_{22}x_2)$ for every (x_1, x_2) in $\mathcal{X}_1 \oplus \mathcal{X}_2$ as in (a). This establishes a one-to-one correspondence between each T in $\mathcal{B}[\mathcal{X}_1 \oplus \mathcal{X}_2,\ \mathcal{Y}_1 \oplus \mathcal{Y}_2]$ and a 2×2 matrix of bounded linear transformations $[T_{ij}]$, called the *operator matrix* for T, which we shall represent by the same symbol T (instead of, for instance, $[T]$) and write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

If $\mathcal{Y}_i = \mathcal{X}_i$ for $i = 1, 2$, then T is the direct sum $T_{11} \oplus T_{22}$ in $\mathcal{B}[\mathcal{X}_1 \oplus \mathcal{X}_2]$ of the previous problem if and only if $T_{12} = O$ in $\mathcal{B}[\mathcal{X}_2, \mathcal{X}_1]$ and $T_{21} = O$ in $\mathcal{B}[\mathcal{X}_1, \mathcal{X}_2]$.

Problem 4.18. Let T be an operator on a normed space $\mathcal{X}$ (i.e., $T \in \mathcal{B}[\mathcal{X}]$). A subset A of $\mathcal{X}$ is T-invariant (or A is an invariant subset for T) if $T(A) \subseteq A$ (i.e., if $Tx \in A$ for every $x \in A$). If $\mathcal{M}$ is a linear manifold (or a subspace) of $\mathcal{X}$ and, as a subset of $\mathcal{X}$, is T-invariant, then we say that $\mathcal{M}$ is an *invariant linear manifold* (or an *invariant subspace*) for T. Prove the following propositions.

(a) If A is T-invariant, then A^- is T-invariant.

(b) If $\mathcal{M}$ is an invariant linear manifold for T, then $\mathcal{M}^-$ is an invariant subspace for T.

(c) $\{0\}$ and $\mathcal{X}$ are invariant subspaces for every T in $\mathcal{B}[\mathcal{X}]$.

Problem 4.19. Let $\mathrm{Lat}(\mathcal{X})$ be the lattice of all subspaces of a normed space $\mathcal{X}$. Since $\{0\}$ and $\mathcal{X}$ are subspaces of $\mathcal{X}$, they lie in $\mathrm{Lat}(\mathcal{X})$. These are the trivial elements of $\mathrm{Lat}(\mathcal{X})$: a *nontrivial subspace* of $\mathcal{X}$ is a proper nonzero subspace of $\mathcal{X}$ (i.e., $\mathcal{M} \in \mathrm{Lat}(\mathcal{X})$ in nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{X}$). Prove (a) and (b).

(a) There are nontrivial subspaces in $\mathrm{Lat}(\mathcal{X})$ if and only if the dimension of $\mathcal{X}$ is greater than 1 (i.e., $\mathrm{Lat}(\mathcal{X}) \neq \big\{\{0\}, \mathcal{X}\big\}$ if and only if $\dim \mathcal{X} > 1$).

Let $\mathcal{B}[\mathcal{X}]$ be the unital algebra of all operators on a normed space $\mathcal{X}$ and let T be an operator in $\mathcal{B}[\mathcal{X}]$. A *nontrivial invariant subspace* for T is a nontrivial element of $\mathrm{Lat}(\mathcal{X})$ which is invariant for T (i.e., a subspace $\mathcal{M} \in \mathrm{Lat}(\mathcal{X})$ such that $\{0\} \neq \mathcal{M} \neq \mathcal{X}$ and $T(\mathcal{M}) \subseteq \mathcal{M}$). An element of $\mathcal{B}[\mathcal{X}]$ is a *scalar operator* if it is a multiple of the identity, say, αI for some scalar α.

(b) Every subspace in $\mathrm{Lat}(\mathcal{X})$ is invariant for any scalar operator in $\mathcal{B}[\mathcal{X}]$, and so every scalar operator has a nontrivial invariant subspace if $\dim \mathcal{X} > 1$.

Problem 4.20. Let $\mathcal{X}$ be a normed space and take $T \in \mathcal{B}[\mathcal{X}]$. Show that

(a) $\mathcal{N}(T)$ and $\mathcal{R}(T)^-$ are invariant subspaces for T,

(b) $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T)^- = \mathcal{X}$ if T has no nontrivial invariant subspace.

Take S and T in $\mathcal{B}[\mathcal{X}]$. We say that S and T *commute* if $ST = TS$. Show that

(c) $\mathcal{N}(S)$, $\mathcal{N}(T)$, $\mathcal{R}(S)^-$, and $\mathcal{R}(T)^-$ are invariant subspaces for both S and T whenever S and T commute.

Problem 4.21. Let $S \in \mathcal{B}[\mathcal{X}]$ and $T \in \mathcal{B}[\mathcal{X}]$ be nonzero operators on a normed space $\mathcal{X}$. Suppose $ST = O$ and show that

(a) $T(\mathcal{N}(S)) \subseteq T(\mathcal{X}) = \mathcal{R}(T) \subseteq \mathcal{N}(S)$,

(b) $\{0\} \neq \mathcal{N}(S) \neq \mathcal{X}$ and $\{0\} \neq \mathcal{R}(T)^- \neq \mathcal{X}$,

(c) $S(\mathcal{R}(T)^-) \subseteq S(\mathcal{R}(T))^- \subseteq \mathcal{R}(T)^-$.

Conclusion: *If $S \neq O$, $T \neq O$, and $ST = O$, then $\mathcal{N}(S)$ and $\mathcal{R}(T)^-$ are nontrivial invariant subspaces for both S and T.*

Problem 4.22. Take $T \in \mathcal{B}[\mathcal{X}]$ on a normed space $\mathcal{X}$. Verify that $p(T) \in \mathcal{B}[\mathcal{X}]$ for every nonzero polynomial $p(T)$ of T. In particular, $T^n \in \mathcal{B}[\mathcal{X}]$ for every integer $n \geq 0$. (*Hint*: $\mathcal{B}[\mathcal{X}]$ is an algebra; see Problems 2.20 and 3.29.)

(a) Show that $\mathcal{N}(p(T))$ and $\mathcal{R}(p(T))^-$ are invariant subspaces for T.

Recall that an operator in $\mathcal{B}[\mathcal{X}]$ is nilpotent if $T^n = O$ for some positive integer n, and algebraic if $p(T) = O$ for some nonzero polynomial p (cf. Problem 2.20).

(b) Show that every nilpotent operator in $\mathcal{B}[\mathcal{X}]$ (with $\dim \mathcal{X} > 1$) has a nontrivial invariant subspace.

(c) Suppose $\mathcal{X}$ is a *complex* normed space and $\dim \mathcal{X} > 1$. Show that every algebraic operator in $\mathcal{B}[\mathcal{X}]$ has a nontrivial invariant subspace.

Hint: Every polynomial (in one complex variable and with complex coefficients) of degree $n \geq 1$ is the product of a polynomial of degree $n-1$ and a polynomial of degree 1.

Problem 4.23. Let $\mathrm{Lat}(T)$ denote the subcollection of $\mathrm{Lat}(\mathcal{X})$ made up of *all invariant subspaces for* $T \in \mathcal{B}[\mathcal{X}]$, where $\mathcal{X}$ is a normed space. It is plain that an operator T has no nontrivial invariant subspace if and only if $\mathrm{Lat}(T) = \big\{\{0\}, \mathcal{X}\big\}$ (see Problems 4.18 and 4.19).

(a) Show that $\mathrm{Lat}(T)$ is a complete lattice in the inclusion ordering.

Hint: Intersection and closure of sums of invariant subspaces are again invariant subspaces. See Section 4.3.

Take an operator T in $\mathcal{B}[\mathcal{X}]$ and a vector x in $\mathcal{X}$. Consider the $\mathcal{X}$-valued power sequence $\{T^n x\}_{n\geq 0}$. The range of $\{T^n x\}_{n\geq 0}$ is called the *orbit of x under T*.

(b) Show that the (linear) span of the orbit of x under T is the set of the images of all nonzero polynomials of T at x; that is,

$$\mathrm{span}\,\{T^n x\}_{n\geq 0} = \big\{p(T)x \in \mathcal{X}\colon\ p \text{ is a nonzero polynomial}\big\}.$$

Since $\text{span}\,\{T^n x\}_{n\geq 0}$ is a linear manifold of $\mathcal{X}$, it follows that its closure, $(\text{span}\,\{T^n x\}_{n\geq 0})^- = \bigvee\{T^n x\}_{n\geq 0}$, is a subspace of $\mathcal{X}$ (Proposition 4.8(b)). That is, $\bigvee\{T^n x\}_{n\geq 0} \in \text{Lat}(\mathcal{X})$.

(c) Show that $\bigvee\{T^n x\}_{n\geq 0} \in \text{Lat}(T)$.

These are the *cyclic subspaces* in $\text{Lat}(T)$: $\mathcal{M} \in \text{Lat}(T)$ is cyclic for T if $\mathcal{M} = \bigvee\{T^n x\}_{n\geq 0}$ for some $x \in \mathcal{X}$. If $\bigvee\{T^n x\}_{n\geq 0} = \mathcal{X}$, then x is said to be a *cyclic vector* for T. We say that a linear manifold $\mathcal{M}$ of $\mathcal{X}$ is *totally cyclic* for T if every nonzero vector in $\mathcal{M}$ is cyclic for T.

(d) Verify that T has no nontrivial invariant subspace if and only if $\mathcal{X}$ is totally cyclic for T.

Problem 4.24. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. A bounded linear transformation $X \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ *intertwines* an operator $T \in \mathcal{B}[\mathcal{X}]$ to an operator $S \in \mathcal{B}[\mathcal{Y}]$ if

$$XT = SX.$$

If there exists an X intertwining T to S, then we say T is *intertwined* to S. Suppose $XT = SX$. Show by induction that

(a) $$XT^n = S^n X$$

for every positive integer n. Thus verify that

(b) $$Xp(T) = p(S)X$$

for every polynomial p. Now use Problem 4.23(b) to prove that

(c) $$X\big(\text{span}\,\{T^n x\}_{n\geq 0}\big) = \text{span}\,\{S^n X x\}_{n\geq 0}$$

for each $x \in \mathcal{X}$, and therefore (see Problem 3.46(a))

(d) $$X\big(\bigvee\{T^n x\}_{n\geq 0}\big) \subseteq \bigvee\{S^n X x\}_{n\geq 0}$$

for every $x \in \mathcal{X}$. An operator $T \in \mathcal{B}[\mathcal{X}]$ is *densely intertwined* to an operator $S \in \mathcal{B}[\mathcal{Y}]$ if there is a bounded linear transformation $X \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ with dense range intertwining T to S. If $XT = SX$ and $\mathcal{R}(X)^- = \mathcal{Y}$, then show that

(e) $$\bigvee\{T^n x\}_{n\geq 0} = \mathcal{X} \quad \text{implies} \quad \mathcal{Y} = \bigvee\{S^n X x\}_{n\geq 0}.$$

Conclusion: *Suppose T in $\mathcal{B}[\mathcal{X}]$ is densely intertwined to S in $\mathcal{B}[\mathcal{Y}]$. Let X in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ be a transformation with dense range intertwining T to S. If $x \in \mathcal{X}$ is a cyclic vector for T, then $Xx \in \mathcal{Y}$ is a cyclic vector for S. Thus, if a linear manifold $\mathcal{M}$ of $\mathcal{X}$ is totally cyclic for T, then the linear manifold $X(\mathcal{M})$ of $\mathcal{Y}$ is totally cyclic for S.*

Problem 4.25. Here is a sufficient condition for transferring nontrivial invariant subspaces from S to T whenever T is densely intertwined to S. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and take $T \in \mathcal{B}[\mathcal{X}]$, $S \in \mathcal{B}[\mathcal{Y}]$, and $X \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ such that

$$XT = SX.$$

Prove the following assertions.

(a) If $\mathcal{M} \subseteq \mathcal{Y}$ is an invariant subspace for S, then the inverse image of $\mathcal{M}$ under X, $X^{-1}(\mathcal{M}) \subseteq \mathcal{X}$, is an invariant subspace for T.

(b) If, in addition, $\mathcal{M} \neq \mathcal{Y}$ (i.e., $\mathcal{M}$ is a proper subspace), $\mathcal{R}(X) \cap \mathcal{M} \neq \{0\}$, and $\mathcal{R}(X)^- = \mathcal{Y}$, then $\{0\} \neq X^{-1}(\mathcal{M}) \neq \mathcal{X}$.

Hint: Problems 1.2 and 2.11, and Theorem 3.23.

Conclusion: *If T is densely intertwined to S, then the inverse image under the intertwining transformation X of a nontrivial invariant subspace $\mathcal{M}$ for S is a nontrivial invariant subspace for T, provided that the range of X is not (algebraically) disjoint with $\mathcal{M}$.*

Show that the condition $\mathcal{R}(X) \cap \mathcal{M} \neq \{0\}$ in (b) is not redundant. That is, if $\mathcal{M}$ is a subspace of $\mathcal{Y}$, then show that

(c) $\{0\} \neq \mathcal{M} \neq \mathcal{Y}$ and $\mathcal{R}(X)^- = \mathcal{Y}$ does not imply $\mathcal{R}(X) \cap \mathcal{M} \neq \{0\}$.

However, if X is surjective, then the condition $\mathcal{R}(X) \cap \mathcal{M} \neq \{0\}$ in (b) is trivially satisfied whenever $\mathcal{M} \neq \{0\}$. Actually, with the assumption $XT = SX$ still in force, check the proposition below.

(d) If S has a nontrivial invariant subspace, and if $\mathcal{R}(X) = \mathcal{Y}$, then T has a nontrivial invariant subspace.

Problem 4.26. Let $\mathcal{X}$ be a normed space. The *commutant* of an operator T in $\mathcal{B}[\mathcal{X}]$ is the set $\{T\}'$ of all operators in $\mathcal{B}[\mathcal{X}]$ that commute with T. That is,

$$\{T\}' = \big\{C \in \mathcal{B}[\mathcal{X}]\colon\ CT = TC\big\}.$$

In other words, the commutant of an operator is the set of all operators intertwining it to itself.

(a) Show that $\{T\}'$ is an operator algebra that contains the identity (i.e., $\{T\}'$ is a unital subalgebra of the normed algebra $\mathcal{B}[\mathcal{X}]$).

A linear manifold (or a subspace) of $\mathcal{X}$ is *hyperinvariant* for $T \in \mathcal{B}[\mathcal{X}]$ if it is invariant for every $C \in \{T\}'$; that is, if it is an invariant linear manifold (or an invariant subspace) for every operator in $\mathcal{B}[\mathcal{X}]$ that commutes with T. As $T \in \{T\}'$, every hyperinvariant linear manifold (subspace) for T obviously is an invariant linear manifold (subspace) for T. Take an arbitrary $T \in \mathcal{B}[\mathcal{X}]$ and, for each $x \in \mathcal{X}$, set

$$\mathcal{T}_x = \bigcup_{C \in \{T\}'} Cx = \big\{y \in \mathcal{X}\colon\ y = Cx \text{ for some } C \in \{T\}'\big\}.$$

$\mathcal{T}_x$ is never empty (for instance, $x \in \mathcal{T}_x$ because $I \in \{T\}'$). In fact, $0 \in \mathcal{T}_x$ for every $x \in \mathcal{X}$, and $\mathcal{T}_x = \{0\}$ if and only if $x = 0$. Prove the next proposition.

(b) For each $x \in \mathcal{X}$, $\mathcal{T}_x^-$ is a hyperinvariant subspace for T.

Hint: As an algebra, $\{T\}'$ is a linear space. This implies that $\mathcal{T}_x$ is a linear manifold of $\mathcal{X}$. If $y = C_0 x$ for some $C_0 \in \{T\}'$, then $Cy = CC_0 x \in \mathcal{T}_x$ for every $C \in \{T\}'$ (i.e., $\mathcal{T}_x$ is hyperinvariant for T because $\{T\}'$ is an algebra). See Problem 4.18(b).

Problem 4.27. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. Take $T \in \mathcal{B}[\mathcal{X}]$, $S \in \mathcal{B}[\mathcal{Y}]$, $X \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$, and $Y \in \mathcal{B}[\mathcal{Y}, \mathcal{X}]$ such that

$$XT = SX \quad \text{and} \quad YS = TY.$$

Show that if $C \in \mathcal{B}[\mathcal{X}]$ commutes with T, then XCY commutes with S. That is (see Problem 4.26), show that

(a) $XCY \in \{S\}'$ for every $C \in \{T\}'$.

Now consider the subspace $\mathcal{T}_x^-$ of $\mathcal{X}$ that, according to Problem 4.26, is nonzero and hyperinvariant for T for every nonzero x in $\mathcal{X}$. Under the above assumptions on T and S, prove the following propositions.

(b) Suppose $\mathcal{M}$ is a nontrivial hyperinvariant subspace for S. If $\mathcal{R}(X)^- = \mathcal{Y}$ and $\mathcal{N}(Y) \cap \mathcal{M} = \{0\}$, then $Y(\mathcal{M}) \neq \{0\}$ and $\mathcal{T}_x^- \neq \mathcal{X}$ for every nonzero x in $Y(\mathcal{M})$. Consequently, $\mathcal{T}_x^-$ is a nontrivial hyperinvariant subspace for T whenever x is a nonzero vector in $Y(\mathcal{M})$.

Hint: Since $\mathcal{M}$ is hyperinvariant for S, it follows from (a) that $\mathcal{M}$ is invariant for XCY whenever $C \in \{T\}'$. Use this fact to show that $X(\mathcal{T}_x) \subseteq \mathcal{M}$ for every $x \in Y(\mathcal{M})$, and hence $X(\mathcal{T}_x^-) \subseteq \mathcal{M}^- = \mathcal{M}$ (Problem 3.46(a)). Now verify that $\mathcal{T}_x^- = \mathcal{X}$ implies $\mathcal{R}(X)^- = X(\mathcal{X})^- = X(\mathcal{T}_x^-)^- \subseteq \mathcal{M}$. Thus, if $\mathcal{M} \neq \mathcal{Y}$ and $\mathcal{R}(X)^- = \mathcal{Y}$, then $\mathcal{T}_x^- \neq \mathcal{X}$ for every vector x in $Y(\mathcal{M})$. Next observe that if $Y(\mathcal{M}) = \{0\}$ (i.e., if $\mathcal{M} \subseteq \mathcal{N}(Y)$), then $\mathcal{N}(Y) \cap \mathcal{M} = \mathcal{M}$. Conclude: If $\mathcal{M} \neq \{0\}$ and $\mathcal{N}(Y) \cap \mathcal{M} = \{0\}$, then $Y(\mathcal{M}) \neq \{0\}$. Finally recall that $\mathcal{T}_x \neq \{0\}$ for every $x \neq 0$ in $\mathcal{X}$, and so $\{0\} \neq \mathcal{T}_x^- \neq \mathcal{X}$ for every nonzero vector x in $Y(\mathcal{M})$.

(c) If S has a nontrivial hyperinvariant subspace, and if $\mathcal{R}(X)^- = \mathcal{Y}$ and $\mathcal{N}(Y) = \{0\}$, then T has a nontrivial hyperinvariant subspace.

Problem 4.28. A bounded linear transformation X of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ is *quasiinvertible* (or a *quasiaffinity*) if it is injective and has a dense range (i.e., $\mathcal{N}(X) = \{0\}$ and $\mathcal{R}(X)^- = \mathcal{Y}$). An operator $T \in \mathcal{B}[\mathcal{X}]$ is a *quasiaffine transform* of an operator $S \in \mathcal{B}[\mathcal{Y}]$ if there exists a quasiinvertible transformation $X \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ intertwining T to S. Two operators are *quasisimilar* if they are quasiaffine transforms of each other. In other words, $T \in \mathcal{B}[\mathcal{X}]$ and $S \in \mathcal{B}[\mathcal{Y}]$ are quasisimilar (notation: $T \sim S$) if there exists $X \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ and $Y \in \mathcal{B}[\mathcal{Y}, \mathcal{X}]$ such that

$$\mathcal{N}(X) = \{0\}, \quad \mathcal{R}(X)^- = \mathcal{Y}, \quad \mathcal{N}(Y) = \{0\}, \quad \mathcal{R}(Y)^- = \mathcal{X},$$

$$XT = SX \quad \text{and} \quad YS = TY.$$

Prove the following propositions.

(a) Quasisimilarity has the defining properties of an equivalence relation.

(b) If two operators are quasisimilar and if one of them has a nontrivial hyperinvariant subspace, then so has the other.

Problem 4.29. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. Two operators $T \in \mathcal{B}[\mathcal{X}]$ and $S \in \mathcal{B}[\mathcal{Y}]$ are *similar* (notation: $T \approx S$) if there exists an injective and surjective bounded linear transformation X of $\mathcal{X}$ onto $\mathcal{Y}$, with a bounded inverse X^{-1} of $\mathcal{Y}$ onto $\mathcal{X}$, that intertwines T to S. That is, $T \in \mathcal{B}[\mathcal{X}]$ and $S \in \mathcal{B}[\mathcal{Y}]$ are similar if there exists $X \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ such that $\mathcal{N}(X) = \{0\}$, $\mathcal{R}(X) = \mathcal{Y}$, $X^{-1} \in \mathcal{B}[\mathcal{Y}, \mathcal{X}]$, and $XT = SX$.

(a) Let T be an operator on $\mathcal{X}$ and let S be an operator on $\mathcal{Y}$. If X is a bounded linear transformation of $\mathcal{X}$ onto $\mathcal{Y}$ with a bounded inverse X^{-1} of $\mathcal{Y}$ onto $\mathcal{X}$ (which is always linear), then check that

$$XT = SX \iff T = X^{-1}SX \iff S = XTX^{-1} \iff X^{-1}S = TX^{-1}.$$

Now prove the following assertions.

(b) If T and S are similar, then they are quasisimilar.

(c) Similarity has the defining properties of an equivalence relation.

(d) If two operators are similar, and if one of them has a nontrivial invariant subspace, then so has the other. (*Hint*: Problem 4.25.)

Note that we are using the same terminology of Section 2.7, namely, "similar", but now with a different meaning. The linear transformation $X\colon \mathcal{X} \to \mathcal{Y}$ in fact is a (linear) isomorphism so that $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic linear spaces, and hence the concept of similarity defined above implies the purely algebraic homonymous concept defined in Section 2.7. However, we are now imposing that all linear transformations involved are continuous (equivalently, that all of them are bounded), viz., T, S, X and also the inverse X^{-1} of X.

Problem 4.30. Let $\{x_k\}_{k=1}^{\infty}$ be a Schauder basis for a (separable) Banach space $\mathcal{X}$ (see Problem 4.11) so that every $x \in \mathcal{X}$ has a unique expansion

$$x \;=\; \sum_{k=1}^{\infty} \alpha_k(x)\, x_k$$

with respect to $\{x_k\}_{k=1}^{\infty}$. For each $k \geq 1$ consider the functional $\varphi_k\colon \mathcal{X} \to \mathbb{F}$ that assigns to each $x \in \mathcal{X}$ its *unique* coefficient $\alpha_k(x)$ in the above expansion:

$$\varphi_k(x) \;=\; \alpha_k(x)$$

for every $x \in \mathcal{X}$. Show that φ_k is a bounded linear functional (i.e., $\varphi_k \in \mathcal{B}[\mathcal{X}, \mathbb{F}]$ for each $k \geq 1$). In other words, each coefficient in a Schauder basis expansion for a vector x in a Banach space $\mathcal{X}$ is a bounded linear functional on $\mathcal{X}$.

Hint: Let $\mathcal{A}_{\boldsymbol{x}}$ be the Banach space defined in Problem 4.10. Consider the mapping $\Phi\colon \mathcal{A}_{\boldsymbol{x}} \to \mathcal{X}$ given by

$$\Phi(a) \;=\; \sum_{k=1}^{\infty} \alpha_k x_k$$

for every $a = \{\alpha_k\}_{k=1}^{\infty}$ in $\mathcal{A}_{\boldsymbol{x}}$. Verify that Φ is linear, injective, surjective, and bounded (actually, Φ is a contraction: $\Phi(a) \leq \|a\|$ for every $a \in \mathcal{A}_{\boldsymbol{x}}$). Now apply Theorem 4.22 to conclude that $\Phi \in \mathcal{G}[\mathcal{A}_{\boldsymbol{x}}, \mathcal{X}]$. For each integer $k \geq 1$ consider the functional $\psi_k\colon \mathcal{A}_{\boldsymbol{x}} \to \mathbb{F}$ given by

$$\psi_k(a) \;=\; \alpha_k$$

for every $a = \{\alpha_k\}_{k=1}^{\infty}$ in $\mathcal{A}_{\boldsymbol{x}}$. Show that each ψ_k is linear and bounded. Finally, observe that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\;\Phi^{-1}\;} & \mathcal{A}_{\boldsymbol{x}} \\ & {\scriptstyle\varphi_k}\searrow & \downarrow{\scriptstyle\psi_k} \\ & & \mathbb{F} \end{array}$$

Problem 4.31. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces (over the same scalar field) and let $\mathcal{M}$ be a linear manifold of $\mathcal{X}$. Equip the direct sum of $\mathcal{M}$ and $\mathcal{Y}$ with any of the norms of Example 4.E and consider the normed space $\mathcal{M} \oplus \mathcal{Y}$. A linear transformation $L\colon \mathcal{M} \to \mathcal{Y}$ is *closed* if its graph is closed in $\mathcal{M} \oplus \mathcal{Y}$. Since a subspace means a closed linear manifold, and recalling that the graph of a linear transformation of $\mathcal{M}$ into $\mathcal{Y}$ is a linear manifold of the linear space $\mathcal{M} \oplus \mathcal{Y}$, such a definition can be rewritten as follows. A linear transformation $L\colon \mathcal{M} \to \mathcal{Y}$ is closed if its graph is a subspace of the normed space $\mathcal{M} \oplus \mathcal{Y}$. Take an arbitrary $L \in \mathcal{L}[\mathcal{M}, \mathcal{Y}]$ and prove that the assertions below are equivalent.

(i) L is closed.

(ii) If $\{u_n\}$ is an $\mathcal{M}$-valued sequence that converges in $\mathcal{X}$, and if its image under L converges in $\mathcal{Y}$, then

$$\lim u_n \in \mathcal{M} \quad \text{and} \quad \lim Lu_n = L \lim u_n.$$

Symbolically, L *is closed if and only if*

$$\left\{ \begin{array}{c} u_n \in \mathcal{M} \to u \in \mathcal{X} \\ Lu_n \to y \in \mathcal{Y} \end{array} \right\} \quad \Longrightarrow \quad \left\{ \begin{array}{c} u \in \mathcal{M} \\ y = Lu \end{array} \right\}.$$

Hint: Apply the Closed Set Theorem. Use the norm $\|\ \|_1$ on $\mathcal{M} \oplus \mathcal{Y}$ (i.e., $\|(u,y)\|_1 = \|u\|_X + \|y\|_Y$ for every $(u,y) \in \mathcal{M} \oplus \mathcal{Y}$).

Problem 4.32. Consider the setup of the previous problem and prove the following propositions.

(a) If $L \in \mathcal{B}[\mathcal{M}, \mathcal{Y}]$ and $\mathcal{M}$ is closed in $\mathcal{X}$, then L is closed.

Every bounded linear transformation defined on a subspace of a normed space is closed. In particular (set $\mathcal{M} = \mathcal{X}$), if $L \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ then L is closed.

(b) If $\mathcal{M}$ and $\mathcal{Y}$ are Banach spaces and if $L \in \mathcal{L}[\mathcal{M}, \mathcal{Y}]$ is closed, then $L \in \mathcal{B}[\mathcal{M}, \mathcal{Y}]$.

Every closed linear transformation between Banach spaces is bounded.

(c) If $\mathcal{Y}$ is a Banach space and $L \in \mathcal{B}[\mathcal{M}, \mathcal{Y}]$ is closed, then $\mathcal{M}$ is closed in $\mathcal{X}$.

Every closed and bounded linear transformation into a Banach space has a closed domain.

Hint: Closed Graph Theorem and Closed Set Theorem.

Recall that continuity means convergence preservation in the sense of Theorem 3.7, and also that the notions of "bounded" and "continuous" coincide for a linear transformation between normed spaces (Theorem 4.14). Now compare Corollary 3.8 with Problem 4.31 and prove the next proposition.

(d) If $\mathcal{M}$ and $\mathcal{Y}$ are Banach spaces, then $L \in \mathcal{L}[\mathcal{M}, \mathcal{Y}]$ is continuous if and only if it is closed.

Problem 4.33. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $\mathcal{M}$ be a linear manifold of $\mathcal{X}$. Take $L \in \mathcal{L}[\mathcal{M}, \mathcal{Y}]$ and consider the following assertions.

(i) $\mathcal{M}$ is closed in $\mathcal{X}$ (so that $\mathcal{M}$ is a Banach space).
(ii) L is a closed linear transformation.
(iii) L is bounded (i.e., L is continuous).

According to Problem 4.32 these three assertions are related as follows: *each pair of them implies the other*.

(a) Exhibit a bounded linear transformation that is not closed.

Hint: ℓ_+^1 is a dense linear manifold of $(\ell_+^2, \|\ \|_2)$. Take the inclusion map of $(\ell_+^1, \|\ \|_2)$ into $(\ell_+^2, \|\ \|_2)$.

The classical example of a closed linear transformation that is not bounded is the differential mapping $D\colon C'[0,1] \to C[0,1])$ defined in Problem 3.18. It is easy to show that $C'[0,1]$, the set of all differentiable functions in $C[0,1]$ whose derivatives lie in $C[0,1]$, is a linear manifold of the Banach space $C[0,1]$

equipped with the sup-norm. It is also easy to show that D is linear. Moreover, according to Problem 3.18(a), D is not continuous (and so unbounded). However, if $\{u_n\}$ is a uniformly convergent sequence of continuously differentiable functions whose derivative sequence $\{Du_n\}$ also converges uniformly, then $\lim Du_n = D(\lim u_n)$. This is a standard result from advanced calculus. Thus D is closed by Problem 4.31.

(b) Give another example of an unbounded closed linear transformation.

Hint: $\mathcal{X} = \mathcal{Y} = \ell^1_+$, $\mathcal{M} = \big\{x = \{\xi_k\}_{k=1}^\infty \in \ell^1_+ \colon \sum_{k=1}^\infty k|\xi_k| < \infty\big\}$, and $D = \operatorname{diag}(\{k\}_{k=1}^\infty) = \operatorname{diag}(1, 2, 3, \ldots)\colon \mathcal{M} \to \ell^1_+$. Verify that $\mathcal{M}$ is a linear manifold of ℓ^1_+. Use $x_n = \frac{1}{n^2}(1, \ldots, 1, 0, 0, 0, \ldots) \in \mathcal{M}$ (the first n entries are all equal to $\frac{1}{n^2}$; the rest is zero) to show that D is not continuous (Corollary 3.8). Suppose $u_n \to u \in \ell^1_+$, with $u_n = \{\xi_{n(k)}\}_{k=1}^\infty$ in $\mathcal{M}$, and $Du_n \to y = \{\upsilon_{(k)}\}_{k=1}^\infty \in \ell^1_+$. Set $\xi_{(k)} = \frac{1}{k}\upsilon_{(k)}$ so that $x = \{\xi_{(k)}\}_{k=1}^\infty$ lies in $\mathcal{M}$. Now show that $\|u_n - x\|_1 \le \|Du_n - y\|_1$, and so $u_n \to x$. Thus $u = x \in \mathcal{M}$ (uniqueness of the limit) and $y = Du$ (since $y = Dx$). Apply Problem 4.31 to conclude that D is closed. Generalize to injective diagonal mappings with unbounded entries.

Problem 4.34. Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of a normed space $\mathcal{X}$. If $\mathcal{M}$ and $\mathcal{N}$ are algebraic complements of each other (i.e., $\mathcal{M} + \mathcal{N} = \mathcal{X}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$), then we say that $\mathcal{M}$ and $\mathcal{N}$ are *complementary subspaces* in $\mathcal{X}$. According to Theorem 2.14 the natural mapping $\Phi\colon \mathcal{M} \oplus \mathcal{N} \to \mathcal{M} + \mathcal{N}$, defined by $\Phi((u, v)) = u + v$ for every $(u, v) \in \mathcal{M} \oplus \mathcal{N}$, is an isomorphism between the linear spaces $\mathcal{M} \oplus \mathcal{N}$ and $\mathcal{M} + \mathcal{N}$ if $\mathcal{M} \cap \mathcal{N} = \{0\}$. Consider the direct sum $\mathcal{M} \oplus \mathcal{N}$ equipped with any of the norms of Example 4.E. Prove the statement:

> If $\mathcal{M}$ and $\mathcal{N}$ are complementary subspaces in a Banach space $\mathcal{X}$, then the natural mapping $\Phi\colon \mathcal{M} \oplus \mathcal{N} \to \mathcal{M} + \mathcal{N}$ is a topological isomorphism.

Hint: Show that the isomorphism Φ is a contraction when $\mathcal{M} \oplus \mathcal{N}$ is equipped with the norm $\|\ \|_1$. Recall that $\mathcal{M}$ and $\mathcal{N}$ are Banach spaces (Proposition 4.7) and conclude that $\mathcal{M} \oplus \mathcal{N}$ is again a Banach space (Example 4.E). Apply the Inverse Mapping Theorem to prove that Φ is a topological isomorphism when $\mathcal{M} \oplus \mathcal{N}$ is equipped with the norm $\|\ \|_1$. Also recall that the norms of Example 4.E are equivalent (see the remarks that follow Proposition 4.26).

Problem 4.35. Prove the following propositions.

(a) If $P\colon \mathcal{X} \to \mathcal{X}$ is a continuous projection on a normed space $\mathcal{X}$, then $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are complementary subspaces in $\mathcal{X}$.

Hint: $\mathcal{R}(P) = \mathcal{N}(I - P)$. Apply Theorem 2.19 and Proposition 4.13.

(b) Conversely, if $\mathcal{M}$ and $\mathcal{N}$ are complementary subspaces in a *Banach* space $\mathcal{X}$, then the unique projection $P\colon \mathcal{X} \to \mathcal{X}$ with $\mathcal{R}(P) = \mathcal{M}$ and $\mathcal{N}(P) = \mathcal{N}$ of Theorem 2.20 is continuous and $\|P\| \ge 1$.

Hint: Consider the natural mapping $\Phi\colon \mathcal{M} \oplus \mathcal{N} \to \mathcal{M} + \mathcal{N}$ of the direct sum $\mathcal{M} \oplus \mathcal{N}$ (equipped with any of the norms of Example 4.E) onto $\mathcal{X} = \mathcal{M} + \mathcal{N}$. Let $P_M\colon \mathcal{M} \oplus \mathcal{N} \to \mathcal{M} \subseteq \mathcal{X}$ be the map defined by $P_M(u, v) = u$ for every $(u, v) \in \mathcal{M} \oplus \mathcal{N}$, which is a contraction (indeed, $\|P_M\| = 1$, see Example 4.I). Apply the previous problem to verify that the diagram

$$\begin{array}{ccc} \mathcal{M} \oplus \mathcal{N} & \xleftarrow{\Phi^{-1}} & \mathcal{M} + \mathcal{N} \\ & {\scriptstyle P_M}\searrow \quad & \downarrow {\scriptstyle P} \\ & & \mathcal{X} \end{array}$$

commutes. Thus show that P is continuous (note that $Pu = u$ for every $u \in \mathcal{M} = \mathcal{R}(P)$).

Remarks: P_M is, in fact, a continuous projection of $\mathcal{M} \oplus \mathcal{N}$ into itself whose range is $\mathcal{R}(P_M) = \mathcal{M} \oplus \{0\}$. If we identify $\mathcal{M} \oplus \{0\}$ with $\mathcal{M}$ (as we did in Example 4.I), then $P_M\colon \mathcal{M} \oplus \mathcal{N} \to \mathcal{M} \oplus \{0\} \subseteq \mathcal{M} \oplus \mathcal{N}$ can be viewed as a map from $\mathcal{M} \oplus \mathcal{N}$ onto $\mathcal{M}$, and hence we wrote $P_M\colon \mathcal{M} \oplus \mathcal{N} \to \mathcal{M} \subseteq \mathcal{X}$; the continuous natural projection of $\mathcal{M} \oplus \mathcal{N}$ onto $\mathcal{M}$. Also notice that the above propositions hold for the *complementary projection* $E = (I - P)\colon \mathcal{X} \to \mathcal{X}$ as well, since $\mathcal{N}(E) = \mathcal{R}(P)$ and $\mathcal{R}(E) = \mathcal{N}(P)$.

Problem 4.36. Consider a bounded linear transformation $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ of a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$. Let $\mathcal{M}$ be a complementary subspace of $\mathcal{N}(T)$ in $\mathcal{X}$. That is, $\mathcal{M}$ is a subspace of $\mathcal{X}$ that is also an algebraic complement of the null space $\mathcal{N}(T)$ of T:

$$\mathcal{M} = \mathcal{M}^-, \quad \mathcal{X} = \mathcal{M} + \mathcal{N}(T), \quad \text{and} \quad \mathcal{M} \cap \mathcal{N}(T) = \{0\}.$$

Set $T_M = T|_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{Y}$, the restriction of T to $\mathcal{M}$, and verify the following propositions.

(a) $T_M \in \mathcal{B}[\mathcal{M}, \mathcal{Y}]$, $\mathcal{R}(T_M) = \mathcal{R}(T)$, and $\mathcal{N}(T_M) = \{0\}$.

Hint: Problems 2.14 and 3.30.

(b) $\mathcal{R}(T_M) = \mathcal{R}(T_M)^-$ if and only if there exists ${T_M}^{-1} \in \mathcal{B}[\mathcal{R}(T_M), \mathcal{M}]$.

Hint: Proposition 4.7 and Corollary 4.24.

(c) If $A \subseteq \mathcal{R}(T)$ and ${T_M}^{-1}(A)^- = \mathcal{M}$, then $T^{-1}(A)^- = \mathcal{X}$.

Hint: Take an arbitrary $x = u + v \in \mathcal{X} = \mathcal{M} + \mathcal{N}(T)$, with $u \in \mathcal{M}$ and $v \in \mathcal{N}(T)$. Verify that there exists a ${T_M}^{-1}(A)$-valued sequence $\{u_n\}$ that converges to u. Set $x_n = u_n + v$ in $\mathcal{X}$ and show that $\{x_n\}$ is a ${T_M}^{-1}(A)$-valued sequence that converges to x. Apply Proposition 3.32.

Now use the above results to prove the following assertion.

(d) If $A \subseteq \mathcal{R}(T)$ and $A^- = \mathcal{R}(T) = \mathcal{R}(T)^-$, then $T^{-1}(A)^- = \mathcal{X}$.

That is, the inverse image under T of a dense subset of the range of T is dense in $\mathcal{X}$ whenever $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ has a closed range and a null space with a complementary subspace in $\mathcal{X}$. This can be viewed as a converse to Problem 3.46(c).

Problem 4.37. Prove the following propositions.

(a) Every finite-dimensional normed space is a separable Banach space.

Hint: Example 3.P, Problem 3.48, and Corollaries 4.28 and 4.31.

(b) If $\mathcal{X}$ and $\mathcal{Y}$ are topologically isomorphic normed spaces and if one of them is a (separable) Banach space, then so is the other.

Hint: Theorems 3.44 and 4.14.

Problem 4.38. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and take $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$. If either $\mathcal{X}$ or $\mathcal{Y}$ is finite dimensional, then T is of finite rank (Problems 2.6 and 2.17). $\mathcal{R}(T)$ is a subspace of $\mathcal{Y}$ whenever T is of finite rank (Corollary 4.29). If T is injective and of finite rank, then $\mathcal{X}$ is finite dimensional (Theorem 2.8 and Problems 2.6 and 2.17). Use Problem 2.7 and Corollaries 4.24 and 4.28 to prove the following assertions.

(a) If $\mathcal{Y}$ is a Banach space and $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is of finite rank and injective, then T has a bounded inverse on its range.

(b) If $\mathcal{X}$ is finite dimensional, then an injective operator in $\mathcal{B}[\mathcal{X}]$ is invertible.

(c) If $\mathcal{X}$ is finite dimensional and $T \in \mathcal{L}[\mathcal{X}]$, then $\mathcal{N}(T) = \{0\}$ if and only if $T \in \mathcal{G}[\mathcal{X}]$.

This means that a linear transformation of a finite-dimensional normed space into itself is a topological isomorphism if and only if it is injective.

(d) If $\mathcal{X}$ is finite dimensional, then every linear isometry of $\mathcal{X}$ into itself is an isometric isomorphism.

That is, every linear isometry of a finite-dimensional normed space into itself is surjective.

Problem 4.39. The previous problem says that nonsurjective isometries in $\mathcal{B}[\mathcal{X}]$ may exist only if the normed space $\mathcal{X}$ is infinite dimensional. Here is an example. Let $(\ell_+, \|\ \|)$ denote either the normed space $(\ell_+^p, \|\ \|_p)$ for some $p \geq 1$ or $(\ell_+^\infty, \|\ \|_\infty)$. Consider the mapping $S_+ \colon \ell_+ \to \ell_+$ defined by

$$S_+ x = \{\upsilon_k\}_{k=0}^\infty \qquad \text{with} \qquad \upsilon_k = \begin{cases} 0, & k = 0, \\ \xi_{k-1}, & k \geq 1, \end{cases}$$

for every $x = \{\xi_k\}_{k=0}^\infty \in \ell_+$. That is, $S_+(\xi_0, \xi_1, \xi_2, \ldots) = (0, \xi_0, \xi_1, \xi_2, \ldots)$ for every $(\xi_0, \xi_1, \xi_2, \ldots)$ in ℓ_+, which is also represented by the infinite matrix

$$S_+ = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \\ & & & \end{pmatrix},$$

where every entry immediately below the main diagonal is equal to 1 and the remaining entries are all zero. This is the *unilateral shift* on ℓ_+.

(a) Show that S_+ is a linear nonsurjective isometry.

Since S_+ is a linear isometry, it follows by Proposition 4.37 that $\|S_+^n x\| = \|x\|$ for every $x \in \ell_+$ and all $n \geq 1$, and hence

$$S_+ \in \mathcal{B}[\ell_+] \quad \text{with} \quad \|S_+^n\| = 1 \text{ for all } n \geq 0.$$

Consider the backward unilateral S_- of Example 4.L, now acting either on ℓ_+^p or on ℓ^∞. Recall that $S_- \in \mathcal{B}[\ell_+]$ and $\|S_-^n\| = 1$ for all $n \geq 0$ (this has been verified in Example 4.L for $(\ell_+, \|\ \|) = (\ell_+^p, \|\ \|_p)$, but the same argument ensures that it holds for $(\ell_+, \|\ \|) = (\ell^\infty, \|\ \|_\infty)$ as well).

(b) Show that $S_- S_+ = I\colon \ell_+ \to \ell_+$, the identity on ℓ_+.

Therefore, $S_- \in \mathcal{B}[\ell_+]$ is a left inverse of $S_+ \in \mathcal{B}[\ell_+]$.

(c) Conclude that S_- is surjective but not injective.

Problem 4.40. Let $(\ell, \|\ \|)$ denote either the normed space $(\ell^p, \|\ \|_p)$ for some $p \geq 1$ or $(\ell^\infty, \|\ \|_\infty)$. Consider the mapping $S\colon \ell \to \ell$ defined by

$$Sx = \{\xi_{k-1}\}_{k=-\infty}^{\infty} \quad \text{for every} \quad x = \{\xi_k\}_{k=-\infty}^{\infty} \in \ell$$

(i.e., $S(\ldots, \xi_{-2}, \xi_{-1}, (\xi_0), \xi_1, \xi_2, \ldots) = (\ldots, \xi_{-3}, \xi_{-2}, (\xi_{-1}), \xi_0, \xi_1, \ldots)$), which is also represented by the (doubly) infinite matrix

$$S = \begin{pmatrix} \ddots & & & & & \\ 1 & 0 & & & & \\ & 1 & (0) & & & \\ & & 1 & 0 & & \\ & & & & \ddots & \end{pmatrix}$$

(with the inner parenthesis indicating the zero-zero position), where every entry immediately below the main diagonal is equal to 1 and the remaining entries are all zero. This is the *bilateral shift* on ℓ.

(a) Show that S is a linear surjective isometry.

That is, S is an isometric isomorphism, and hence

$$S \in \mathcal{G}[\,\ell\,] \quad \text{with} \quad \|S^n\| = 1 \ \text{ for all } \ n \geq 0.$$

Its inverse S^{-1} is then again an isometric isomorphism, so that

$$S^{-1} \in \mathcal{G}[\,\ell\,] \quad \text{with} \quad \|(S^{-1})^n\| = 1 \ \text{ for all } \ n \geq 0.$$

(b) Verify that the inverse S^{-1} of S is given by the formula

$$S^{-1}x \,=\, \{\xi_{k+1}\}_{k=-\infty}^{\infty} \quad \text{for every} \quad x = \{\xi_k\}_{k=-\infty}^{\infty} \in \ell$$

(that is, $S^{-1}(\ldots,\xi_{-2},\xi_{-1},(\xi_0),\xi_1,\xi_2,\ldots) = (\ldots,\xi_{-1},\xi_0,(\xi_1),\xi_2,\xi_3,\ldots)$), which is also represented by a (doubly) infinite matrix

$$S^{-1} \,=\, \begin{pmatrix} \ddots & 1 & & & & \\ & 0 & 1 & & & \\ & & (0) & 1 & & \\ & & & 0 & & \\ & & & & \ddots & \end{pmatrix},$$

where every entry immediately above the main diagonal is equal to 1 and the remaining entries are all zero. This is the *backward bilateral shift* on ℓ.

Problem 4.41. Use Proposition 4.37 to prove the following assertions.

(a) Let $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be normed spaces (over the same scalar field) and take $T \in \mathcal{B}[\mathcal{Y},\mathcal{Z}]$ and $S \in \mathcal{B}[\mathcal{W},\mathcal{X}]$. If $V \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ is an isometry, then

$$\|TV\| = \|T\| \quad \text{and} \quad \|VS\| = \|S\|.$$

(b) The product of two isometries is again an isometry.

Hint: If T is an isometry, then $\|TVx\| = \|Vx\| = \|x\|$ for every $x \in \mathcal{X}$.

(c) $\|V^n\| = 1$ for every $n \geq 1$ whenever V is an isometry in $\mathcal{B}[\mathcal{X}]$.

(d) A linear isometry of a Banach space into a normed space has closed range.

Hint: Propositions 4.20 and 4.37.

Problem 4.42. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. Verify that $T \in \mathcal{B}[\mathcal{X}]$ and $S \in \mathcal{B}[\mathcal{Y}]$ are similar (in the sense of Problem 4.29 — notation: $T \approx S$) if and only if there exists a topological isomorphism intertwining T to S; that is, if and only if there exists $W \in \mathcal{G}[\mathcal{X},\mathcal{Y}]$ such that

$$WT \,=\, SW.$$

Thus $\mathcal{X}$ and $\mathcal{Y}$ are topologically isomorphic normed spaces if there are similar operators in $\mathcal{B}[\mathcal{X}]$ and $\mathcal{B}[\mathcal{Y}]$. A stronger form of similarity is obtained when there is an isometric isomorphism, say U in $\mathcal{G}[\mathcal{X},\mathcal{Y}]$, intertwining T to S; i.e.,

$$UT \,=\, SU.$$

If this happens, then we say that T and S are *isometrically equivalent* (notation: $T \cong S$). Again, $\mathcal{X}$ and $\mathcal{Y}$ are isometrically isomorphic normed spaces if there are isometrically equivalent operators in $\mathcal{B}[\mathcal{X}]$ and $\mathcal{B}[\mathcal{Y}]$. As in the case of similarity, show that isometric equivalence has the defining properties of an equivalence relation. An important difference between similarity and isometric equivalence is that isometric equivalence is norm-preserving: if T and S are isometrically equivalent, then $\|T\| = \|S\|$. Prove this identity and show that it may fail if T and S are simply similar. Now let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Show that, in this case, $T \in \mathcal{B}[\mathcal{X}]$ and $S \in \mathcal{B}[\mathcal{Y}]$ are similar if and only if there exists an injective and surjective bounded linear transformation in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ intertwining T to S.

Problem 4.43. Let $\mathcal{X}$ be a normed space. Verify that the following three conditions are pairwise equivalent.

(a) $\mathcal{X}$ is separable (as a metric space).

(b) There exists a countable subset of $\mathcal{X}$ that spans $\mathcal{X}$.

(c) There exists a dense linear manifold $\mathcal{M}$ of $\mathcal{X}$ such that $\dim \mathcal{M} \leq \aleph_0$.

Hint: Proposition 4.9.

(d) Moreover, show also that a completion $\widehat{\mathcal{X}}$ of a separable normed space $\mathcal{X}$ is itself separable.

Problem 4.44. In many senses barreled spaces in a locally convex-space setting plays a role similar to Banach spaces in a normed-space setting. In fact, as we saw in Problem 4.4, a Banach space is barreled. Barreled spaces actually are the spaces where the Banach–Steinhaus Theorem holds in a locally convex-space setting: *Every pointwise bounded collection of continuous linear transformations of a barreled space into a locally convex space is equicontinuous*. To see that this is exactly the locally convex-space version of Theorem 4.43, we need the notion of *equicontinuity* in a locally convex space. Let $\mathcal{X}$ and $\mathcal{Y}$ be topological vector spaces. A subset Θ of $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is *equicontinuous* if for each neighborhood N_Y of the origin of $\mathcal{Y}$ there exists a neighborhood N_X of the origin of $\mathcal{X}$ such that $T(N_X) \subseteq N_Y$ for all $T \in \Theta$.

(a) Show that if $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces, then $\Theta \subseteq \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is equicontinuous if and only if $\Theta \subseteq \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ and $\sup_{T \in \Theta} \|T\| < \infty$.

The notion of a bounded set in a topological vector space (and, in particular, in a locally convex space) was defined in Problem 4.2. Moreover, it was shown in Problem 4.5(b) that this in fact is the natural extension to topological vector spaces of the usual notion of a bounded set in a normed space.

(b) Show that the Banach–Steinhaus Theorem (Theorem 4.43) can be stated as follows: *Every pointwise bounded collection of continuous linear transformations of a Banach space into a normed space is equicontinuous*.

Problem 4.45. Let $\{T_n\}$ be a sequence in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$, where $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces. Prove the following results.

(a) If $T_n \xrightarrow{s} T$ for some $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$, then $\|T_n x\| \to \|Tx\|$ for every $x \in \mathcal{X}$ and $\|T\| \le \liminf_n \|T_n\|$.

(b) If $\sup_n \|T_n\| < \infty$ and $\{T_n a\}$ is a Cauchy sequence in $\mathcal{Y}$ for every a in a dense set A in $\mathcal{X}$, then $\{T_n x\}$ is a Cauchy sequence in $\mathcal{Y}$ for every x in $\mathcal{X}$.

Hint: $T_n x - T_m x = T_n x - T_n a_k + T_n a_k - T_m a_k + T_m a_k - T_m x$.

(c) If there exists $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ such that $T_n a \to Ta$ for every a in a dense set A in $\mathcal{X}$, and if $\sup_n \|T_n\| < \infty$, then $T_n \xrightarrow{s} T$.

Hint: $(T_n - T)x = (T_n - T)(x - a_\varepsilon) + (T_n - T)a_\varepsilon$.

(d) If $\mathcal{X}$ is a Banach space and $\{T_n x\}$ is a Cauchy sequence for every $x \in \mathcal{X}$, then $\sup_n \|T_n\| < \infty$.

(e) If $\mathcal{X}$ and $\mathcal{Y}$ is a Banach space and $\{T_n x\}$ is a Cauchy sequence for every $x \in \mathcal{X}$, then $T_n \xrightarrow{s} T$ for some $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$.

Problem 4.46. Let $\{T_n\}$ be a sequence in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ and let $\{S_n\}$ be a sequence in $\mathcal{B}[\mathcal{Y},\mathcal{Z}]$, where $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are normed spaces. Suppose

$$T_n \xrightarrow{s} T \quad \text{and} \quad S_n \xrightarrow{s} S$$

for $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ and $S \in \mathcal{B}[\mathcal{Y},\mathcal{Z}]$. Prove the following propositions.

(a) If $\sup_n \|S_n\| < \infty$, then $S_n T_n \xrightarrow{s} ST$.

(b) If $\mathcal{Y}$ is a Banach space, then $S_n T_n \xrightarrow{s} ST$.

(c) If $S_n \xrightarrow{u} S$, then $S_n T_n \xrightarrow{s} ST$.

(d) If $S_n \xrightarrow{u} S$ and $T_n \xrightarrow{u} T$, then $S_n T_n \xrightarrow{u} ST$.

Finally, show that addition of strongly (uniformly) convergent sequences of bounded linear transformations is again a strongly (uniformly) convergent sequence of bounded linear transformations whose strong (uniform) limit is the sum of the strong (uniform) limits of each summand.

Problem 4.47. Let $\mathcal{X}$ be a Banach space and let T be an operator in $\mathcal{B}[\mathcal{X}]$. If λ is any scalar such that $\|T\| < |\lambda|$, then $\lambda I - T$ is an invertible element of $\mathcal{B}[\mathcal{X}]$ (i.e., $(\lambda I - T) \in \mathcal{G}[\mathcal{X}]$ — see the paragraph that follows Theorem 4.22) and the series $\sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}$ converges in $\mathcal{B}[\mathcal{X}]$ to $(\lambda I - T)^{-1}$. That is,

$$\|T\| < |\lambda| \quad \text{implies} \quad (\lambda I - T)^{-1} = \tfrac{1}{\lambda} \sum_{k=0}^{\infty} \left(\tfrac{T}{\lambda}\right)^k \in \mathcal{B}[\mathcal{X}].$$

This is a rather important result, known as the *von Neumann expansion*. The purpose of this problem is to prove it. Take $T \in \mathcal{B}[\mathcal{X}]$ and $0 \neq \lambda \in \mathbb{F}$ arbitrary. Show by induction that, for each integer $n \geq 0$,

(a) $$\|T^n\| \leq \|T\|^n, \quad \text{and}$$

(b) $$(\lambda I - T)\tfrac{1}{\lambda}\sum_{i=0}^{n}\big(\tfrac{T}{\lambda}\big)^i = \tfrac{1}{\lambda}\sum_{i=0}^{n}\big(\tfrac{T}{\lambda}\big)^i(\lambda I - T) = I - \big(\tfrac{T}{\lambda}\big)^{n+1}.$$

From now on suppose $\|T\| < |\lambda|$ and consider the power sequence $\big\{\big(\frac{T}{\lambda}\big)^n\big\}_{n=0}^{\infty}$ in $\mathcal{B}[\mathcal{X}]$. Use the result in (a) to show that

(c) $$\big\{\big(\tfrac{T}{\lambda}\big)^n\big\}_{n=0}^{\infty} \text{ is absolutely summable.}$$

Thus conclude that (cf. Problem 3.12)

(d) $$\big(\tfrac{T}{\lambda}\big)^n \xrightarrow{u} O$$

$\big($i.e., $\frac{T}{\lambda}$ is uniformly stable$\big)$, and also that (see Proposition 4.4)

(e) $$\big\{\big(\tfrac{T}{\lambda}\big)^k\big\}_{k=0}^{\infty} \text{ is summable.}$$

This means that the infinite series $\sum_{k=0}^{\infty}\big(\frac{T}{\lambda}\big)^k$ converges in $\mathcal{B}[\mathcal{X}]$ or, equivalently, that there exists an operator in $\mathcal{B}[\mathcal{X}]$, say $\sum_{k=0}^{\infty}\big(\frac{T}{\lambda}\big)^k$, such that

$$\sum_{k=0}^{n}\big(\tfrac{T}{\lambda}\big)^k \xrightarrow{u} \sum_{k=0}^{\infty}\big(\tfrac{T}{\lambda}\big)^k.$$

Apply the results in (b) and (d) to check the following convergences.

(f) $$(\lambda I - T)\tfrac{1}{\lambda}\sum_{k=0}^{n}\big(\tfrac{T}{\lambda}\big)^k \xrightarrow{u} I \quad \text{and} \quad \tfrac{1}{\lambda}\sum_{k=0}^{n}\big(\tfrac{T}{\lambda}\big)^k(\lambda I - T) \xrightarrow{u} I.$$

Now use (e) and (f) to show that

(g) $$(\lambda I - T)\tfrac{1}{\lambda}\sum_{k=0}^{\infty}\big(\tfrac{T}{\lambda}\big)^k = \tfrac{1}{\lambda}\sum_{k=0}^{\infty}\big(\tfrac{T}{\lambda}\big)^k(\lambda I - T) = I.$$

Then $\frac{1}{\lambda}\sum_{k=0}^{\infty}\big(\frac{T}{\lambda}\big)^k \in \mathcal{B}[\mathcal{X}]$ is the inverse of $(\lambda I - T) \in \mathcal{B}[\mathcal{X}]$ (Problem 1.7). So $\lambda I - T$ is an invertible element of $\mathcal{B}[\mathcal{X}]$ (i.e., $(\lambda I - T) \in \mathcal{G}[\mathcal{X}]$) whose inverse $(\lambda I - T)^{-1}$ is the (uniform) limit of the sequence $\{\sum_{i=0}^{n}\frac{T^i}{\lambda^{i+1}}\}_{n=0}^{\infty}$. That is,

$$(\lambda I - T)^{-1} = \tfrac{1}{\lambda}\sum_{k=0}^{\infty}\big(\tfrac{T}{\lambda}\big)^k \in \mathcal{B}[\mathcal{X}].$$

Finally, verify that

(h) $$\|(\lambda I - T)^{-1}\| \leq \tfrac{1}{|\lambda|} \sum_{k=0}^{\infty} \left(\tfrac{\|T\|}{|\lambda|}\right)^k = (|\lambda| - \|T\|)^{-1}.$$

Remark: Exactly the same proof applies if, instead of $\mathcal{B}[\mathcal{X}]$, we were working on an abstract unital Banach algebra $\mathcal{A}$.

Problem 4.48. If T is a strict contraction on a Banach space $\mathcal{X}$, then

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k \in \mathcal{B}[\mathcal{X}].$$

This is the special case of Problem 4.47 for $\lambda = 1$. Use it to prove assertion (a) below. Then prove the next three assertions.

(a) Every operator in the open unit ball centered at the identity I of $\mathcal{B}[\mathcal{X}]$ is invertible. That is, if $\|I - S\| < 1$, then $S \in \mathcal{G}[\mathcal{X}]$.

(b) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Centered at each invertible transformation $T \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$ there exists a nonempty open ball $B_\varepsilon(T) \subset \mathcal{G}[\mathcal{X}, \mathcal{Y}]$ such that $\sup_{S \in B_\varepsilon(T)} \|S^{-1}\| < \infty$. In particular, $\mathcal{G}[\mathcal{X}, \mathcal{Y}]$ is open in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$.

Hint: Suppose $\|T - S\| < \varepsilon < \|T^{-1}\|^{-1}$ so that

$$\|I_X - T^{-1}S\| = \|T^{-1}(T - S)\| < \|T^{-1}\|\varepsilon < 1.$$

Thus $T^{-1}S = I_X - (I_X - T^{-1}S)$ lies in $\mathcal{G}[\mathcal{X}]$ by (a), and therefore $S = TT^{-1}S$ also lies in $\mathcal{G}[\mathcal{X}, \mathcal{Y}]$, with (cf. Corollary 4.23 and Proposition 4.16)

$$\|S^{-1}\| = \|S^{-1}TT^{-1}\| \leq \|T^{-1}\|\|S^{-1}T\|.$$

But, according to Problem 4.47(h),

$$\|S^{-1}T\| = \|(T^{-1}S)^{-1}\| = \|\,[I_X - (I_X - T^{-1}S)]^{-1}\| \leq (1 - \|I_X - T^{-1}S\|)^{-1}.$$

Conclude: $\|S^{-1}\| \leq \|T^{-1}\|\,(1 - \|T^{-1}\|\varepsilon)^{-1}$.

(c) Inversion is a continuous mapping. That is, if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, then the map $T \mapsto T^{-1}$ of $\mathcal{G}[\mathcal{X}, \mathcal{Y}]$ into $\mathcal{G}[\mathcal{Y}, \mathcal{X}]$ is continuous.

Hint: $T^{-1} - S^{-1} = T^{-1}(S - T)S^{-1}$. If $T_n \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$ and $\{T_n\}$ converges in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ to $S \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$, then $\sup_n \|T_n^{-1}\| < \infty$, and so $T_n^{-1} \to S^{-1}$.

(d) If $T \in \mathcal{B}[\mathcal{X}]$ is an invertible contraction on a normed space $\mathcal{X}$, then

$$\|T^n x\| \leq \|x\| \leq \|T^{-n}x\| \text{ for every } x \in \mathcal{X} \text{ and every integer } n \geq 0.$$

Hint: Show that T^n is a contraction if T is (cf. Problems 1.10 and 4.22).

Problem 4.49. Let $\mathcal{X}$ be a normed space. Show that the set of all strict contractions in $\mathcal{B}[\mathcal{X}]$ is not closed in $\mathcal{B}[\mathcal{X}]$ (and so not strongly closed in $\mathcal{B}[\mathcal{X}]$).

Hint: Each $D_n = \operatorname{diag}(\frac{1}{2}, \ldots, \frac{n}{n+1}, 0, 0, 0, \ldots)$, with just the first n entries different from zero, is a strict contraction in any $\mathcal{B}[\ell^p_+]$ (and in $\mathcal{B}[\ell^\infty_+]$).

On the other hand, show that the set of all contractions in $\mathcal{B}[\mathcal{X}]$ is strongly closed in $\mathcal{B}[\mathcal{X}]$, and so (uniformly) closed in $\mathcal{B}[\mathcal{X}]$.

Hint: $\big|\|T_n x\| - \|Tx\|\big| \le \|(T_n - T)x\|$.

Problem 4.50. Show that the strong limit of a sequence of linear isometries is again a linear isometry. In other words, the set of all isometries from $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is strongly closed, and so uniformly closed (where $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces).

Hint: Proposition 4.37 and Problem 4.45(a).

Problem 4.51. Take an arbitrary $p \ge 1$ and consider the normed space ℓ^p_+ of Example 4.B. Let $\{D_n\}$ be a sequence of diagonal operators in $\mathcal{B}[\ell^p_+]$. If $\{D_n\}$ converges strongly to $D \in \mathcal{B}[\ell^p_+]$, then D is a diagonal operator. Prove.

Problem 4.52. Let $\{P_k\}_{k=1}^\infty$ be a sequence of diagonal operators in $\mathcal{B}[\ell^p_+]$ for any $p \ge 1$ such that, for each $k \ge 1$,

$$P_k = \operatorname{diag}(\{e_k\}_{k=1}^\infty) = \operatorname{diag}(0, \ldots, 0, 1, 0, 0, 0, \ldots)$$

(the only nonzero entry is equal to 1 and lies at the kth position). Set

$$E_n = \sum_{k=1}^{n} P_k = \operatorname{diag}(1, \ldots, 1, 0, 0, 0, \ldots) \in \mathcal{B}[\ell^p_+]$$

for every integer $n \ge 1$. Show that $E_n \xrightarrow{s} I$, the identity in $\mathcal{B}[\ell^p_+]$, but $\{E_n\}_{n=1}^\infty$ does not converge uniformly because $\|E_n - I\| = 1$ for all $n \ge 1$.

Problem 4.53. Let $a = \{\alpha_k\}_{k=1}^\infty$ be a scalar-valued sequence and consider a sequence $\{D_n\}_{n=1}^\infty$ of diagonal mappings of the Banach space ℓ^p_+ (for any $p \ge 1$) into itself such that, for each integer $n \ge 1$,

$$D_n = \operatorname{diag}(\alpha_1, \ldots, \alpha_n, 0, 0, 0, \ldots),$$

where the entries in the main diagonal are all null except perhaps for the first n entries. It is clear that D_n lies in $\mathcal{B}[\ell^p_+]$ for each $n \ge 1$ (reason: $\mathcal{B}_0[\ell^p_+] \subset \mathcal{B}[\ell^p_+]$). If $a \in \ell^\infty_+$, then consider the diagonal operator $D_a = \operatorname{diag}(\{\alpha_k\}_{k=1}^\infty) \in \mathcal{B}[\ell^p_+]$ of Examples 4.H and 4.K. Prove the following assertions.

(a) If $\sup_k |\alpha_k| < \infty$, then $D_n \xrightarrow{s} D_a$.

Hint: $\|D_n x - D_a x\|^p \le \sup_k |\alpha_k|^p \sum_{k=n}^\infty |\xi_k|^p$ for $x = \{\xi_k\}_{k=1}^\infty \in \ell^p_+$.

(b) Conversely, if $\{D_n x\}_{n=1}^\infty$ converges in ℓ^p_+ for every vector $x \in \ell^p_+$, then $\sup_k |\alpha_k| < \infty$, and hence $D_n \xrightarrow{s} D_a$.

Hint: Proposition 3.39, Theorem 4.43, and Example 4.H.

(c) If $\lim_k |\alpha_k| = 0$, then $D_n \xrightarrow{u} D_a$.

Hint: Verify that $\|(D_n - D_a)x\|^p \le \sup_{n\le k}|\alpha_k|^p\|x\|^p$ and conclude that $\lim_n \|D_n - D_a\| \le \lim_n \sup_{n\le k}|\alpha_k| = \limsup_n|\alpha_k|$.

(d) Conversely, if $\{D_n\}_{n=1}^{\infty}$ converges uniformly, then $\lim_k |\alpha_k| = 0$, and hence $D_n \xrightarrow{u} D_a$.

Hint: Uniform convergence implies strong convergence. Apply (c). Compute $(D_n - D_a)e_k$ for every k, n.

Problem 4.54. Take any $\alpha \in \mathbb{C}$ such that $\alpha \ne 1$. Consider the operators A and P in $\mathcal{B}[\mathbb{C}^2]$ identified with the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -\alpha & 1+\alpha \end{pmatrix} \quad \text{and} \quad P = \tfrac{1}{\alpha-1}\begin{pmatrix} \alpha & -1 \\ \alpha & -1 \end{pmatrix}$$

(i.e., these matrices are the representations of A and P with respect to the canonical basis for $\mathbb{C}^2$).

(a) Show that $PA = AP = P = P^2$.

(b) Prove by induction that

$$A^{n+1} = \alpha A^n + (1-\alpha)P,$$

and hence (see Problem 2.19 or supply another induction)

$$A^n = \alpha^n(I - P) + P,$$

for every integer $n \ge 0$.

(c) Finally, show that

$$|\alpha| < 1 \quad \text{implies} \quad A^n \xrightarrow{u} P \ \text{ and } \ \|P\| > 1,$$

where $\|\ \|$ denotes the norm on $\mathcal{B}[\mathbb{C}^2]$ induced by any of the norms $\|\ \|_p$ (for $p \ge 1$) or $\|\ \|_\infty$ on the linear space $\mathbb{C}^2$ as in Example 4.A.

Hint: $1 < \|Pe_1\|_\infty \le \|Pe_1\|_p$ (cf. Problem 3.33).

Problem 4.55. Take a linear transformation $T \in \mathcal{L}[\mathcal{X}]$ on a normed space $\mathcal{X}$. Suppose the power sequence $\{T^n\}$ is pointwise convergent, which means that there exists $P \in \mathcal{L}[\mathcal{X}]$ such that $T^n x \to Px$ in $\mathcal{X}$ for every $x \in \mathcal{X}$. Show that

(a) $PT^k = T^k P = P = P^k$ for every integer $k \ge 1$,

(b) $(T - P)^n = T^n - P$ for every integer $n \ge 1$.

Now suppose T lies in $\mathcal{B}[\mathcal{X}]$ and prove the following propositions.

(c) If $T^n \xrightarrow{s} P \in \mathcal{B}[\mathcal{X}]$, then P is a projection and $(T - P)^n \xrightarrow{s} O$.

(d) If $T^n x \to Px$ for every $x \in \mathcal{X}$, then $P \in \mathcal{L}[\mathcal{X}]$ is a projection. If, in addition, $\mathcal{X}$ is a Banach space, then P is a continuous projection and $T - P$ in $\mathcal{B}[\mathcal{X}]$ is strongly stable.

Problem 4.56. Let $F\colon X \to X$ be a mapping of a set X into itself. Recall that F is injective if and only if it has a left inverse $F^{-1}\colon \mathcal{R}(X) \to X$ on its range $\mathcal{R}(X) = F(X)$. Therefore, if F is injective and idempotent (i.e., $F = F^2$), then $F = F^{-1}FF = F^{-1}F = I$, and hence

(a) the unique idempotent injective mapping is the identity.

This is a purely set-theoretic result (no algebra or topology is involved). Now let X be a metric space and recall that every isometry is injective. Thus,

(b) the unique idempotent isometry is the identity.

In particular, if $\mathcal{X}$ is a normed space and $F\colon \mathcal{X} \to \mathcal{X}$ is a projection (i.e., an idempotent linear transformation) and an isometry, then $F = I$: the identity is the unique isometry that also is a projection. Show that

(c) the unique linear isometry that has a strongly convergent power sequence is the identity.

Hint: Problems 4.50 and 4.55.

Problem 4.57. Let $\{T_n\}$ be a sequence of bounded linear transformations in $\mathcal{B}[\mathcal{Y}, \mathcal{Z}]$, where $\mathcal{Y}$ is a Banach space and $\mathcal{Z}$ is a normed space, and take T in $\mathcal{B}[\mathcal{Y}, \mathcal{Z}]$. Show that, if $\mathcal{M}$ is a finite-dimensional subspace of $\mathcal{Y}$, then

(a) $$T_n \xrightarrow{s} T \quad \text{implies} \quad (T_n - T)|_{\mathcal{M}} \xrightarrow{u} O.$$

(*Hint*: Proposition 4.46.) Now let K be a compact linear transformation of a normed space $\mathcal{X}$ into $\mathcal{Y}$ (i.e., $K \in \mathcal{B}_\infty[\mathcal{X}, \mathcal{Y}]$). Prove that

(b) $$T_n \xrightarrow{s} T \quad \text{implies} \quad T_n K \xrightarrow{u} TK.$$

Hint: Take any $x \in \mathcal{X}$. Use Proposition 4.56 to show that for each $\varepsilon > 0$ there exists a finite-dimensional subspace $\mathcal{R}_\varepsilon$ of $\mathcal{Y}$ and a vector $r_{\varepsilon,x}$ in $\mathcal{R}_\varepsilon$ such that

$$\|Kx - r_{\varepsilon,x}\| < 2\varepsilon\|x\| \quad \text{and} \quad \|r_{\varepsilon,x}\| < (2\varepsilon + \|K\|)\|x\|.$$

Then verify that

$$\begin{aligned} \|(T_n K - TK)x\| &\le \|T_n - T\|\,\|Kx - r_{\varepsilon,x}\| + \big\|(T_n - T)|_{\mathcal{R}_\varepsilon}\big\|\,\|r_{\varepsilon,x}\| \\ &< \Big(2\varepsilon\|T_n - T\| + \big(2\varepsilon + \|K\|\big)\big\|(T_n - T)|_{\mathcal{R}_\varepsilon}\big\|\Big)\|x\|. \end{aligned}$$

Finally, apply the Banach–Steinhaus Theorem (Theorem 4.43) to ensure that $\sup_n \|T_n - T\| < \infty$ and conclude from item (a): for every $\varepsilon > 0$

$$\limsup_n \|T_n K - TK\| < \big(2 \sup_n \|T_n - T\|\big)\varepsilon.$$

Problem 4.58. Prove the converse of Corollary 4.55 under the assumption that the Banach space $\mathcal{Y}$ has a Schauder basis. In other words, prove the following proposition.

> If $\mathcal{Y}$ is a Banach space with a Schauder basis and $\mathcal{X}$ is a normed space, then every compact linear transformation in $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$ is the uniform limit of a sequence of finite-rank linear transformations in $\mathcal{B}_0[\mathcal{X},\mathcal{Y}]$.

Hint: Suppose $\mathcal{Y}$ is infinite dimensional (otherwise the result is trivial) and has a Schauder basis. Take an arbitrary K in $\mathcal{B}_\infty[\mathcal{X},\mathcal{Y}]$. $\mathcal{R}(K)^-$ is a Banach space possessing a Schauder basis, say $\{y_i\}_{i=1}^\infty$, so that every $y \in \mathcal{R}(K)^-$ has a unique expansion $y = \sum_{i=1}^\infty \alpha_i(y)\, y_i$ (Problem 4.11). For each nonnegative integer n consider the mapping $E_n\colon \mathcal{R}(K)^- \to \mathcal{R}(K)^-$ defined by $E_n y = \sum_{i=1}^n \alpha_i(y)\, y_i$. Show that each E_n is bounded and linear (Problem 4.30), and of finite rank (since $\mathcal{R}(E_n) \subseteq \bigvee\{y_i\}_{i=1}^n$). Also show that $\{E_n\}_{n=1}^\infty$ converges strongly to the identity operator I on $\mathcal{R}(K)^-$ (Problem 4.9(b)). Use the previous problem to conclude that $E_n K \xrightarrow{u} K$. Finally, check that $E_n K \in \mathcal{B}_0[\mathcal{X},\mathcal{Y}]$for each n.

Remark: Consider the remark in Problem 4.11. There we commented on the construction of a separable Banach space that has no Schauder basis. Such a breakthrough was achieved by Enflo in 1973 when he exhibited a separable (and reflexive) Banach space $\mathcal{X}$ for which $\mathcal{B}_0[\mathcal{X}]$ is not dense in $\mathcal{B}_\infty[\mathcal{X}]$, so that there exist compact operators on $\mathcal{X}$ that are not the (uniform) limit of finite-rank operators (and so the converse of Corollary 4.55 fails in general). Thus, according to the above proposition, such an $\mathcal{X}$ is a separable Banach space without a Schauder basis.

Problem 4.59. Recall that the concepts of strong and uniform convergence coincide in a finite-dimensional space (Proposition 4.46). Consider the Banach space ℓ_+^p for any $p \geq 1$ (which has a Schauder basis — Problem 4.12). Exhibit a sequence of finite-rank (compact) operators on ℓ_+^p that converges strongly to a finite-rank (compact) operator but is not uniformly convergent.

Hint: Let P_k be the diagonal operator defined in Problem 4.52.

Problem 4.60. Let $\mathcal{M}$ be a subspace of an infinite-dimensional Banach space $\mathcal{X}$. An extension over $\mathcal{X}$ of a compact operator on $\mathcal{M}$ may not be compact.

Hint: Let $\mathcal{M}$ and $\mathcal{N}$ be Banach spaces over the same field. Suppose $\mathcal{N}$ is infinite dimensional. Set $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$ and consider the direct sum $T = K \oplus I$ in $\mathcal{B}[\mathcal{X}]$, where K is a compact operator in $\mathcal{B}_\infty[\mathcal{M}]$ and I is the identity operator in $\mathcal{B}[\mathcal{N}]$, as in Problem 4.16.

Problem 4.61. Let $\mathcal{X}$ and $\mathcal{Y}$ be nonzero normed spaces over the same field. Let $\mathcal{M}$ be a proper subspace of $\mathcal{X}$. Show that there exists $O \neq T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ such that $\mathcal{M} \subseteq \mathcal{N}(T)$ (i.e., such that $T(\mathcal{M}) = \{0\}$). (*Hint*: Corollary 4.63.)

Problem 4.62. Let $\mathcal{X}$ be a normed space. Since $\big|\|x\| - \|y\|\big| \leq \|x - y\|$ for every $x, y \in \mathcal{X}$, it follows that the norm on $\mathcal{X}$ is a real-valued contraction that takes each vector of $\mathcal{X}$ to its norm. Show that for each vector in $\mathcal{X}$ there exists a real-valued *linear* contraction on $\mathcal{X}$ that takes that vector to its norm.

Problem 4.63. Let A be an arbitrary subset of a normed space $\mathcal{X}$. The *annihilator* of A is the following subset of the dual space $\mathcal{X}^*$:

$$A^\perp = \big\{f \in \mathcal{X}^*\colon\ A \subseteq \mathcal{N}(f)\big\}.$$

(a) If $A \neq \varnothing$, then show that

$$A^\perp = \big\{f \in \mathcal{X}^*\colon\ f(A) = \{0\}\big\}.$$

(b) Show that $\varnothing^\perp = \{0\}^\perp = \mathcal{X}$, $\mathcal{X}^\perp = \{0\}$, and $B^\perp \subseteq A^\perp$ whenever $A \subseteq B$.

(c) Show that $A^\perp$ is a subspace of $\mathcal{X}^*$.

(d) Show that $A^- \subseteq \bigcap_{f \in A^\perp} \mathcal{N}(f)$.

Hint: If $f \in A^\perp$, then $A \subseteq \mathcal{N}(f)$. Thus conclude that $A^- \subseteq \mathcal{N}(f)$ for every $f \in A^\perp$ (Proposition 4.13).

Now let $\mathcal{M}$ be a linear manifold of $\mathcal{X}$ and prove the following assertions.

(e) $\mathcal{M}^- = \bigcap_{f \in \mathcal{M}^\perp} \mathcal{N}(f)$.

Hint: if $x_0 \in \mathcal{X} \backslash \mathcal{M}^-$, then there exists an $f \in \mathcal{M}^\perp$ such that $f(x_0) = 1$ (Corollary 4.63), and therefore $x_0 \notin \bigcap_{f \in \mathcal{M}^\perp} \mathcal{N}(f)$. Thus conclude that $\bigcap_{f \in \mathcal{M}^\perp} \mathcal{N}(f) \subseteq \mathcal{M}^-$. Next use item (d).

(f) $\mathcal{M}^- = \mathcal{X}$ if and only if $\mathcal{M}^\perp = \{0\}$.

Problem 4.64. Let $\{e_i\}_{i=1}^n$ be a Hamel basis for a finite-dimensional normed space $\mathcal{X}$. Verify the following propositions.

(a) For each $i = 1, \ldots, n$ there exists $f_i \in \mathcal{X}^*$ such that $f_i(e_j) = \delta_{ij}$ for every $j = 1, \ldots, n$.

Hint: Set $f_i(x) = \xi_i$ for every $x = \sum_{i=1}^n \xi_i e_i \in \mathcal{X}$.

(b) $\{f_i\}_{i=1}^n$ is a Hamel basis for $\mathcal{X}^*$.

Hint: If $f \in \mathcal{X}^*$, then $f = \sum_{i=1}^n f(e_i) f_i$.

Now conclude that $\dim \mathcal{X} = \dim \mathcal{X}^*$ whenever $\dim \mathcal{X} < \infty$.

Problem 4.65. Let $J\colon \mathcal{Y} \to \mathcal{X}$ be an isometric isomorphism of a normed space $\mathcal{Y}$ onto a normed space $\mathcal{X}$, and consider the mapping $J^*\colon \mathcal{X}^* \to \mathcal{Y}^{\mathbb{F}}$ defined by $J^* f = f \circ J$ for every $f \in \mathcal{X}^*$. Show that

(a) $J^*(\mathcal{X}^*) = \mathcal{Y}^*$ so that $J^*\colon \mathcal{X}^* \to \mathcal{Y}^*$ is surjective,

(b) $J^*\colon \mathcal{X}^* \to \mathcal{Y}^*$ is linear, and

(c) $\|J^* f\| = \|f\|$ for every $f \in \mathcal{X}^*$. (*Hint*: Problem 4.41(a).)

Conclude: If $\mathcal{X}$ and $\mathcal{Y}$ are isometrically isomorphic normed spaces, then their duals $\mathcal{X}^*$ and $\mathcal{Y}^*$ are isometric isomorphic too. That is,

$$\mathcal{X} \cong \mathcal{Y} \quad \text{implies} \quad \mathcal{X}^* \cong \mathcal{Y}^*.$$

Problem 4.66. Consider the normed space ℓ_+^∞ equipped with its usual sup-norm. Recall that $\ell_+^c \subseteq \ell_+^\infty$ (Problem 3.59). Let $S_- \in \mathcal{B}[\ell_+^\infty]$ be the backward unilateral shift on ℓ_+^∞ (Example 4.L and Problem 4.39). A bounded linear functional $f\colon \ell_+^\infty \to \mathbb{F}$ is a *Banach limit* if it satisfies the following conditions.

(i) $\|f\| = 1$,

(ii) $f(x) = f(S_- x)$ for every $x \in \ell_+^\infty$,

(iii) If $x = \{\xi_k\}$ lies in ℓ_+^c, then $f(x) = \lim_k \xi_k$,

(iv) If $x = \{\xi_k\}$ in ℓ_+^∞ is such that $\xi_k \geq 0$ for all k, then $f(x) \geq 0$.

Condition (iii) says that Banach limits extend to ℓ_+^∞ the limit function on ℓ_+^c (i.e., f is defined on ℓ_+^∞ and its restriction to ℓ_+^c, $f|_{\ell_+^c}$, assigns to each convergent sequence its own limit). The remaining conditions represent fundamental properties of a limit function. Condition (i) ensures that $\lim_k |\xi_k| \leq \sup_k |\xi_k|$, and condition (ii) that $\lim_k \xi_k = \lim_k \xi_{k+n}$ for every positive integer n, whenever $\{\xi_k\} \in \ell_+^c$. Condition (iv) says that f is order-preserving for real-valued sequences in ℓ_+^∞ (i.e., if $x = \{\xi_k\}$ and $y = \{\upsilon_k\}$ are real-valued sequences in ℓ_+^∞, then $f(x), f(y) \in \mathbb{R}$ — why? — and $f(x) \leq f(y)$ if $\xi_k \leq \upsilon_k$ for every k). The purpose of this problem is to show how the Hahn–Banach Theorem ensures the existence of Banach limits on ℓ_+^∞.

(a) Suppose $\mathbb{F} = \mathbb{R}$ (so that the sequences in ℓ_+^∞ are all real valued). Let e be the constant sequence in ℓ_+^∞ whose entries are all ones, $e = (1, 1, 1, \ldots)$, and set $\mathcal{M} = \mathcal{R}(I - S_-)$. Show that $d(e, \mathcal{M}) = 1$.

Hint: Verify that $d(e, \mathcal{M}) \leq 1$ (for $\|e\|_\infty = 1$ and $0 \in \mathcal{M}$). Now take an arbitrary $u = \{\upsilon_k\}$ in $\mathcal{M}$. If $\upsilon_{k_0} \leq 0$ for some integer k_0, then show that $1 \leq \|e - u\|_\infty$. But $u \in \mathcal{R}(I - S_-)$, and so $\upsilon_k = \xi_k - \xi_{k+1}$ for some $x = \{\xi_k\}$ in ℓ_+^∞. If $\upsilon_k \geq 0$ for all k, then $\{\xi_k\}$ is decreasing and bounded. Check that $\{\xi_k\}$ converges in $\mathbb{R}$ (Problem 3.10), show that $\upsilon_k \to 0$ (Problem 3.51), and conclude that $1 \leq \|e - u\|_\infty$ whenever $\upsilon_k \geq 0$ for all k. Hence $1 \leq \|e - u\|_\infty$ for every $u \in \mathcal{M}$ so that $d(e, \mathcal{M}) \geq 1$.

Therefore (it does not matter whether $\mathcal{M}$ is closed or not), $\mathcal{M}^-$ is a subspace of ℓ_+^∞ (Proposition 4.9(a)) and $d(e, \mathcal{M}^-) = 1$ (Problem 3.43(b)). Then, by Corollary 4.63, there exists a bounded linear functional $\varphi\colon \ell_+^\infty \to \mathbb{R}$ such that

$$\varphi(e) = 1, \quad \varphi(\mathcal{M}^-) = \{0\}, \quad \text{and} \quad \|\varphi\| = 1.$$

(b) Show that $\varphi(x) = \varphi(S_-^n x)$ for every $x \in \ell_+^\infty$ and all $n \geq 1$.

Hint: $\varphi((I - S_-)x) = 0$ because $\varphi(\mathcal{M}) = \{0\}$. This leads to $\varphi(x) = \varphi(S_- x)$ for every x in ℓ_+^∞. Conclude the proof by induction.

(c) Show that φ satisfies condition (iii).

Hint: Take an arbitrary $x = \{\xi_k\}$ in ℓ_+^c so that $\xi_k \to \xi$ in $\mathbb{R}$ for some $\xi \in \mathbb{R}$. Observe that $|\xi_{k+n} - \xi| \leq |\xi_{k+n} - \xi_n| + |\xi_n - \xi|$ for every pair of positive integers n, k. Now Use Problem 3.51(a) to show that $\|S_-^n x - \xi e\|_\infty \to 0$. That is, $S_-^n x \to \xi e$ in ℓ_+^∞. Next verify that $\varphi(x) = \xi\varphi(e)$.

(d) Show that φ satisfies condition (iv).

Hint: Take any $0 \neq x = \{\xi_k\}$ in ℓ_+^c such that $\xi_k \geq 0$ for all k, and set $x' = (\|x\|_\infty^{-1})x = \{\xi_k'\}$. Verify that $0 \leq \xi_k' \leq 1$ for all k, and so $\|e - x'\|_\infty \leq 1$. Finally, show that $1 - \varphi(x') = \varphi(e - x') \leq 1$, and conclude: $\varphi(x) \geq 0$.

Thus, in the real case, $\varphi : \ell_+^\infty \to \mathbb{R}$ is a Banach limit on ℓ_+^∞.

(e) Now suppose $\mathbb{F} = \mathbb{C}$ (so that complex-valued sequences are allowed in ℓ_+^∞). For each x in ℓ_+^∞ write $x = x_1 + ix_2$, where x_1 and x_2 are real-valued sequences in ℓ_+^∞, and set

$$f(x) \;=\; \varphi(x_1) + i\varphi(x_2).$$

Show that this defines a bounded linear functional $f : \ell_+^\infty \to \mathbb{C}$.

Hint: $\|f\| \leq 2$.

(f) Verify that f satisfies conditions (ii), (iii) and (iv).

(g) Prove that $\|f\| = 1$.

Hint: Let $\ell_+^\#$ be the set of all scalar-valued sequences that take on only a finite number of values (i.e., that have a finite range). Clearly, $\ell_+^\# \subset \ell_+^\infty$. If $y \in \ell_+^\#$ with $\|y\|_\infty \leq 1$, then there is a finite partition of $\mathbb{N}$, say $\{N_j\}_{j=1}^m$, and a finite set of scalars $\{\alpha_j\}_{j=1}^m$ with $|\alpha_j| \leq 1$ for all j such that $y = \sum_{j=1}^m \alpha_j \chi_{N_j}$. Here χ_{N_j} is the characteristic function of N_j which, in fact, is an element of $\ell_+^\#$ (i.e., $\chi_{N_j} = \{\chi_{N_j}(k)\}_{k\in\mathbb{N}}$, where $\chi_{N_j}(k) = 1$ if $k \in N_j$ and $\chi_{N_j}(k) = 0$ if $k \in \mathbb{N}\backslash N_j$). Verify: $f(y) = \sum_{j=1}^m \alpha_j f(\chi_{N_j}) = \sum_{j=1}^m \alpha_j \varphi(\chi_{N_j})$ and $\sum_{j=1}^m \varphi(\chi_{N_j}) = \varphi\big(\sum_{j=1}^m \chi_{N_j}\big) = \varphi(\chi_{\mathbb{N}}) = \varphi(e)$. Recall that φ satisfies condition (iv) and show that $|f(y)| \leq (\sup_j |\alpha_j|)\varphi(e)$.

Conclusion 1. If $y \in \ell_+^\#$ and $\|y\|_\infty \leq 1$, then $|f(y)| \leq 1$.

The closed unit ball B with center at the origin of $\mathbb{C}$ is compact. For each positive integer n, take a finite $\frac{1}{n}$-net for B, say $B_n \subset B$. If $x = \{\xi_k\} \in \ell_+^\infty$ is such that $\|x\|_\infty \leq 1$, then $\xi_k \in B$ for all k. Thus for each k there exists $\upsilon_n(k) \in B_n$ such that $|\upsilon_n(k) - \xi_k| < \frac{1}{n}$. This defines for each n a B_n-valued sequence $y_n = \{\upsilon_n(k)\}$ with $\|y_n - x\|_\infty < \frac{1}{n}$, which defines an $\ell_+^\#$-valued sequence $\{y_n\}$ with $\|y_n\|_\infty \leq 1$ for all n that converges in ℓ_+^∞ to x.

Conclusion 2. Every $x \in \ell_+^\infty$ with $\|x\|_\infty \le 1$ is the limit in ℓ_+^∞ of an $\ell_+^\#$-valued sequence $\{y_n\}$ with $\sup_n \|y_n\|_\infty \le 1$.

Recall that $f\colon \ell_+^\infty \to \mathbb{C}$ is continuous. Apply Conclusion 2 to show that $f(y_n) \to f(x)$, and so $|f(y_n)| \to |f(x)|$. Since $|f(y_n)| \le 1$ for every n (by Conclusion 1), it follows that $|f(x)| \le 1$ for every $x \in \ell_+^\infty$ with $\|x\|_\infty \le 1$. Then $\|f\| \le 1$. Verify that $\|f\| \ge 1$ (since $f(e) = \varphi(e)$ and $\|e\|_\infty = 1$).

Thus, in the complex case, $f\colon \ell_+^\infty \to \mathbb{C}$ is a Banach limit on ℓ_+^∞.

Problem 4.67. Let $\mathcal{X}$ be a normed space. An $\mathcal{X}$-valued sequence $\{x_n\}$ is said to be *weakly convergent* if there exists $x \in \mathcal{X}$ such that $\{f(x_n)\}$ converges in $\mathbb{F}$ to $f(x)$ for every $f \in \mathcal{X}^*$. In this case we say that $\{x_n\}$ *converges weakly to* $x \in \mathcal{X}$ (notation: $x_n \xrightarrow{w} x$) and x is said to be the *weak limit* of $\{x_n\}$. Prove the following assertions.

(a) The weak limit of a weakly convergent sequence is unique.

Hint: $f(x) = 0$ for all $f \in \mathcal{X}^*$ implies $x = 0$.

(b) $\{x_n\}$ converges weakly to x if and only if every subsequence of $\{x_n\}$ converges weakly to x.

Hint: Proposition 3.5.

(c) If $\{x_n\}$ converges in the norm topology to x, then it converges weakly to x (i.e., $x_n \to x \implies x_n \xrightarrow{w} x$).

Hint: $|f(x_n - x)| \le \|f\|\|x_n - x\|$.

(d) If $\{x_n\}$ converges weakly, then it is bounded in the norm topology (i.e., $x_n \xrightarrow{w} x \implies \sup_n \|x_n\| < \infty$).

Hint: For each $x \in \mathcal{X}$ there exists $\varphi_x \in \mathcal{X}^{**}$ such that $\varphi_x(f) = f(x)$ for every $f \in \mathcal{X}^*$ and $\|\varphi_x\| = \|x\|$. This is the natural embedding of $\mathcal{X}$ into $\mathcal{X}^{**}$ (Theorem 4.66). Verify that $\sup_n |f(x_n)| < \infty$ for every $f \in \mathcal{X}^*$ whenever $x_n \xrightarrow{w} x$, and show that $\sup_n |\varphi_{x_n}(f)| < \infty$ for every $f \in \mathcal{X}^*$. Now use the Banach–Steinhaus Theorem (recall: $\mathcal{X}^*$ is a Banach space).

Problem 4.68. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. A $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$-valued sequence $\{T_n\}$ *converges weakly* in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ if $\{T_n x\}$ converges weakly in $\mathcal{Y}$ for every $x \in \mathcal{X}$. In other words, $\{T_n\}$ *converges weakly* in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ if $\{f(T_n x)\}$ converges in $\mathbb{F}$ for every $f \in \mathcal{Y}^*$ and every $x \in \mathcal{X}$. Prove the following assertions.

(a) If $\{T_n\}$ converges weakly in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$, then there exists a unique $T \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ (called the *weak limit* of $\{T_n\}$) such that $T_n x \xrightarrow{w} T x$ in $\mathcal{Y}$ for every $x \in \mathcal{X}$.

Hint: That there exists such a unique mapping $T\colon \mathcal{X} \to \mathcal{Y}$ follows by Problem 4.67(a). This mapping is linear because every f in $\mathcal{Y}^*$ is linear.

Notation: $T_n \xrightarrow{w} T$ (or $T_n - T \xrightarrow{w} O$). If $\{T_n\}$ does not converge weakly to T, then we write $T_n \xrightarrow{w} \!\!\!\!\!/\;\; T$.

(b) If $\mathcal{X}$ is a Banach space and $T_n \xrightarrow{w} T$, then $\sup_n \|T_n\| < \infty$ and $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$.

Hint: Apply the Banach–Steinhaus Theorem and Problem 4.67(d) to prove (uniform) boundedness for $\{T_n\}$. Show that $|f(Tx)| \le \|f\|(\sup_n \|T_n\|)\|x\|$ for every $x \in \mathcal{X}$ and every $f \in \mathcal{Y}^*$. Thus conclude that, for every $x \in \mathcal{X}$, $\|Tx\| = \sup_{f \in \mathcal{Y}^*,\, \|f\|=1} |f(Tx)| \le \sup_n \|T_n\|\, \|x\|$.

(c) $T_n \xrightarrow{s} T$ implies $T_n \xrightarrow{w} T$.

Hint: $|f\big((T_n - T)x\big)| \le \|f\|\, \|(T_n - T)x\|$.

Take $T \in \mathcal{B}[\mathcal{X}]$ and consider the power sequence $\{T^n\}$ in the normed algebra $\mathcal{B}[\mathcal{X}]$. The operator T is *weakly stable* if $T^n \xrightarrow{w} O$.

(d) Strong stability implies weak stability,

$$T^n \xrightarrow{s} O \quad \Longrightarrow \quad T^n \xrightarrow{w} O,$$

which in turn implies power boundedness if $\mathcal{X}$ is a Banach space:

$$T^n \xrightarrow{w} O \quad \Longrightarrow \quad \sup_n \|T^n\| < \infty \ \text{ if } \ \mathcal{X} \ \text{ is Banach.}$$

Problem 4.69. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. Consider the setup of the previous problem and prove the following propositions.

(a) If $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$, then $Tx_n \xrightarrow{w} Tx$ in $\mathcal{Y}$ whenever $x_n \xrightarrow{w} x$ in $\mathcal{X}$. That is, a continuous linear transformation of $\mathcal{X}$ into $\mathcal{Y}$ takes weakly convergent sequences in $\mathcal{X}$ into weakly convergent sequences in $\mathcal{Y}$.

Hint: If $f \in \mathcal{Y}^*$, then $f \circ T \in \mathcal{X}^*$.

(b) If $T \in \mathcal{B}_\infty[\mathcal{X}, \mathcal{Y}]$, then $\|Tx_n - Tx\| \to 0$ whenever $x_n \xrightarrow{w} x$ in $\mathcal{X}$. That is, a compact linear transformation of $\mathcal{X}$ into $\mathcal{Y}$ takes weakly convergent sequences in $\mathcal{X}$ into convergent sequences in $\mathcal{Y}$.

Hint: Suppose $x_n \xrightarrow{w} x$ in $\mathcal{X}$ and take $T \in \mathcal{B}_\infty[\mathcal{X}, \mathcal{Y}]$. Use Theorem 4.49, part (a) of this problem, and Problem 4.67(d) to show that

$$(\mathrm{b}_1) \qquad Tx_n \xrightarrow{w} Tx \ \text{ in } \ \mathcal{Y} \quad \text{and} \quad \sup_n \|x_n\| < \infty.$$

Suppose $\{Tx_n\}$ does not converge (in the norm topology of $\mathcal{Y}$) to Tx. Use Proposition 3.5 to show that $\{Tx_n\}$ has a subsequence, say $\{Tx_{n_k}\}$, that does not converge to Tx. Conclude: There exists $\varepsilon_0 > 0$ and a positive integer k_{ε_0} such that

$$(\mathrm{b}_2) \qquad \|T(x_{n_k} - x)\| > \varepsilon_0 \quad \text{for every} \quad k \ge k_{\varepsilon_0}.$$

Verify from (b_1) that $\sup_k \|x_{n_k}\| < \infty$. Apply Theorem 4.52 to show that $\{Tx_{n_k}\}$ has a subsequence, say $\{Tx_{n_{k_j}}\}$, that converges in the norm topology of $\mathcal{Y}$. Now use the weak convergence in (b_1) and Problem 4.67(b) to show that $\{Tx_{n_{k_j}}\}$ in fact converges to Tx (i.e., $Tx_{n_{k_j}} \to Tx$ in $\mathcal{Y}$). Therefore, for each $\varepsilon > 0$ there exists a positive integer j_ε such that

(b_3) $\qquad \|T(x_{n_{k_j}} - x)\| < \varepsilon \quad$ for every $\quad j \geq j_\varepsilon$.

Finally, verify that (b_3) contradicts (b_2) and conclude that $\{Tx_n\}$ must converge in $\mathcal{Y}$ to Tx.

Problem 4.70. Let $\mathcal{X}$ be a normed space. An $\mathcal{X}^*$-valued sequence $\{f_n\}$ is weakly convergent if there exists $f \in \mathcal{X}^*$ such that $\{\varphi(f_n)\}$ converges in $\mathbb{F}$ to $\varphi(f)$ for every $\varphi \in \mathcal{X}^{**}$ (cf. Problem 4.67). In this case we write $f_n \xrightarrow{w} f$ in $\mathcal{X}^*$. An $\mathcal{X}^*$-valued sequence $\{f_n\}$ is *weakly* convergent* if there exists $f \in \mathcal{X}^*$ such that $\{f_n(x)\}$ converges in $\mathbb{F}$ to $f(x)$ for every $x \in \mathcal{X}$ (notation: $f_n \xrightarrow{w^*} f$). Thus *weak* convergence* in $\mathcal{X}^*$ means pointwise convergence of $\mathcal{B}[\mathcal{X}, \mathbb{F}]$-valued sequences to an element of $\mathcal{B}[\mathcal{X}, \mathbb{F}]$.

(a) Show that weak convergence in $\mathcal{X}^*$ implies weak* convergence in $\mathcal{X}^*$ (i.e., $f_n \xrightarrow{w} f \implies f_n \xrightarrow{w^*} f$).

Hint: According to the natural embedding of $\mathcal{X}$ into $\mathcal{X}^{**}$ (Theorem 4.66), for each $x \in \mathcal{X}$ there exists $\varphi_x \in \mathcal{X}^{**}$ such that $\varphi_x(f) = f(x)$ for every $f \in \mathcal{X}^*$. Verify that, for each $x \in \mathcal{X}$, $f_n(x) \to f(x)$ if $\varphi_x(f_n) \to \varphi_x(f)$.

(b) If $\mathcal{X}$ is reflexive, then the concepts of weak convergence in $\mathcal{X}^*$ and weak* convergence in $\mathcal{X}^*$ coincide. Prove.

Problem 4.71. Let K be a compact (thus totally bounded — see Corollary 3.81) subset of a normed space $\mathcal{X}$. Take an arbitrary $\varepsilon > 0$ and let A_ε be a finite ε-net for K (Definition 3.68). Take the closed ball $B_\varepsilon[a]$ of radius ε centered at each $a \in A_\varepsilon$, and consider the functional $\psi_a \colon K \to [0, \varepsilon]$ defined by

$$\psi_a(x) = \begin{cases} \varepsilon - \|x - a\|, & x \in B_\varepsilon[a], \\ 0, & x \notin B_\varepsilon[a]. \end{cases}$$

Define the function $\Phi_{A_\varepsilon} \colon K \to \mathcal{X}$ by the formula

$$\Phi_{A_\varepsilon}(x) = \frac{\sum_{a \in A_\varepsilon} a\, \psi_a(x)}{\sum_{a \in A_\varepsilon} \psi_a(x)} \quad \text{for every} \quad x \in K.$$

Prove that Φ_{A_ε} is continuous and $\|\Phi_{A_\varepsilon}(x) - x\| < \varepsilon$ for every $x \in K$. This is a technical result that will be needed in the next problem.

Hint: Verify that $\sum_{a \in A_\varepsilon} \psi_a(x) > 0$ for every $x \in K$ so that the function Φ_{A_ε} is well defined. Show that each ψ_a is continuous, and infer that Φ_{A_ε} is continuous as well. Take any $x \in K$. If $\psi_a(x) \neq 0$ for some $a \in A_\varepsilon$, then $\|x - a\| < \varepsilon$. Thus

$$\|\Phi_{A_\varepsilon}(x) - x\| \leq \frac{\sum_{a \in A_\varepsilon} \|a - x\|\, \psi_a(x)}{\sum_{a \in A_\varepsilon} \psi_a(x)} < \varepsilon.$$

Problem 4.72. An important classical result in topology reads as follows. Let $B_1[0]$ be the closed unit ball (radius 1 with center at the origin) in $\mathbb{R}^n$. Recall that all norms in $\mathbb{R}^n$ are equivalent (Theorem 4.27).

(i) If $F: B_1[0] \to B_1[0]$ is a continuous function, then it has a fixed point in $B_1[0]$ (i.e., there exists $x \in \mathbb{R}^n$ with $\|x\| \leq 1$ such that $F(x) = x$).

This is the *Browder Fixed Point Theorem*. A useful corollary extends it from closed unit balls (which are compact and convex in $\mathbb{R}^n$) to compact and convex sets in a finite-dimensional normed space as follows.

(ii) Let K be a nonempty compact and convex subset of a finite-dimensional normed space. If $F: K \to K$ is a continuous function, then it has a fixed point in K (i.e., there exists $x \in K$ such that $F(x) = x$).

We borrow the notion of a *compact mapping* from nonlinear functional analysis. Let D be a nonempty subset of a normed space $\mathcal{X}$. A mapping $F: D \to \mathcal{X}$ is *compact* if it is continuous and $F(B)^-$ is compact in $\mathcal{X}$ whenever B is a bounded subset of D. Recall that *a continuous image of any compact set is a compact set* (Theorem 3.64). Thus, if D is a compact subset of $\mathcal{X}$, then every continuous mapping $F: D \to \mathcal{X}$ is compact. However, we are now concerned with the case where D (the domain of F) is not compact but bounded. In this case, if F is continuous and $F(D)^-$ is compact, then F is a compact mapping (for $F(B)^- \subseteq F(D)^-$ if $B \subseteq D$). The next result is the *Schauder Fixed Point Theorem*. It is a generalization of (ii) to infinite-dimensional spaces. Prove it.

(iii) Let D be a nonempty closed bounded convex subset of a normed space $\mathcal{X}$, and let $F: D \to \mathcal{X}$ be a compact mapping. If D is F-invariant, then F has a fixed point (i.e., if $F(D) \subseteq D$, then $F(x) = x$ for some $x \in D$).

Hint: Set $K = F(D)^- \subseteq D$, which is compact. For each $n \geq 1$ let A_n be a finite $\frac{1}{n}$-net for K, and take $\Phi_{A_n}: K \to \mathcal{X}$ as in Problem 4.71. Verify by the definition of Φ_{A_n} that $\Phi_{A_n}(K) \subseteq \text{co}(K) \subseteq D$ since D is convex (Problem 2.2). So infer that D is $(\Phi_{A_n} \circ F)$-invariant. Set $F_n = (\Phi_{A_n} \circ F): D \to D$. Use Problem 4.71 to conclude that, for each $n \geq 1$ and every $x \in D$,

$$\|F_n(x) - F(x)\| < \tfrac{1}{n}.$$

Set $D_n = D \cap \text{span}\,(A_n)$, each D_n being a nonempty compact and convex subset of the finite-dimensional space $\text{span}\,(A_n)$. Also, D_n is F_n-invariant and $F_n' = F_n|_{D_n}: D_n \to D_n$ is continuous. Use (ii) to conclude that each F_n has a fixed point in D_n (say, $x_n \in D_n$ such that $F_n(x_n) = x_n$). Take the sequence $\{F(x_n)\}$ of points in the sequentially compact set K so that there is a point $x_0 \in K \subseteq D$ and a subsequence $\{F(x_{n_k})\}$ of $\{F(x_n)\}$ such that $F(x_{n_k}) \to x_0$ (Definition 3.76 and Theorem 3.80). Since $F_{n_k}(x_{n_k}) = x_{n_k}$ and $\|F_{n_k}(x_{n_k}) - F(x_{n_k})\| < \frac{1}{n_k}$, it follows that

$$\|x_{n_k} - x_0\| \leq \|F_{n_k}(x_{n_k}) - F(x_{n_k})\| + \|F(x_{n_k}) - x_0\| \to 0.$$

Since F is continuous, Corollary 3.8 ensures that $F(x_0) = \lim F(x_{n_k}) = x_0$.

5

Hilbert Spaces

What is missing? The algebraic structure of a normed space allowed us to operate with vectors (addition and scalar multiplication), and its topological structure (the one endowed by the norm) gave us a notion of closeness (by means of the metric generated by the norm), which interacts harmoniously with the algebraic operations. In particular, it provided the notion of length of a vector. So what is missing if algebra and topology have already been properly laid on the same underlying set? A full geometric structure is still missing. Algebra and topology are not enough to extend to abstract spaces the geometric concept of relative direction (or angle) between vectors that is familiar in Euclidean geometry. The keyword here is *orthogonality*, a concept that emerges when we equip a linear space with an *inner product*. This supplies a tremendously rich structure that leads to remarkable simplifications.

5.1 Inner Product Spaces

We assume throughout this chapter (as we did in Chapter 4) that $\mathbb{F}$ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, both equipped with their usual topologies induced by their usual metrics. Recall that an upper bar stands for complex conjugate in $\mathbb{C}$. That is, for each complex number $\lambda = \alpha + i\beta$ in standard form the real numbers $\alpha = \mathrm{Re}\,\lambda$ and $\beta = \mathrm{Im}\,\lambda$ are the *real* and *imaginary parts* of λ, respectively, and the complex number $\overline{\lambda} = \alpha - i\beta$ is the *conjugate* of λ. The following are basic properties of conjugates: for every λ, μ in $\mathbb{C}$, $\overline{(\overline{\mu})} = \mu$, $\overline{(\lambda + \mu)} = \overline{\lambda} + \overline{\mu}$, $\overline{(\lambda\mu)} = \overline{\lambda}\,\overline{\mu}$, $\lambda + \overline{\lambda} = 2\mathrm{Re}\,\lambda$, $\lambda - \overline{\lambda} = 2i\,\mathrm{Im}\,\lambda$, $\overline{\lambda}\lambda = |\lambda|^2 = (\mathrm{Re}\,\lambda)^2 + (\mathrm{Im}\,\lambda)^2$, and $\lambda = \overline{\lambda}$ if and only if $\lambda \in \mathbb{R}$.

Let $\mathcal{X}$ be a linear space over $\mathbb{F}$. Consider a functional $\sigma\colon \mathcal{X} \times \mathcal{X} \to \mathbb{F}$ defined on the Cartesian product of the linear space $\mathcal{X}$ with itself. If $\sigma(\,\cdot\,, v)\colon \mathcal{X} \to \mathbb{F}$ and $\sigma(u, \,\cdot\,)\colon \mathcal{X} \to \mathbb{F}$ are linear functionals on $\mathcal{X}$ for every $u, v \in \mathcal{X}$ (i.e., if σ is linear in both arguments), then σ is called a *bilinear form* (or a *bilinear functional*) on $\mathcal{X}$. If $\sigma(\,\cdot\,, v)\colon \mathcal{X} \to \mathbb{F}$ is a linear functional on $\mathcal{X}$ for each $v \in \mathcal{X}$, and if $\overline{\sigma(u, \,\cdot\,)}\colon \mathcal{X} \to \mathbb{F}$ is a linear functional on $\mathcal{X}$ for each $u \in \mathcal{X}$, then σ is said to be a *sesquilinear form* (or a *sesquilinear functional*) on $\mathcal{X}$. Equivalently, σ is a sesquilinear form on $\mathcal{X}$ if, for every $u, v, x, y \in \mathcal{X}$ and every $\alpha, \beta \in \mathbb{F}$,

C.S. Kubrusly, *The Elements of Operator Theory*, DOI 10.1007/978-0-8176-4998-2_5,

© Springer Science+Business Media, LLC 2011

$$\begin{aligned}\sigma(\alpha u + \beta x, v) &= \alpha\sigma(u, v) + \beta\sigma(x, v),\\ \sigma(u, \alpha v + \beta y) &= \overline{\alpha}\sigma(u, v) + \overline{\beta}\sigma(u, y).\end{aligned}$$

In other words, a sesquilinear form is *additive* in both arguments, *homogeneous* in the first argument, and *conjugate homogeneous* in the second argument ("sesqui" means "one and a half"). A functional $\sigma\colon \mathcal{X}\times\mathcal{X} \to \mathbb{F}$ is *symmetric* if $\sigma(x, y) = \sigma(y, x)$, and *Hermitian symmetric* if

$$\sigma(x, y) = \overline{\sigma(y, x)},$$

for every $x, y \in \mathcal{X}$. If $\mathbb{F} = \mathbb{R}$, then it is clear that σ is symmetric if and only if it is Hermitian symmetric, and the notions of bilinear and sesquilinear forms coincide in the real case. It is also readily verified that a functional $\sigma\colon \mathcal{X}\times\mathcal{X} \to \mathbb{F}$ that is linear in the first argument (i.e., such that $\sigma(\,\cdot\,, v)\colon \mathcal{X} \to \mathbb{F}$ is a linear functional for every $v \in \mathcal{X}$) and Hermitian symmetric is a sesquilinear form. Thus a Hermitian symmetric sesquilinear form is precisely a Hermitian symmetric functional on $\mathcal{X}\times\mathcal{X}$ that is linear in the first argument. If σ is a sesquilinear form on $\mathcal{X}$, then the functional $\phi\colon \mathcal{X} \to \mathbb{F}$ defined by $\phi(x) = \sigma(x, x)$ for every $x \in \mathcal{X}$ is called a *quadratic form* on $\mathcal{X}$ induced (or generated) by σ. If σ is Hermitian symmetric, then the induced quadratic form ϕ is real valued (i.e., $\sigma(x, x) \in \mathbb{R}$ for every $x \in \mathcal{X}$ if σ is Hermitian symmetric). Note that if σ is a sesquilinear form, then $\sigma(0, v) = \sigma(u, 0) = 0$ for every $u, v \in \mathcal{X}$, and so $\sigma(0, 0) = 0$. A quadratic form induced by a Hermitian symmetric sesquilinear form σ is *nonnegative* or *positive* if

$$\begin{aligned}&\sigma(x, x) \geq 0 \quad \text{for every } x \in \mathcal{X}, \qquad \text{or}\\ &\sigma(x, x) > 0 \quad \text{for every nonzero } x \in \mathcal{X},\end{aligned}$$

respectively. Therefore, a quadratic form ϕ is positive if it is nonnegative and $\sigma(x, x) = 0$ only if $x = 0$. An *inner product* (or a *scalar product*) on a linear space $\mathcal{X}$ is a Hermitian symmetric sesquilinear form that induces a positive quadratic form. In other words, an inner product on a linear space $\mathcal{X}$ is a functional on the Cartesian product $\mathcal{X}\times\mathcal{X}$ that satisfies the following properties, called the *inner product axioms*.

Definition 5.1. Let $\mathcal{X}$ be a linear space over $\mathbb{F}$. A functional

$$\langle\ ;\ \rangle\colon \mathcal{X}\times\mathcal{X} \to \mathbb{F}$$

is an *inner product* on $\mathcal{X}$ if the following conditions are satisfied for all vectors x, y, and z in $\mathcal{X}$ and all scalars α in $\mathbb{F}$.

(i)	$\langle x + y\,;z\rangle = \langle x\,;z\rangle + \langle y\,;z\rangle$	(*additivity*),
(ii)	$\langle \alpha x\,;y\rangle = \alpha\langle x\,;y\rangle$	(*homogeneity*),
(iii)	$\langle x\,;y\rangle = \overline{\langle y\,;x\rangle}$	(*Hermitian symmetry*),
(iv)	$\langle x\,;x\rangle \geq 0$	(*nonnegativeness*),
(v)	$\langle x\,;x\rangle = 0$ only if $x = 0$	(*positiveness*).

A linear space $\mathcal{X}$ equipped with an inner product on it is an *inner product space* (or a *pre-Hilbert space*). If $\mathcal{X}$ is a real or complex linear space (so that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) equipped with an inner product on it, then it is referred to as a *real* or *complex inner product space*, respectively.

Observe that $\langle\ ;\ \rangle\colon \mathcal{X}\times\mathcal{X} \to \mathbb{F}$ is actually a sesquilinear form. In fact,

$$\text{(i}'\text{)} \qquad \langle x+y\,;z\rangle \,=\, \langle x\,;z\rangle + \langle y\,;z\rangle, \qquad \langle x\,;w+z\rangle \,=\, \langle x\,;w\rangle + \langle x\,;z\rangle,$$

$$\text{(ii}'\text{)} \qquad \langle \alpha x\,;y\rangle \,=\, \alpha\langle x\,;y\rangle, \quad \text{and} \quad \langle x\,;\alpha y\rangle \,=\, \overline{\alpha}\langle x\,;y\rangle,$$

for all vectors x, y, w, z in $\mathcal{X}$ and all scalars α in $\mathbb{F}$. Properties (i$'$) and (ii$'$) define a sesquilinear form. For an inner product, (i$'$) and (ii$'$) are obtained by axioms (i), (ii), and (iii), and are enough by themselves to ensure that

$$\Big\langle \sum_{i=1}^{n} \alpha_i x_i\,;\beta_0 y_0\Big\rangle \,=\, \sum_{i=1}^{n} \alpha_i\overline{\beta_0}\langle x_i\,;y_0\rangle, \quad \Big\langle \alpha_0 x_0\,;\sum_{i=1}^{n} \beta_i y_i\Big\rangle \,=\, \sum_{i=1}^{n} \alpha_0\overline{\beta_i}\langle x_0\,;y_i\rangle,$$

and so

$$\Big\langle \sum_{i=0}^{n} \alpha_i x_i\,;\sum_{j=0}^{n} \beta_j y_j\Big\rangle \,=\, \sum_{i,j=0}^{n} \alpha_i\overline{\beta}_j\langle x_i\,;y_j\rangle,$$

for every $\alpha_i, \beta_i \in \mathbb{F}$ and every $x_i, y_i \in \mathcal{X}$, with $i = 0, \ldots, n$, for each integer $n \geq 1$. Let $\|\ \|^2\colon \mathcal{X} \to \mathbb{F}$ denote the quadratic form induced by the sesquilinear form $\langle\ ;\ \rangle$ on $\mathcal{X}$ (the notation $\|\ \|^2$ for the quadratic form induced by an inner product is certainly not a mere coincidence as we shall see shortly); that is,

$$\|x\|^2 \,=\, \langle x\,;x\rangle$$

for every $x \in \mathcal{X}$. The preceding identities ensure that, for every $x, y \in \mathcal{X}$,

$$\|x+y\|^2 \,=\, \|x\|^2 + \langle x\,;y\rangle + \langle y\,;x\rangle + \|y\|^2$$

and

$$\langle x\,;0\rangle \,=\, \langle 0\,;x\rangle \,=\, \langle 0\,;0\rangle \,=\, 0.$$

The above results hold for every sesquilinear form. Now, since $\langle\ ;\ \rangle$ is also Hermitian symmetric (i.e., since $\langle\ ;\ \rangle$ also satisfies axiom (iii)), it follows that $\langle x\,;y\rangle + \langle y\,;x\rangle = \langle x\,;y\rangle + \overline{\langle x\,;y\rangle} = 2\,\mathrm{Re}\,\langle x\,;y\rangle$ for every $x, y \in \mathcal{X}$, and hence

$$\|x+y\|^2 \,=\, \|x\|^2 + 2\,\mathrm{Re}\,\langle x\,;y\rangle + \|y\|^2$$

by axioms (i) and (iii). Moreover, by using axioms (ii) and (v) we get

$$\langle x\,;y\rangle = 0 \ \text{ for all } \ y \in \mathcal{X} \quad \text{ if and only if } \quad x = 0.$$

The next result is of fundamental importance. It is referred to as the *Schwarz* (or *Cauchy–Schwarz*, or even *Cauchy–Bunyakovski–Schwarz*) *inequality*.

Lemma 5.2. *Let $\langle\ ;\ \rangle\colon \mathcal{X}\times\mathcal{X} \to \mathbb{F}$ be an inner product on a linear space $\mathcal{X}$. Set $\|x\| = \langle x\,;x\rangle^{\frac{1}{2}}$ for each $x \in \mathcal{X}$. If $x, y \in \mathcal{X}$, then*

$$|\langle x\,;y\rangle| \le \|x\|\,\|y\|.$$

Proof. Take an arbitrary pair of vectors x and y in $\mathcal{X}$, and consider just the first four axioms of Definition 5.1, viz., axioms (i), (ii), (iii), and (iv). Thus

$$0 \le \langle x-\alpha y\,;x-\alpha y\rangle = \langle x\,;x\rangle - \overline{\alpha}\langle x\,;y\rangle - \alpha\overline{\langle x\,;y\rangle} + |\alpha|^2\langle y\,;y\rangle$$

for every $\alpha \in \mathbb{F}$. Note that $\langle z\,;z\rangle \ge 0$ by axiom (iv), and so it has a square root $\|z\| = \langle z\,;z\rangle^{\frac{1}{2}}$, for every $z \in \mathcal{X}$. Now set $\alpha = \frac{\langle x\,;y\rangle}{\beta}$ for any $\beta > 0$ so that

$$0 \le \|x\|^2 - \tfrac{1}{\beta}\Big(2 - \tfrac{\|y\|^2}{\beta}\Big)|\langle x\,;y\rangle|^2.$$

If $\|y\| \ne 0$, then set $\beta = \|y\|^2$ to get the Schwarz inequality. If $\|y\| = 0$, then $0 \le 2|\langle x\,;y\rangle|^2 \le \beta\|x\|^2$ for all $\beta > 0$, and hence $|\langle x\,;y\rangle| = 0$ (which trivially satisfies the Schwarz inequality). □

Proposition 5.3. *If* $\langle\ ;\ \rangle\colon \mathcal{X}\times\mathcal{X} \to \mathbb{F}$ *is an inner product on a linear space* $\mathcal{X}$*, then the function* $\|\ \|\colon \mathcal{X} \to \mathbb{R}$*, defined by*

$$\|x\| = \langle x\,;x\rangle^{\frac{1}{2}}$$

for each $x \in \mathcal{X}$*, is a norm on* $\mathcal{X}$*.*

Proof. Axioms (ii), (iii), (iv), and (v) in Definition 5.1 imply the norm axioms (i), (ii), and (iii) of Definition 4.1. The triangle inequality (axiom (iv) of Definition 4.1) is a consequence of the Schwarz inequality:

$$0 \le \|x+y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\,\langle x\,;y\rangle + \|y\|^2 \le \big(\|x\| + \|y\|\big)^2$$

for every x and y in $\mathcal{X}$ (reason: $\mathrm{Re}\,\langle x\,;y\rangle \le |\langle x\,;y\rangle| \le \|x\|\,\|y\|$). □

A word on notation and terminology. An inner product space in fact is an ordered pair $(\mathcal{X}, \langle\ ;\ \rangle)$, where $\mathcal{X}$ is a linear space and $\langle\ ;\ \rangle$ is an inner product on $\mathcal{X}$. We shall often refer to an inner product space by simply saying that "$\mathcal{X}$ is an inner product space" without explicitly mentioning the inner product $\langle\ ;\ \rangle$ that equips the linear space $\mathcal{X}$. However, there may be occasions when the role played by different inner products should be emphasized and, in these cases, we shall insert a subscript on the inner products (e.g., $(\mathcal{X}, \langle\ ;\ \rangle_X)$ and $(\mathcal{Y}, \langle\ ;\ \rangle_Y)$). If a linear space $\mathcal{X}$ can be equipped with more than one inner product, say $\langle\ ;\ \rangle_1$ and $\langle\ ;\ \rangle_2$, then $(\mathcal{X}, \langle\ ;\ \rangle_1)$ and $(\mathcal{X}, \langle\ ;\ \rangle_2)$ will represent *different* inner product spaces with the same linear space $\mathcal{X}$. The norm $\|\ \|$ of Proposition 5.3 is the *norm induced* (or defined, or generated) by the inner product $\langle\ ;\ \rangle$, so that every inner product space is a special kind of normed space (and hence a very special kind of linear metric space). Whenever we refer to the topological structure of an inner product space $(\mathcal{X}, \langle\ ;\ \rangle)$, it will always be understood that such a topology on $\mathcal{X}$ is that defined by the metric d that is generated by the norm $\|\ \|$, which in turn is the one induced by the inner product $\langle\ ;\ \rangle$. That is,

$$d(x,y) \;=\; \|x-y\| \;=\; \langle x-y\,;x-y\rangle^{\frac{1}{2}}$$

for every $x, y \in \mathcal{X}$ (cf. Propositions 4.2 and 5.3). This is the *norm topology* on $\mathcal{X}$ *induced by the inner product*. Since every inner product on a linear space induces a norm, it follows that every inner product space is a normed space (equipped with the induced norm). However, an arbitrary norm on a linear space may not be induced by any inner product on it (so that an arbitrary normed space may not be an inner product space). The next result leads to a necessary and sufficient condition that a norm be induced by an inner product.

Proposition 5.4. *Let* $\langle\ ;\ \rangle$ *be an inner product on a linear space* $\mathcal{X}$ *and let* $\|\ \|$ *be the induced norm on* $\mathcal{X}$*. Then*

$$\|x+y\|^2 + \|x-y\|^2 \;=\; 2\big(\|x\|^2 + \|y\|^2\big)$$

for every $x, y \in \mathcal{X}$*. This is called the* parallelogram law*. If* $(\mathcal{X}, \langle\ ;\ \rangle)$ *is a complex inner product space, then*

$$\langle x\,;y\rangle \;=\; \tfrac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\big)$$

for every $x, y \in \mathcal{X}$*. If* $(\mathcal{X}, \langle\ ;\ \rangle)$ *is a real inner product space, then*

$$\langle x\,;y\rangle \;=\; \tfrac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big)$$

for every $x, y \in \mathcal{X}$*. The above two expressions are referred to as the* complex and real polarization identities*, respectively.*

Proof. Axioms (i), (ii), and (iii) in Definition 5.1 lead to properties (i′) and (ii′), which in turn, by setting $\|x\|^2 = \langle x\,;x\rangle$ for every $x \in \mathcal{X}$, ensure that

$$\|x+\alpha y\|^2 \;=\; \|x\|^2 + \overline{\alpha}\langle x\,;y\rangle + \alpha\langle y\,;x\rangle + |\alpha|^2\|y\|^2$$

for every $x, y \in \mathcal{X}$ and every $\alpha \in \mathbb{F}$. For the parallelogram law, set $\alpha = 1$ and $\alpha = -1$. For the *complex* polarization identity, also set $\alpha = i$ and $\alpha = -i$. For the *real* polarization identity, set $\alpha = 1$, $\alpha = -1$, and use axiom (iii). □

Remark: The parallelogram law and the complex polarization identity hold for every sesquilinear form.

Theorem 5.5. (von Neumann). *Let* $\mathcal{X}$ *be a linear space. A norm on* $\mathcal{X}$ *is induced by an inner product on* $\mathcal{X}$ *if and only if it satisfies the parallelogram law. Moreover, if a norm on* $\mathcal{X}$ *satisfies the parallelogram law, then the unique inner product that induces it is given by the polarization identity.*

Proof. Proposition 5.4 ensures that if a norm on $\mathcal{X}$ is induced by an inner product, then it satisfies the parallelogram law, and the inner product on $\mathcal{X}$ can be written in terms of this norm according to the polarization identity. Conversely, suppose a norm $\|\ \|$ on $\mathcal{X}$ satisfies the parallelogram law and consider the mapping $\langle\ ;\ \rangle\colon \mathcal{X}\times\mathcal{X} \to \mathbb{F}$ defined by the polarization identity. Take x, y, and z arbitrary in $\mathcal{X}$. Note that

$$x+z=\Big(\frac{x+y}{2}+z\Big)+\frac{x-y}{2}\quad\text{ and }\quad y+z=\Big(\frac{x+y}{2}+z\Big)-\frac{x-y}{2}.$$

Thus, by the parallelogram law,

$$\|x+z\|^2+\|y+z\|^2\;=\;2\Big(\big\|\tfrac{x+y}{2}+z\big\|^2+\big\|\tfrac{x-y}{2}\big\|^2\Big).$$

Suppose $\mathbb{F}=\mathbb{R}$ so that $\langle\ ;\ \rangle\colon\mathcal{X}\times\mathcal{X}\to\mathbb{R}$ is the mapping defined by the real polarization identity (on the real normed space $\mathcal{X}$). Hence

$$\begin{aligned}\langle x\,;z\rangle+\langle y\,;z\rangle\;&=\;\tfrac14\big(\|x+z\|^2-\|x-z\|^2+\|y+z\|^2-\|y-z\|^2\big)\\&=\;\tfrac14\big[\big(\|x+z\|^2+\|y+z\|^2\big)-\big(\|x-z\|^2+\|y-z\|^2\big)\big]\\&=\;\tfrac12\big[\big(\big\|\tfrac{x+y}{2}+z\big\|^2+\big\|\tfrac{x-y}{2}\big\|^2\big)-\big(\big\|\tfrac{x+y}{2}-z\big\|^2+\big\|\tfrac{x-y}{2}\big\|^2\big)\big]\\&=\;\tfrac12\big(\big\|\tfrac{x+y}{2}+z\big\|^2-\big\|\tfrac{x+y}{2}-z\big\|^2\big)\;=\;2\big\langle\tfrac{x+y}{2}\,;z\big\rangle.\end{aligned}$$

The above identity holds for arbitrary $x,y,z\in\mathcal{X}$, and so it holds for $y=0$. Moreover, the polarization identity ensures that $\langle 0\,;z\rangle=0$ for every $z\in\mathcal{X}$. Thus, by setting $y=0$ above, we get $\langle x\,;z\rangle=2\langle\frac{x}{2}\,;z\rangle$ for every $x,z\in\mathcal{X}$. Then

$$\text{(i)}\qquad\qquad\langle x\,;z\rangle+\langle y\,;z\rangle\;=\;\langle x+y\,;z\rangle$$

for arbitrary x, y, and z in $\mathcal{X}$. It is readily verified (using exactly the same argument) that such an identity still holds if $\mathbb{F}=\mathbb{C}$, where the mapping $\langle\ ;\ \rangle\colon\mathcal{X}\times\mathcal{X}\to\mathbb{C}$ now satisfies the complex polarization identity (on the complex normed space $\mathcal{X}$). This is axiom (i) of Definition 5.1 (additivity). To verify axiom (ii) of Definition 5.1 (homogeneity in the first argument), proceed as follows. Take x and y arbitrary in $\mathcal{X}$. The polarization identity ensures that

$$\langle -x\,;y\rangle\;=\;-\langle x\,;y\rangle.$$

Since (i) holds true, it follows by a trivial induction that

$$\langle nx\,;y\rangle\;=\;n\langle x\,;y\rangle,$$

and hence $\langle x\,;y\rangle=\langle n\frac{x}{n}\,;y\rangle=n\langle\frac{x}{n}\,;y\rangle$ so that

$$\langle\tfrac{x}{n}\,;y\rangle\;=\;\tfrac1n\langle x\,;y\rangle,$$

for every positive integer n. The above three expressions imply that

$$\langle qx\,;y\rangle\;=q\langle x\,;y\rangle$$

for every rational number q (since $\langle 0\,;y\rangle=0$ by the polarization identity). Take an arbitrary $\alpha\in\mathbb{R}$ and recall that $\mathbb{Q}$ is dense in $\mathbb{R}$. Thus there exists a rational-valued sequence $\{q_n\}$ that converges in $\mathbb{R}$ to α. Moreover, according to (i) and recalling that $-\langle\alpha x\,;y\rangle=\langle-\alpha x\,;y\rangle$,

$$|\langle q_nx\,;y\rangle-\langle\alpha x\,;y\rangle|\;=\;|\langle(q_n-\alpha)x\,;y\rangle|.$$

The polarization identity ensures that $|\langle \alpha_n x\,;y\rangle| \to 0$ whenever $\alpha_n \to 0$ in $\mathbb{R}$ (because the norm is continuous). Hence $|\langle (q_n - \alpha)x\,;y\rangle| \to 0$, and therefore $|\langle q_n x\,;y\rangle - \langle \alpha x\,;y\rangle| \to 0$, which means $\langle q_n x\,;y\rangle \to \langle \alpha x\,;y\rangle$. This implies that $\langle \alpha x\,;y\rangle = \lim_n \langle q_n x\,;y\rangle = \lim_n q_n\langle x\,;y\rangle = \alpha\langle x\,;y\rangle$. Outcome:

$$\text{(ii(a))} \qquad \langle \alpha x\,;y\rangle \;=\; \alpha\langle x\,;y\rangle$$

for every $\alpha \in \mathbb{R}$. If $\mathbb{F} = \mathbb{C}$, then the complex polarization identity (on the complex space $\mathcal{X}$) ensures that

$$\langle ix\,;y\rangle \;=\; i\langle x\,;y\rangle.$$

Take an arbitrary $\lambda = \alpha + i\beta$ in $\mathbb{C}$ and observe by (i) and (ii(a)) that $\langle \lambda x\,;y\rangle = \langle (\alpha + i\beta)x\,;y\rangle = \langle \alpha x\,;y\rangle + \langle i\beta x\,;y\rangle = (\alpha + i\beta)\langle x\,;y\rangle = \lambda\langle x\,;y\rangle$. Conclusion:

$$\text{(ii(b))} \qquad \langle \lambda x\,;y\rangle \;=\; \lambda\langle x\,;y\rangle$$

for every $\lambda \in \mathbb{C}$. Axioms (iii), (iv), and (v) of Definition 5.1 (Hermitian symmetry and positiveness) emerge as immediate consequences of the polarization identity. Thus the mapping $\langle\ ;\ \rangle\colon \mathcal{X}\times\mathcal{X} \to \mathbb{F}$ defined by the polarization identity is, in fact, an inner product on $\mathcal{X}$. Moreover, this inner product induces the norm $\|\ \|$; that is, $\langle x\,;x\rangle = \|x\|^2$ for every $x \in \mathcal{X}$ (polarization identity again). Finally, if $\langle\ ;\ \rangle_0\colon \mathcal{X}\times\mathcal{X} \to \mathbb{F}$ is an inner product on $\mathcal{X}$ that induces the same norm $\|\ \|$ on $\mathcal{X}$, then it must coincide with $\langle\ ;\ \rangle$. That is, $\langle x\,;y\rangle_0 = \langle x\,;y\rangle$ for every $x, y \in \mathcal{X}$ (polarization identity once again). □

A *Hilbert space* is a complete inner product space. That is, a Hilbert space is an inner product space that is complete as a metric space with respect to the metric generated by the norm induced by the inner product. A *real* or *complex Hilbert space* is a complete real or complex inner product space. In fact, every Hilbert space is a special kind of Banach space: a Hilbert space is a Banach space whose norm is induced by an inner product. By Theorem 5.5, *a Hilbert space is a Banach space whose norm satisfies the parallelogram law*.

5.2 Examples

Theorem 5.5 may suggest that just a few of the classical examples of Section 4.2 survive as inner product spaces. This indeed is the case.

Example 5.A. Consider the linear space $\mathbb{F}^n$ over $\mathbb{F}$ (with either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and set

$$\langle x\,;y\rangle \;=\; \sum_{i=1}^{n} \xi_i\overline{\upsilon_i}$$

for every $x = (\xi_1, \dots, \xi_n)$ and $y = (\upsilon_1, \dots, \upsilon_n)$ in $\mathbb{F}^n$. It is readily verified that this defines an inner product on $\mathbb{F}^n$ (check the axioms in Definition 5.1), which

induces the norm $\|\ \|_2$ on $\mathbb{F}^n$. In particular, it induces the *Euclidean norm* on $\mathbb{R}^n$ so that $(\mathbb{R}^n, \langle\ ;\ \rangle)$ is the n-dimensional *Euclidean space* (see Example 4.A). Since $(\mathbb{F}^n, \|\ \|_2)$ is a Banach space, it follows that

$$(\mathbb{F}^n, \langle\ ;\ \rangle) \text{ is a Hilbert space.}$$

Now consider the norms $\|\ \|_p$ (for $p \geq 1$) and $\|\ \|_\infty$ on $\mathbb{F}^n$ defined in Example 4.A. If $n > 1$, then all of them, except the norm $\|\ \|_2$, are not induced by any inner product on $\mathbb{F}^n$. Indeed, set $x = (1, 0, \ldots, 0)$ and $y = (0, 1, 0, \ldots, 0)$ in $\mathbb{F}^n$ and verify that the parallelogram law fails for every norm $\|\ \|_p$ with $p \neq 2$, as it also fails for the sup-norm $\|\ \|_\infty$. Therefore, if $n > 1$, then $(\mathbb{F}^n, \|\ \|_2)$ is the only Hilbert space among the Banach spaces of Example 4.A.

Example 5.B. Consider the Banach spaces $(\ell^p_+, \|\ \|_p)$ for each $p \geq 1$ and $(\ell^\infty_+, \|\ \|_\infty)$ of Example 4.B. It is easy to show that, except for $(\ell^2_+, \|\ \|_2)$, these are not Hilbert spaces: the norms $\|\ \|_p$ for every $p \neq 2$ and $\|\ \|_\infty$ do not pass the parallelogram law test of Theorem 5.5, and hence are not induced by any possible inner product on ℓ^p_+ $(p \neq 2)$ or on ℓ^∞_+ (e.g., take $x = e_1 = (1, 0, 0, 0, \ldots)$ and $y = e_2 = (0, 1, 0, 0, 0, \ldots)$ in $\ell^p_+ \cap \ell^\infty_+$). On the other hand, the function $\langle\ ;\ \rangle : \ell^2_+ \times \ell^2_+ \to \mathbb{F}$ given by

$$\langle x\,; y\rangle \;= \sum_{k=1}^{\infty} \xi_k \overline{\upsilon}_k$$

for every $x = \{\xi_k\}_{k\in\mathbb{N}}$ and $y = \{\upsilon_k\}_{k\in\mathbb{N}}$ in ℓ^2_+ is well defined (i.e., the above infinite series converges in $\mathbb{F}$ for every $x, y \in \ell^2_+$ by the Hölder inequality for $p = q = 2$ and Proposition 4.4). Moreover, it actually is an inner product on ℓ^2_+ (i.e., it satisfies the axioms of Definition 5.1), which induces the norm $\|\ \|_2$ on ℓ^2_+. Thus, as $(\ell^2_+, \|\ \|_2)$ is a Banach space,

$$(\ell^2_+, \langle\ ;\ \rangle) \text{ is a Hilbert space.}$$

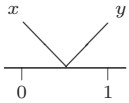
Similarly, the Banach spaces $(\ell^p, \|\ \|_p)$ for any $1 \leq p \neq 2$ and $(\ell^\infty, \|\ \|_\infty)$ are not Hilbert spaces. However, the function $\langle\ ;\ \rangle : \ell^2 \times \ell^2 \to \mathbb{F}$ given by

$$\langle x\,; y\rangle \;= \sum_{k=-\infty}^{\infty} \xi_k \overline{\upsilon}_k$$

for every $x = \{\xi_k\}_{k\in\mathbb{Z}}$ and $y = \{\upsilon_k\}_{k\in\mathbb{Z}}$ in ℓ^2 is an inner product on ℓ^2, which induces the norm $\|\ \|_2$ on ℓ^2. Indeed, the sequence of nonnegative numbers $\{\sum_{k=-n}^{n} |\xi_k \overline{\upsilon}_k|\}_{n\in\mathbb{N}_0}$ converges in $\mathbb{R}$ if the sequences $\{\sum_{k=-n}^{n} |\xi_k|^2\}_{n\in\mathbb{N}_0}$ and $\{\sum_{k=-n}^{n} |\upsilon_k|^2\}_{n\in\mathbb{N}_0}$ of nonnegative numbers converge in $\mathbb{R}$ (the Hölder inequality for $p = q = 2$), and so $\{\sum_{k=-n}^{n} \xi_k \overline{\upsilon}_k\}_{n\in\mathbb{N}_0}$ converges in $\mathbb{F}$ (by Proposition 4.4). Therefore, the function $\langle\ ;\ \rangle$ is well defined and, as it is easy to check, it satisfies the axioms of Definition 5.1. Thus, as $(\ell^2, \|\ \|_2)$ is a Banach space,

$$(\ell^2, \langle\ ;\ \rangle) \text{ is a Hilbert space.}$$

Example 5.C. Consider the linear space $C[0,1]$ equipped with any of the norms $\|\ \|_p$ $(p \geq 1)$ of Example 4.D or with the sup-norm $\|\ \|_\infty$ of Example 4.G. Among these norms on $C[0,1]$, the only one that is induced by an inner product on $C[0,1]$ is the norm $\|\ \|_2$. Indeed, take x and y in $C[0,1]$ such that $xy = 0$ and $\|x\| = \|y\| \neq 0$, where $\|\ \|$ denotes either $\|\ \|_p$ for some $p \geq 1$ or $\|\ \|_\infty$. That is, suppose x and y are nonzero continuous functions on $[0,1]$ of equal norms such that their nonzero values are attained on disjoint subsets of $[0,1]$. For instance,



Observe that $\|x+y\|_p^p = \|x-y\|_p^p = 2\|x\|_p^p$ for every $p \geq 1$ and $\|x+y\|_\infty = \|x-y\|_\infty = 2\|x\|_\infty$. Thus $\|\ \|_p$ for $p \neq 2$ and $\|\ \|_\infty$ do not satisfy the parallelogram law, and so these norms are not induced by any inner product on $C[0,1]$ (Theorem 5.5). Now consider the function $\langle\ ;\ \rangle : C[0,1] \times C[0,1] \to \mathbb{F}$ given by

$$\langle x\,;y\rangle \ = \int_0^1 x(t)\,\overline{y(t)}\,dt$$

for every $x, y \in C[0,1]$. It is readily verified that $\langle\ ;\ \rangle$ is an inner product on $C[0,1]$ that induces the norm $\|\ \|_2$. Hence

$$(C[0,1], \langle\ ;\ \rangle) \text{ is an inner product space.}$$

However, $(C[0,1], \langle\ ;\ \rangle)$ is not a Hilbert space (reason: $(C[0,1], \|\ \|_2)$ is not a Banach space — Example 4.D). As a matter of fact, among the normed spaces $(C[0,1], \|\ \|_p)$ for each $p \geq 1$ and $(C[0,1], \|\ \|_\infty)$, the only Banach space is $(C[0,1], \|\ \|_\infty)$. This leads to a dichotomy: either equip $C[0,1]$ with $\|\ \|_2$ to get an inner product space that is not a Banach space, or equip it with $\|\ \|_\infty$ to get a Banach space whose norm is not induced by an inner product. In any case, $C[0,1]$ cannot be made into a Hilbert space. Roughly speaking, the set of continuous functions on $[0,1]$ is not large enough to be a Hilbert space.

Let $\mathcal{X}$ be a linear space over a field $\mathbb{F}$. A functional $\langle\ ;\ \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{F}$ is a *semi-inner product* on $\mathcal{X}$ if it satisfies the first four axioms of Definition 5.1. The difference between an inner product and a semi-inner product is that a semi-inner product is a Hermitian symmetric sesquilinear form that induces a nonnegative quadratic form which is not necessarily positive (i.e., axiom (v) of Definition 5.1 may not be satisfied by a semi-inner product). A semi-inner product $\langle\ ;\ \rangle$ on $\mathcal{X}$ induces a seminorm $\|\ \|$, which in turn generates a pseudometric d, namely, $d(x,y) = \|x-y\| = \langle x-y\,;x-y\rangle^{\frac{1}{2}}$ for every x, y in $\mathcal{X}$. A *semi-inner product space* is a linear space equipped with a semi-inner product.

Remark: The identity $\|x+y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\,\langle x\,;y\rangle + \|y\|^2$ for every $x, y \in \mathcal{X}$ still holds for a semi-inner product and its induced seminorm. Indeed, *the*

Schwarz inequality, the parallelogram law, and the polarization identities remain valid in a semi-inner product space (i.e., they still hold if we replace "inner product" and "norm" with "semi-inner product" and "seminorm", respectively — cf. proofs of Lemma 5.2 and Proposition 5.4). The same happens with respect to Theorem 5.5.

Proposition 5.6. *Let* $\| \ \|$ *be the seminorm induced by a semi-inner product* $\langle \ ; \ \rangle$ *on a linear space* $\mathcal{X}$*. Consider the quotient space* $\mathcal{X}/\mathcal{N}$*, where* $\mathcal{N} = \{x \in \mathcal{X}\colon \|x\| = 0\}$ *is a linear manifold of* $\mathcal{X}$*. Set*

$$\langle [x]\,;[y]\rangle_{\sim} \,=\, \langle x\,;y\rangle$$

for every $[x]$ *and* $[y]$ *in* $\mathcal{X}/\mathcal{N}$*, where* x *and* y *are arbitrary vectors in* $[x]$ *and* $[y]$*, respectively. This defines an inner product on* $\mathcal{X}/\mathcal{N}$ *so that* $(\mathcal{X}/\mathcal{N}, \langle \ ; \ \rangle_{\sim})$ *is an inner product space.*

Proof. The seminorm $\| \ \|$ is induced by a semi-inner product so that it satisfies the parallelogram law of Proposition 5.4. Consider the norm $\| \ \|_{\sim}$ on $\mathcal{X}/\mathcal{N}$ of Proposition 4.5 and note that

$$\begin{aligned} \|[x]+[y]\|_{\sim}^2 \,+\|[x]-[y]\|_{\sim}^2 \,&=\, \|[x+y]\|_{\sim}^2 \,+\|[x-y]\|_{\sim}^2 \\ &=\, \|x+y\|^2+\|x-y\|^2 \\ &=\, 2\big(\|x\|^2+\|y\|^2\big) \,=\, 2\big(\|[x]\|_{\sim}^2 \,+\|[y]\|_{\sim}^2\,\big) \end{aligned}$$

for every $[x], [y] \in \mathcal{X}/\mathcal{N}$. Thus $\| \ \|_{\sim}$ satisfies the parallelogram law. This means that it is induced by a (unique) inner product $\langle \ ; \ \rangle_{\sim}$ on $\mathcal{X}/\mathcal{N}$, which is given in terms of the norm $\| \ \|_{\sim}$ by the polarization identity (Theorem 5.5). On the other hand, the semi-inner product $\langle \ ; \ \rangle$ on $\mathcal{X}$ also is given in terms of the seminorm $\| \ \|$ through the polarization identity as in Proposition 5.4. Since $\|[x]+\alpha[y]\|_{\sim} = \|x+\alpha y\|$ for every $[x], [y] \in \mathcal{X}/\mathcal{N}$ and every $\alpha \in \mathbb{F}$ (with x and y being arbitrary elements of $[x]$ and $[y]$, respectively), it is readily verified by the polarization identity that $\langle [x]\,;[y]\rangle_{\sim} = \langle x\,;y\rangle$. □

Example 5.D. For each $p \geq 1$ let $r^p(S)$ be the linear space of all scalar-valued Riemann p-integrable functions, on a nondegenerate interval S of the real line, equipped with the seminorm $|\ |_p$ of Example 4.C. Again (see Example 5.C), it is easy to show that, except for the seminorm $|\ |_2$, these seminorms do not satisfy the parallelogram law. Moreover,

$$\langle x\,;y\rangle \,=\, \int_S x(s)\overline{y(s)}\,ds$$

for every $x, y \in r^2(S)$ defines a semi-inner product that induces the seminorm $|\ |_2$ given by $|x|_2 = (\int_S |x(s)|^2\,ds)^{\frac{1}{2}}$ for each $x \in r^2(S)$. Consider the linear manifold $\mathcal{N} = \{x \in r^2(S)\colon |x|_2 = 0\}$ of $r^2(S)$, and let $R^2(S)$ be the quotient space $r^2(S)/\mathcal{N}$ as in Example 4.C. Set

$$\langle [x]\,;[y]\rangle \;=\; \langle x\,;y\rangle$$

for every $[x],[y]\in R^2(S)$, where x and y are arbitrary vectors in $[x]$ and $[y]$, respectively. According to Proposition 5.6, this defines an inner product on $R^2(S)$, which is the one that induces the norm $\|\ \|_2$ of Example 4.C. Since $(R^2(S),\|\ \|_2)$ is not a Banach space, it follows that

$$(R^2(S),\langle\ ;\ \rangle) \text{ is an inner product space}$$

but not a Hilbert space. The completion $(L^2(S),\|\ \|_2)$ of $(R^2(S),\|\ \|_2)$ is a Banach space whose norm is induced by the inner product $\langle\ ;\ \rangle$ so that

$$(L^2(S),\langle\ ;\ \rangle) \text{ is a Hilbert space.}$$

This, in fact, is the completion of the inner product space $(C[0,1],\langle\ ;\ \rangle)$ of Example 5.C (if $S=[0,1]$ — see Examples 4.C and 4.D). We shall discuss the completion of an inner product space in Section 5.6.

Example 5.E. Let $\{(\mathcal{X}_i,\langle\ ;\ \rangle_i)\}_{i=1}^n$ be a finite collection of inner product spaces, where all the linear spaces $\mathcal{X}_i$ are over the same field $\mathbb{F}$, and let $\bigoplus_{i=1}^n\mathcal{X}_i$ be the direct sum of the family $\{\mathcal{X}_i\}_{i=1}^n$. For each $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ in $\bigoplus_{i=1}^n\mathcal{X}_i$, set

$$\langle x\,;y\rangle \;=\; \sum_{i=1}^n\langle x_i\,;y_i\rangle_i.$$

It is easy to check that this defines an inner product on $\bigoplus_{i=1}^n\mathcal{X}_i$ that induces the norm $\|\ \|_2$ of Example 4.E. Indeed, if $\|\ \|_i$ is the norm on each $\mathcal{X}_i$ induced by the inner product $\langle\ ;\ \rangle_i$, then $\langle x\,;x\rangle=\sum_{i=1}^n\langle x_i\,;x_i\rangle_i=\sum_{i=1}^n\|x_i\|_i^2=\|x\|_2^2$ for every $x=(x_1,\dots,x_n)$ in $\bigoplus_{i=1}^n\mathcal{X}_i$. Since $\big(\bigoplus_{i=1}^n\mathcal{X}_i,\|\ \|_2\big)$ is a Banach space if and only if each $(\mathcal{X}_i,\|\ \|_i)$ is a Banach space, it follows that

$$\big(\textstyle\bigoplus_{i=1}^n\mathcal{X}_i,\langle\ ;\ \rangle\big) \text{ is a Hilbert space}$$

if and only if each $(\mathcal{X}_i,\langle\ ;\ \rangle_i)$ is a Hilbert space. If the inner product spaces $(\mathcal{X}_i,\langle\ ;\ \rangle_i)$ coincide with a fixed inner product space $(\mathcal{X},\langle\ ;\ \rangle_X)$, then $\langle x\,;y\rangle=\sum_{i=1}^n\langle x_i\,;y_i\rangle_X$ defines an inner product on $\mathcal{X}^n=\bigoplus_{i=1}^n\mathcal{X}$ and

$$(\mathcal{X}^n,\langle\ ;\ \rangle) \text{ is a Hilbert space}$$

whenever $(\mathcal{X},\langle\ ;\ \rangle_X)$ is a Hilbert space. This generalizes Example 5.A.

Example 5.F. Let $\{(\mathcal{X}_k,\langle\ ;\ \rangle_k)\}$ be a countably infinite collection of inner product spaces indexed by $\mathbb{N}$ (or by $\mathbb{N}_0$), where all the linear spaces $\mathcal{X}_k$ are over the same field $\mathbb{F}$. Consider the full direct sum $\bigoplus_{k=1}^\infty\mathcal{X}_k$ of $\{\mathcal{X}_k\}_{k=1}^\infty$, which is a linear space over $\mathbb{F}$. Let $\big[\bigoplus_{k=1}^\infty\mathcal{X}_k\big]_2$ be the linear manifold of $\bigoplus_{k=1}^\infty\mathcal{X}_k$ made up of all *square-summable sequences* $\{x_k\}_{k=1}^\infty$ in $\bigoplus_{k=1}^\infty\mathcal{X}_k$. That is (see Example 4.F),

$$\big[\bigoplus_{k=1}^{\infty}\mathcal{X}_k\big]_2 \;=\; \big\{\{x_k\}_{k=1}^{\infty} \in \bigoplus_{k=1}^{\infty}\mathcal{X}_k\colon\; \textstyle\sum_{k=1}^{\infty}\|x_k\|_k^2 < \infty\big\},$$

where each $\|\ \|_k$ is the norm on $\mathcal{X}_k$ induced by the inner product $\langle\ ;\ \rangle_k$. Take arbitrary sequences $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ in $\big[\bigoplus_{k=1}^{\infty}\mathcal{X}_k\big]_2$ so that the real-valued sequences $\{\|x_k\|_k\}_{k=1}^{\infty}$ and $\{\|y_k\|_k\}_{k=1}^{\infty}$ lie in ℓ_+^2. Write $\langle\ ;\ \rangle_{\ell_+^2}$ and $\|\ \|_{\ell_+^2}$ for inner product and norm on ℓ_+^2 (see Example 5.B). Use the Schwarz inequality in each inner product space $\mathcal{X}_k$ and also in the Hilbert space ℓ_+^2 to get

$$\begin{aligned}\sum_{k=1}^{\infty}|\langle x_k\,;y_k\rangle_k| \;\le\; \sum_{k=1}^{\infty}\|x_k\|_k\|y_k\|_k \;&=\; \big\langle\{\|x_k\|_k\}_{k=1}^{\infty}\,;\{\|y_k\|_k\}_{k=1}^{\infty}\big\rangle_{\ell_+^2}\\ &\le\; \big\|\{\|x_k\|_k\}_{k=1}^{\infty}\big\|_{\ell_+^2}\,\big\|\{\|y_k\|_k\}_{k=1}^{\infty}\big\|_{\ell_+^2}.\end{aligned}$$

Therefore $\sum_{k=1}^{\infty}|\langle x_k\,;y_k\rangle_k| < \infty$, and so the infinite series $\sum_{k=1}^{\infty}\langle x_k\,;y_k\rangle_k$ is absolutely convergent in the Banach space $(\mathbb{F},|\ |)$, which implies that it converges in $(\mathbb{F},|\ |)$ by Proposition 4.4. Set

$$\langle x\,;y\rangle \;=\; \sum_{k=1}^{\infty}\langle x_k\,;y_k\rangle_k$$

for every $x=\{x_k\}_{k=1}^{\infty}$ and $y=\{y_k\}_{k=1}^{\infty}$ in $\big[\bigoplus_{k=1}^{\infty}\mathcal{X}_k\big]_2$. It is easy to show that this defines an inner product on $\big[\bigoplus_{k=1}^{\infty}\mathcal{X}_k\big]_2$ that induces the norm $\|\ \|_2$ of Example 4.F. Moreover, since $\big(\big[\bigoplus_{k=1}^{\infty}\mathcal{X}_k\big]_2,\|\ \|_2\big)$ is a Banach space if and only if each $(\mathcal{X}_k,\|\ \|_k)$ is a Banach space, it follows that

$$\big(\big[\textstyle\bigoplus_{k=1}^{\infty}\mathcal{X}_k\big]_2,\langle\ ;\ \rangle\big)\ \text{ is a Hilbert space}$$

if and only if each $(\mathcal{X}_k,\langle\ ;\ \rangle_k)$ is a Hilbert space. A similar argument holds if the collection $\{(\mathcal{X}_k,\langle\ ;\ \rangle_k)\}$ is indexed by $\mathbb{Z}$. Indeed, if we set

$$\big[\bigoplus_{k=-\infty}^{\infty}\mathcal{X}_k\big]_2 \;=\; \big\{\{x_k\}_{k=-\infty}^{\infty} \in \bigoplus_{k=-\infty}^{\infty}\mathcal{X}_k\colon\; \textstyle\sum_{k=-\infty}^{\infty}\|x_k\|_k^2 < \infty\big\},$$

the linear manifold of the full direct sum $\bigoplus_{k=-\infty}^{\infty}\mathcal{X}_k$ of $\{\mathcal{X}_k\}_{k=-\infty}^{\infty}$ made up of all *square-summable nets* $\{x_k\}_{k=-\infty}^{\infty}$ in $\bigoplus_{k=-\infty}^{\infty}\mathcal{X}_k$, then

$$\langle x\,;y\rangle \;=\; \sum_{k=-\infty}^{\infty}\langle x_k\,;y_k\rangle_k$$

for every $x=\{x_k\}_{k=-\infty}^{\infty}$ and $y=\{y_k\}_{k=-\infty}^{\infty}$ in $\big[\bigoplus_{k=-\infty}^{\infty}\mathcal{X}_k\big]_2$ defines the inner product on $\big[\bigoplus_{k=-\infty}^{\infty}\mathcal{X}_k\big]_2$ that induces the norm $\|\ \|_2$ of Example 4.F. Again, if each $(\mathcal{X}_k,\langle\ ;\ \rangle_k)$ is a Hilbert space, then

$$\big(\big[\textstyle\bigoplus_{k=-\infty}^{\infty}\mathcal{X}_k\big]_2,\langle\ ;\ \rangle\big)\ \text{ is a Hilbert space.}$$

If the inner product spaces $(\mathcal{X}_k,\langle\ ;\ \rangle_k)$ coincide with a fixed inner product space $(\mathcal{X},\langle\ ;\ \rangle_X)$, then set

$$\ell^2_+(\mathcal{X}) = \left[\bigoplus_{k=1}^{\infty}\mathcal{X}\right]_2 \quad \text{and} \quad \ell^2(\mathcal{X}) = \left[\bigoplus_{k=-\infty}^{\infty}\mathcal{X}\right]_2$$

as in Example 4.F. If $(\mathcal{X}, \langle\ ;\ \rangle_X)$ is a Hilbert space, then

$$(\ell^2_+(\mathcal{X}), \langle\ ;\ \rangle) \text{ and } (\ell^2(\mathcal{X}), \langle\ ;\ \rangle) \text{ are Hilbert spaces.}$$

5.3 Orthogonality

Let a and b be nonzero vectors in the Euclidean plane $\mathbb{R}^2$, and let θ_{ab} be the angle between the line segments joining these points to the origin (this is usually called the *angle between a and b*). Set $a' = \|a\|^{-1}a = (\alpha_1, \alpha_2)$ and $b' = \|b\|^{-1}b = (\beta_1, \beta_2)$ in the unit circle about the origin. It is an exercise of elementary plane geometry to verify that $\cos\theta_{ab} = \alpha_1\beta_1 + \alpha_2\beta_2 = \langle a' ; \beta' \rangle = \|a\|^{-1}\|b\|^{-1}\langle a ; b\rangle$. We shall be particularly concerned with the notion of *orthogonal* (or *perpendicular*) vectors a and b. The line segments joining a and b to the origin are perpendicular if $\theta_{ab} = \frac{\pi}{2}$ (equivalently, if $\cos\theta_{ab} = 0$), which means that $\langle a ; b\rangle = 0$. The notions of angle and orthogonality can be extended from the Euclidean plane to a *real* inner product space $(\mathcal{X}, \langle\ ;\ \rangle)$ by setting

$$\cos\theta_{xy} \;=\; \frac{\langle x ; y\rangle}{\|x\|\,\|y\|}$$

whenever x and y are nonzero vectors in $\mathcal{X} \neq \{0\}$. Note that $-1 \leq \cos\theta_{xy} \leq 1$ by the Schwarz inequality, and also that $\cos\theta_{xy} = 0$ if and only if $\langle x ; y\rangle = 0$.

Definition 5.7. Two vectors x and y in any (real or complex) inner product space $(\mathcal{X}, \langle\ ;\ \rangle)$ are said to be *orthogonal* (notation: $x \perp y$) if $\langle x ; y\rangle = 0$. A vector x in $\mathcal{X}$ is orthogonal to a subset A of $\mathcal{X}$ (notation: $x \perp A$) if it is orthogonal to every vector in A (i.e., if $\langle x ; y\rangle = 0$ for every $y \in A$). Two subsets A and B of $\mathcal{X}$ are orthogonal (notation: $A \perp B$) if every vector in A is orthogonal to every vector in B (i.e., if $\langle x ; y\rangle = 0$ for every $x \in A$ and every $y \in B$).

Thus A and B are orthogonal if there is no x in A and no y in B such that $\langle x ; y\rangle \neq 0$. In this sense the empty set $\varnothing$ is orthogonal to every subset of $\mathcal{X}$. Clearly, $x \perp y$ if and only if $y \perp x$, and hence $A \perp B$ if and only if $B \perp A$, so that $\perp$ is a symmetric relation both on $\mathcal{X}$ and on the power set $\wp(\mathcal{X})$. We write $x \not\perp y$ if $x \in \mathcal{X}$ and $y \in \mathcal{X}$ are not orthogonal. Similarly, $A \not\perp B$ means that $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{X}$ are not orthogonal. Note that if there exists a nonzero vector x in $A \cap B$, then $\langle x ; x\rangle = \|x\|^2 \neq 0$, and hence $A \not\perp B$. Therefore,

$$A \perp B \quad \text{implies} \quad A \cap B \subseteq \{0\}.$$

We shall say that a subset A of an inner product space $\mathcal{X}$ is an *orthogonal set* (or a *set of pairwise orthogonal vectors*) if $x \perp y$ for every pair $\{x, y\}$ of *distinct* vectors in A. Similarly, an $\mathcal{X}$-valued sequence $\{x_k\}$ is an *orthogonal sequence* (or a *sequence of pairwise orthogonal vectors*) if $x_k \perp x_j$ whenever

$k \neq j$. Since $\|x+y\|^2 = \|x\|^2 + 2\mathrm{Re}\langle x\,;y\rangle + \|y\|^2$ for every x and y in $\mathcal{X}$, it follows as an immediate consequence of the definition of orthogonality that

$$x \perp y \quad \text{implies} \quad \|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

This is the *Pythagorean Theorem*. The next result is a generalization of it for a finite orthogonal set.

Proposition 5.8. *If $\{x_i\}_{i=0}^n$ is a finite set of pairwise orthogonal vectors in a inner product space, then*

$$\Big\|\sum_{i=0}^n x_i\Big\|^2 = \sum_{i=0}^n \|x_i\|^2.$$

Proof. We have already seen that the result holds for $n=1$ (i.e., it holds for *every* pair of distinct orthogonal vectors). Suppose it holds for some $n \geq 1$ (i.e., suppose $\|\sum_{i=0}^n x_i\|^2 = \sum_{i=0}^n \|x_i\|^2$ for *every* orthogonal set $\{x_i\}_{i=0}^n$ with $n+1$ elements). Let $\{x_i\}_{i=0}^{n+1}$ be an *arbitrary* orthogonal set with $n+2$ elements. Since $x_{n+1} \perp \{x_i\}_{i=0}^n$, it follows that $x_{n+1} \perp \sum_{i=0}^n x_i$ (since $\langle x_{n+1}\,;\sum_{i=0}^n x_i\rangle = \sum_{i=0}^n \langle x_{n+1}\,;x_i\rangle$). Hence

$$\Big\|\sum_{i=0}^{n+1} x_i\Big\|^2 = \Big\|\sum_{i=0}^n x_i + x_{n+1}\Big\|^2 = \Big\|\sum_{i=0}^n x_i\Big\|^2 + \|x_{n+1}\|^2 = \sum_{i=0}^{n+1} \|x_i\|^2,$$

so that the result holds for $n+1$ (i.e., it holds for every orthogonal set with $n+2$ elements whenever it holds for every orthogonal set with $n+1$ elements), which completes the proof by induction. □

Recall that an $\mathcal{X}$-valued sequence $\{x_k\}_{k=1}^\infty$ (where $\mathcal{X}$ is any normed space) is *square-summable* if $\sum_{k=1}^\infty \|x_k\|^2 < \infty$. Here is a countably infinite version of the Pythagorean Theorem.

Corollary 5.9. *Let $\{x_k\}_{k=1}^\infty$ be a sequence of pairwise orthogonal vectors in a inner product space $\mathcal{X}$.*

(a) *If the infinite series $\sum_{k=1}^\infty x_k$ converges in $\mathcal{X}$, then $\{x_k\}_{k=1}^\infty$ is a square-summable sequence and $\|\sum_{k=1}^\infty x_k\|^2 = \sum_{k=1}^\infty \|x_k\|^2$.*

(b) *If $\mathcal{X}$ is a Hilbert space and $\{x_k\}_{k=1}^\infty$ is a square-summable sequence, then the infinite series $\sum_{k=1}^\infty x_k$ converges in $\mathcal{X}$.*

Proof. Let $\{x_k\}_{k=1}^\infty$ be an orthogonal sequence in $\mathcal{X}$.

(a) If the infinite series $\sum_{k=1}^\infty x_k$ converges in $\mathcal{X}$; that is, if $\sum_{k=1}^n x_k \to \sum_{k=1}^\infty x_k$ in $\mathcal{X}$ as $n \to \infty$, then $\|\sum_{k=1}^n x_k\|^2 \to \|\sum_{k=1}^\infty x_k\|^2$ as $n \to \infty$ (reason: norm and squaring are continuous mappings). Proposition 5.8 says that $\|\sum_{k=1}^n x_k\|^2 = \sum_{k=1}^n \|x_k\|^2$ for every $n \geq 1$, and hence $\sum_{k=1}^n \|x_k\|^2 \to \|\sum_{k=1}^\infty x_k\|^2$ as $n \to \infty$.

(b) Consider the $\mathcal{X}$-valued sequence $\{y_n\}_{n=1}^{\infty}$ of partial sums of $\{x_k\}_{k=1}^{\infty}$; that is, set $y_n = \sum_{k=1}^{n} x_k$ for each integer $n \geq 1$. By Proposition 5.8 we know that $\|y_{n+m} - y_n\|^2 = \sum_{j=n+1}^{n+m} \|x_{k_j}\|^2$ for every $m, n \geq 1$. If $\sum_{k=1}^{\infty} \|x_k\|^2 < \infty$, then $\sup_{m \geq 1} \|y_{n+m} - y_n\|^2 = \sum_{k=n+1}^{\infty} \|x_k\|^2 \to 0$ as $n \to \infty$ (Problem 3.11), and hence $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{X}$ (Problem 3.51). If $\mathcal{X}$ is Hilbert, then $\{y_n\}_{n=1}^{\infty}$ converges in $\mathcal{X}$, which means that the infinite series $\sum_{k=1}^{\infty} x_k$ converges in $\mathcal{X}$. □

Therefore, *if* $\{x_k\}_{k=1}^{\infty}$ *is an orthogonal sequence in a Hilbert space* $\mathcal{H}$*, then* $\sum_{k=1}^{\infty} \|x_k\|^2 < \infty$ *if and only if the infinite series* $\sum_{k=1}^{\infty} x_k$ *converges in* $\mathcal{H}$ *and, in this case,* $\|\sum_{k=1}^{\infty} x_k\|^2 = \sum_{k=1}^{\infty} \|x_k\|^2$.

Example 5.G. Let $\{(\mathcal{X}_k, \langle\ ;\ \rangle_k)\}_{k=1}^{\infty}$ be a sequence of Hilbert spaces. Consider the Hilbert space $\big(\big[\bigoplus_{k=1}^{\infty} \mathcal{X}_k\big]_2, \langle\ ;\ \rangle\big)$, where $\big[\bigoplus_{k=1}^{\infty} \mathcal{X}_k\big]_2$ is the linear space of all square-summable sequences in the full direct sum $\bigoplus_{k=1}^{\infty} \mathcal{X}_k$ and $\langle\ ;\ \rangle$ is the inner product of Example 5.F; that is,

$$\langle \boldsymbol{x}\,;\boldsymbol{y}\rangle \;=\; \sum_{k=1}^{\infty} \langle x_k\,; y_k\rangle_k$$

for every $\boldsymbol{x} = \{x_k\}_{k=1}^{\infty}$ and $\boldsymbol{y} = \{y_k\}_{k=1}^{\infty}$ in $\big[\bigoplus_{k=1}^{\infty} \mathcal{X}_k\big]_2$. This is referred to as an (external) *orthogonal direct sum*. An *orthogonal* direct sum actually deserves its name. Indeed, if we identify each linear space $\mathcal{X}_i$ with the linear manifold $\bigoplus_{k=1}^{\infty} \mathcal{X}_{i(k)}$ of $\big[\bigoplus_{k=1}^{\infty} \mathcal{X}_k\big]_2$ such that $\mathcal{X}_{i(k)} = \{0_k\} \subseteq \mathcal{X}_k$ for $k \neq i$ and $\mathcal{X}_{i(i)} = \mathcal{X}_i$ (as in Example 4.I), then it is clear that

$$\mathcal{X}_i \perp \mathcal{X}_j \quad \text{ whenever } \quad i \neq j,$$

where such an orthogonality is interpreted as $\bigoplus_{k=1}^{\infty} \mathcal{X}_{i(k)} \perp \bigoplus_{k=1}^{\infty} \mathcal{X}_{j(k)}$ with respect to the inner product $\langle\ ;\ \rangle$ on $\big[\bigoplus_{k=1}^{\infty} \mathcal{X}_k\big]_2$. Observe that the norm $\|\ \|$ induced on $\big[\bigoplus_{k=1}^{\infty} \mathcal{X}_k\big]_2$ by this inner product is given by

$$\|\boldsymbol{x}\|^2 \;=\; \langle \boldsymbol{x}\,;\boldsymbol{x}\rangle \;=\; \sum_{k=1}^{\infty} \langle x_k\,; x_k\rangle_k \;=\; \sum_{k=1}^{\infty} \|x_k\|_k^2$$

for every $\boldsymbol{x} = \{x_k\}_{k=1}^{\infty}$ in $\big[\bigoplus_{k=1}^{\infty} \mathcal{X}_k\big]_2$. This can also be verified via Corollary 5.9 as follows. Take an arbitrary vector $\boldsymbol{x} = \{x_k\}_{k=1}^{\infty}$ in $\big[\bigoplus_{k=1}^{\infty} \mathcal{X}_k\big]_2$ (i.e., take an arbitrary square-summable sequence from $\bigoplus_{k=1}^{\infty} \mathcal{X}_k$). Set $x_{i(k)} = \delta_{ik} x_k$ in $\mathcal{X}_k$ for every $k, i \geq 1$ (i.e., $x_{i(k)} = 0_k$ if $k \neq i$ and $x_{i(i)} = x_i$). For each $i \geq 1$ consider the vector $\boldsymbol{x}_i = \{x_{i(k)}\}_{k=1}^{\infty}$ so that $\boldsymbol{x}_1 = (x_1, 0_2, 0_3, \ldots)$ and, for $i \geq 2$,

$$\boldsymbol{x}_i = \{x_{i(k)}\}_{k=1}^{\infty} \;=\; (0_1, \ldots, 0_{i-1}, x_i, 0_{i+1}, \ldots) \quad \text{in} \quad \big[\bigoplus_{k=1}^{\infty} \mathcal{X}_k\big]_2.$$

Note that $\{\boldsymbol{x}_i\}_{i=1}^{\infty}$ is an orthogonal square-summable sequence. (Indeed, $\boldsymbol{x}_i$ lies in $\bigoplus_{k=1}^{\infty} \mathcal{X}_{i(k)}$, $\|\boldsymbol{x}_i\| = \|x_i\|_i$, and $\{x_i\}_{i=1}^{\infty}$ is square-summable.) Thus Corollary

5.9 ensures that (i) the infinite series $\bigoplus_{i=1}^{\infty}\boldsymbol{x}_i$ converges in the Hilbert space $\big(\big[\bigoplus_{k=1}^{\infty}\mathcal{X}_k\big]_2, \langle\ ;\ \rangle\big)$, and (ii) $\|\bigoplus_{i=1}^{\infty}\boldsymbol{x}_i\|^2 = \sum_{i=1}^{\infty}\|\boldsymbol{x}_i\|^2$. A word on notation: We are denoting vector addition in the linear space $\bigoplus_{k=1}^{\infty}\mathcal{X}_k$ by $\oplus$ and vector subtraction by $\ominus$, as usual. From (i), $\boldsymbol{x} = \bigoplus_{i=1}^{\infty}\boldsymbol{x}_i$ (because $\boldsymbol{x} \ominus (\bigoplus_{i=1}^{n}\boldsymbol{x}_i) = \bigoplus_{i=n+1}^{\infty}\boldsymbol{x}_i$ for each $n \geq 1$) and from (ii), $\|\boldsymbol{x}\|^2 = \sum_{i=1}^{\infty}\|\boldsymbol{x}_i\|^2$. Therefore,

$$\|\boldsymbol{x}\|^2 \;=\; \sum_{i=1}^{\infty}\|\boldsymbol{x}_i\|^2 \;=\; \sum_{i=1}^{\infty}\|x_i\|_i^2.$$

If $(\mathcal{X}, \langle\ ;\ \rangle)$ is an inner product space, and if $\mathcal{M}$ is a linear manifold of the linear space $\mathcal{X}$, then it is easy to show that the restriction $\langle\ ;\ \rangle_M\colon \mathcal{M}\times\mathcal{M} \to \mathbb{F}$ of the inner product $\langle\ ;\ \rangle\colon \mathcal{X}\times\mathcal{X} \to \mathbb{F}$ to $\mathcal{M}\times\mathcal{M}$ is an inner product on $\mathcal{M}$, so that $(\mathcal{M}, \langle\ ;\ \rangle_M)$ is an inner product space. Moreover, the norm $\|\ \|_M\colon \mathcal{M} \to \mathbb{R}$ induced by the inner product $\langle\ ;\ \rangle_M$ on $\mathcal{M}$ coincides with the restriction to $\mathcal{M}$ of the norm $\|\ \|\colon \mathcal{X} \to \mathbb{R}$ induced by the inner product $\langle\ ;\ \rangle$ on $\mathcal{X}$. Thus $(\mathcal{M}, \|\ \|_M)$ is a linear manifold of the normed space $(\mathcal{X}, \|\ \|)$. Whenever a linear manifold of an inner product space is regarded as an inner product space, it will always be understood that the inner product on it is the restricted inner product $\langle\ ;\ \rangle_M$. We shall drop the subscript and write $(\mathcal{M}, \langle\ ;\ \rangle)$ instead of $(\mathcal{M}, \langle\ ;\ \rangle_M)$, and refer to the inner product space $(\mathcal{M}, \langle\ ;\ \rangle)$ by simply saying that "$\mathcal{M}$ is a linear manifold of $\mathcal{X}$". Recall that a subspace of a normed space is a *closed* linear manifold of it. Hence a subspace of an inner product space $\mathcal{X}$ is a linear manifold of the linear space $\mathcal{X}$ that is closed in the inner product topology. That is, a subspace $\mathcal{M}$ of an inner product space $\mathcal{X}$ is a linear manifold of $\mathcal{X}$ that is closed as a subset of $\mathcal{X}$ when $\mathcal{X}$ is regarded as a metric space whose metric is that generated by the norm that is induced by the inner product. According to Proposition 4.7, a linear manifold of a Hilbert space $\mathcal{H}$ is a Hilbert space if and only if it is a subspace of $\mathcal{H}$. We observed in Section 4.3 that the sum of subspaces is not necessarily a subspace (it is a linear manifold but may not be closed). An important consequence of the Pythagorean Theorem is that the sum of *orthogonal* subspaces of a Hilbert space is a subspace.

Theorem 5.10. (a) *If $\mathcal{M}$ and $\mathcal{N}$ are complete orthogonal linear manifolds of an inner product space $\mathcal{X}$, then $\mathcal{M}+\mathcal{N}$ is a complete linear manifold of $\mathcal{X}$.*

(b) *If $\mathcal{M}$ and $\mathcal{N}$ are orthogonal subspaces of a Hilbert space $\mathcal{H}$, then the sum $\mathcal{M}+\mathcal{N}$ is a subspace of $\mathcal{H}$.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be orthogonal linear manifolds of an inner product space $\mathcal{X}$. Take an arbitrary Cauchy sequence $\{x_n\}$ in $\mathcal{M}+\mathcal{N}$ so that $x_n = u_n + v_n$ with u_n in $\mathcal{M}$ and v_n in $\mathcal{N}$ for each n. Since $\mathcal{M}$ and $\mathcal{N}$ are linear manifolds of $\mathcal{X}$, it follows that $u_m - u_n$ lies in $\mathcal{M}$ and $v_m - v_n$ lies in $\mathcal{N}$, and hence $u_m - u_n \perp v_m - v_n$, for every pair of integers m and n (since $\mathcal{M} \perp \mathcal{N}$). Writing $x_m - x_n = (u_m - u_n) + (v_m - v_n)$ the Pythagorean Theorem ensures that

$$\|x_m - x_n\|^2 \;=\; \|u_m - u_n\|^2 + \|v_m - v_n\|^2$$

for every m and n. This implies that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in $\mathcal{M}$ and $\mathcal{N}$, respectively (because $\{x_n\}$ is a Cauchy sequence).

(a) If $\mathcal{M}$ and $\mathcal{N}$ are complete, then $\{u_n\}$ converges in $\mathcal{M}$ and $\{v_n\}$ converges in $\mathcal{N}$. Recalling that addition is a continuous operation (Problem 4.1), we get from Corollary 3.8 that $\{x_n\}$ converges in $\mathcal{M}+\mathcal{N}$. Conclusion: Every Cauchy sequence in $\mathcal{M}+\mathcal{N}$ converges in $\mathcal{M}+\mathcal{N}$. Thus $\mathcal{M}+\mathcal{N}$ is complete (and hence closed in $\mathcal{X}$ by Theorem 3.40(a)).

(b) If $\mathcal{M}$ and $\mathcal{N}$ are closed linear manifolds of a complete inner product space $\mathcal{H}$, then they are complete (Theorem 3.40(b)), and the linear manifold $\mathcal{M}+\mathcal{N}$ of $\mathcal{H}$ is complete (and therefore closed in $\mathcal{H}$) according to item (a). □

Remark: Theorem 5.10(a) fails if $\mathcal{M}$ and $\mathcal{N}$ are either (1) orthogonal but not complete or (2) complete but not orthogonal. Equivalently, Theorem 5.10(b) fails if $\mathcal{M}$ and $\mathcal{N}$ are either (1) orthogonal subspaces of an incomplete inner product space or (2) not orthogonal subspaces of a Hilbert space. That is, completeness and orthogonality are both crucial assumptions in the statement of Theorem 5.10. This will be verified in Problems 5.12 and 5.13.

Let $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary nonempty indexed family of subspaces of a Hilbert space $\mathcal{H}$ (i.e., an arbitrary nonempty subcollection of $\mathrm{Lat}(\mathcal{H})$). Recall that the sum $\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma$ of $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ is the linear manifold of $\mathcal{H}$ consisting of *all finite sums* of vectors in $\mathcal{H}$ with each summand being a vector in one of the subspaces $\mathcal{M}_\gamma$. That is,

$$\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma \;=\; \mathrm{span}\Big(\bigcup_{\gamma\in\Gamma}\mathcal{M}_\gamma\Big).$$

Corollary 5.11. *Every finite sum of pairwise orthogonal subspaces of a Hilbert space $\mathcal{H}$ is itself a subspace of $\mathcal{H}$.*

Proof. Let $\mathcal{H}$ be a Hilbert space. Theorem 5.10(b) says that the sum of every pair of orthogonal subspaces of $\mathcal{H}$ is again a subspace of $\mathcal{H}$. Take an arbitrary integer $n\geq 2$. Suppose the sum of every set of n pairwise orthogonal subspaces of $\mathcal{H}$ is a subspace of $\mathcal{H}$. Now take an arbitrary collection of $n+1$ pairwise orthogonal subspaces of $\mathcal{H}$, say, $\{\mathcal{M}_i\}_{i=1}^{n+1}$. If $x\in\sum_{i=1}^{n}\mathcal{M}_i$, then $x=\sum_{i=1}^{n}x_i$ with each x_i in $\mathcal{M}_i$, and hence $\langle x\,;x_{n+1}\rangle=\sum_{i=1}^{n}\langle x_i\,;x_{n+1}\rangle=0$ whenever x_{n+1} lies in $\mathcal{M}_{n+1}$ (because $\mathcal{M}_i\perp\mathcal{M}_{n+1}$ for every $i=1,\ldots,n$). Thus $\sum_{i=1}^{n}\mathcal{M}_i\perp\mathcal{M}_{n+1}$. As $\sum_{i=1}^{n}\mathcal{M}_i$ is assumed to be a subspace of $\mathcal{H}$, and since

$$\sum_{i=1}^{n+1}\mathcal{M}_i \;=\; \mathrm{span}\Big(\bigcup_{i=1}^{n+1}\mathcal{M}_i\Big) \;=\; \mathrm{span}\Big(\bigcup_{i=1}^{n}\mathcal{M}_i\cup\mathcal{M}_{n+1}\Big) \;=\; \sum_{i=1}^{n}\mathcal{M}_i+\mathcal{M}_{n+1},$$

it follows by Theorem 5.10(b) that $\sum_{i=1}^{n+1}\mathcal{M}_i$ is a subspace of $\mathcal{H}$. This completes the proof by induction. □

5.4 Orthogonal Complement

If A is a subset of an inner product space $\mathcal{X}$, then the *orthogonal complement* of A is the set

$$A^\perp = \{x \in \mathcal{X}\colon\ x \perp A\} = \{x \in \mathcal{X}\colon\ \langle x\,;y\rangle = 0 \text{ for every } y \in A\}$$

consisting of all vectors in $\mathcal{X}$ that are orthogonal to every vector in A. If A is the empty set $\varnothing$, then for every x in $\mathcal{X}$ there is no vector y in A for which $\langle x\,;y\rangle \neq 0$, and hence $\varnothing^\perp = \mathcal{X}$. It is plain that $x \perp \{0\}$ for every $x \in \mathcal{X}$, and $x \perp \mathcal{X}$ if and only if $x = 0$. Hence

$$\{0\}^\perp = \mathcal{X} \quad \text{and} \quad \mathcal{X}^\perp = \{0\}.$$

Let A and B be nonempty subsets of $\mathcal{X}$. The next results are immediate consequences of the definition of orthogonal complement.

$$A \perp A^\perp, \quad A \cap A^\perp \subseteq \{0\}, \quad \text{and} \quad A \cap A^\perp = \{0\} \text{ whenever } 0 \in A$$

(reason: if there exists $x \in A \cap A^\perp$, then $\langle x\,;x\rangle = 0$), and

$$A \perp B \quad \text{if and only if} \quad A \subseteq B^\perp.$$

Since $\perp$ is a symmetric relation (i.e., $A \perp B$ if and only if $B \perp A$), the above equivalent assertions also are equivalent to $B \subseteq A^\perp$. Moreover,

$$A \perp B \quad \text{implies} \quad A \cap B \subseteq \{0\}$$

(if $A \subseteq B^\perp$, then $A \cap B \subseteq B^\perp \cap B \subseteq \{0\}$). It is readily verified that

$$A \subseteq B \quad \text{implies} \quad B^\perp \subseteq A^\perp \quad \text{and so} \quad A^{\perp\perp} \subseteq B^{\perp\perp},$$

where $A^{\perp\perp} = (A^\perp)^\perp$. Since $A \perp A^\perp$ and $A^\perp \perp A^{\perp\perp}$, we get $A \subseteq A^{\perp\perp}$ (so that $A^{\perp\perp\perp} \subseteq A^\perp$) and $A^\perp \subseteq A^{\perp\perp\perp}$, where $A^{\perp\perp\perp} = (A^{\perp\perp})^\perp$. Therefore,

$$A \subseteq A^{\perp\perp} \quad \text{and} \quad A^\perp = A^{\perp\perp\perp}.$$

Proposition 5.12. *The orthogonal complement $A^\perp$ of every subset A of any inner product space $\mathcal{X}$ is a subspace of $\mathcal{X}$. Moreover,*

$$A^\perp = (A^\perp)^- = (A^-)^\perp = (\operatorname{span} A)^\perp = \textstyle\left(\bigvee A\right)^\perp.$$

The orthogonal complement of every dense subset of $\mathcal{X}$ is the zero space:

$$A^\perp = \{0\} \quad \textit{whenever} \quad A^- = \mathcal{X}.$$

Proof. Suppose $A \neq \varnothing$ (otherwise the results are trivially verified). Since the inner product is linear in the first argument, it follows that $A^\perp$ is a linear manifold of the linear space $\mathcal{X}$. If $x \perp A$, then $x \perp \sum_{i=1}^n \alpha_i y_i$ for every integer $n \geq 1$

whenever $y_i \in A$ and $\alpha_i \in \mathbb{F}$ for each $i = 1, \ldots, n$, and hence $A^\perp \subseteq (\operatorname{span} A)^\perp$. On the other hand, $A \subseteq \operatorname{span} A$ so that $(\operatorname{span} A)^\perp \subseteq A^\perp$. Then

$$A^\perp = (\operatorname{span} A)^\perp.$$

That $A^\perp$ is closed in $\mathcal{X}$ is a consequence of the continuity of the inner product (Problem 5.6). Actually, if $\{x_n\}$ is an $A^\perp$-valued sequence that converges in $\mathcal{X}$ to $x \in \mathcal{X}$, then (cf. Corollary 3.8) $\langle x\,; y\rangle = \langle \lim x_n\,; y\rangle = \lim\langle x_n\,; y\rangle = 0$ for every $y \in A$, which implies that $x \in A^\perp$. Therefore, $A^\perp$ is closed in $\mathcal{X}$ by the Closed Set Theorem (Theorem 3.30); that is,

$$A^\perp = (A^\perp)^-,$$

and so $A^\perp$ is a subspace (i.e., a closed linear manifold) of $\mathcal{X}$. Now take an arbitrary x in $A^\perp$ and an arbitrary y in A^-. By Proposition 3.27 there is an A-valued sequence $\{y_n\}$ that converges in $\mathcal{X}$ to y. Using Corollary 3.8 again, and recalling that the inner product is continuous, we get $\langle y\,; x\rangle = \langle \lim y_n\,; x\rangle = \lim\langle y_n\,; x\rangle = 0$. Thus $A^\perp \perp A^-$ so that $A^\perp \subseteq (A^-)^\perp$. But $(A^-)^\perp \subseteq A^\perp$ because $A \subseteq A^-$. Hence

$$A^\perp = (A^-)^\perp.$$

Since $A^\perp = (\operatorname{span} A)^\perp$ and $A^\perp = (A^-)^\perp$ for every subset A of $\mathcal{X}$,

$$A^\perp = (\operatorname{span} A)^\perp = \big[(\operatorname{span} A)^-\big]^\perp = \big(\bigvee A\big)^\perp.$$

Finally, if $A^- = \mathcal{X}$, then $A^\perp = (A^-)^\perp = \mathcal{X}^\perp = \{0\}$. □

Remark: If $L \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$, where $\mathcal{X}$ is an inner product space, then *the linear transformation* $L|_{\mathcal{N}(L)^\perp}$ *is injective* (i.e., then $\mathcal{N}(L|_{\mathcal{N}(L)^\perp}) = \{0\}$). In fact, if $v \in \mathcal{N}(L)^\perp$ lies in $\mathcal{N}(L|_{N(L)^\perp})$, then $v \in \mathcal{N}(L) \cap \mathcal{N}(L)^\perp = \{0\}$.

The next theorem is of critical importance; it may be thought of as a pivotal result in the theory if Hilbert spaces. Recall that the distance $d(x, M)$ of a point x in a normed space $\mathcal{X}$ to a *nonempty* subset M of $\mathcal{X}$ is the real number

$$d(x, M) = \inf_{u \in M} \|x - u\|.$$

Theorem 5.13. *Let x be an arbitrary vector in a Hilbert space $\mathcal{H}$.*

(a) *If M is a closed convex nonempty subset of $\mathcal{H}$, then there exists a unique vector u_x in M such that*

$$\|x - u_x\| = d(x, M).$$

(b) *Moreover, if $\mathcal{M}$ is a subspace of $\mathcal{H}$, then the unique vector in $\mathcal{M}$ for which $\|x - u_x\| = d(x, \mathcal{M})$ is the unique vector in $\mathcal{M}$ such that the difference $x - u_x$ is orthogonal to $\mathcal{M}$; that is, such that*

$$x - u_x \in \mathcal{M}^\perp.$$

Proof. (a) Let x be an arbitrary vector in $\mathcal{H}$ and let M be a nonempty subset of $\mathcal{H}$ so that $d(x,M) = \inf_{u\in M}\|x-u\|$ exists in $\mathbb{R}$. Therefore, for each integer $n \geq 1$ there exists $u_n \in M$ such that

$$d(x,M) \;\leq\; \|x-u_n\| \;<\; d(x,M) + \tfrac{1}{n}.$$

Consider the M-valued sequence $\{u_n\}$. $\mathcal{H}$ is an inner product space, and so the parallelogram law ensures that

$$\|2x-u_m-u_n\|^2 + \|u_n-u_m\|^2 \;=\; 2\big(\|x-u_m\|^2 + \|x-u_n\|^2\big)$$

for each $m,n \geq 1$. Since M is convex, it follows that $\frac{1}{2}(u_m+u_n) \in M$, and hence $2\,d(x,M) \leq 2\|\frac{1}{2}(u_m+u_n)-x\| = \|2x-u_m-u_n\|$ so that

$$0 \;\leq\; \|u_m-u_n\|^2 \;\leq\; 2\big(\|x-u_m\|^2 + \|x-u_n\|^2 - 2d(x,M)^2\big)$$

for every $m,n \geq 1$. This inequality and the fact that $\|x-u_n\| \to d(x,M)$ as $n \to \infty$ are enough to ensure that $\{u_n\}$ is a Cauchy sequence in $\mathcal{H}$, and therefore it converges in the Hilbert space $\mathcal{H}$ to, say, $u_x \in \mathcal{H}$. But the norm is a continuous function so that (see Corollary 3.8)

$$\|x-u_x\| \;=\; \|x-\lim u_n\| \;=\; \lim\|x-u_n\| \;=\; d(x,M).$$

Moreover, since M is closed in $\mathcal{H}$ and $\{u_n\}$ is an M-valued sequence that converges to u_x in $\mathcal{H}$, it follows by the Closed Set Theorem (Theorem 3.30) that $u_x \in M$. Conclusion: There exists u_x in M such that

$$\|x-u_x\| \;=\; d(x,M).$$

To prove uniqueness take an arbitrary u in M such that $\|x-u\| = d(x,M)$. Observe that $\frac{1}{2}(u_x+u)$ lies in M because M is convex, and hence $d(x,M) \leq \|\frac{1}{2}(u_x+u)-x\|$. Thus $4\,d(x,M)^2 \leq \|u_x+u-2x\|^2$. This inequality and the parallelogram law imply that

$$\begin{aligned} 4\,d(x,M)^2 + \|u_x-u\|^2 \;&\leq\; \|u_x+u-2x\|^2 + \|u_x-u\|^2 \\ &=\; 2\big(\|u_x-x\|^2 + \|u-x\|^2\big) \;=\; 4\,d(x,M)^2. \end{aligned}$$

Outcome: $\|u_x-u\|^2 = 0$; that is, $u = u_x$.

(b) Now let x be an arbitrary vector in $\mathcal{H}$ and suppose $\mathcal{M}$ is a subspace of $\mathcal{H}$, which obviously implies that $\mathcal{M}$ is a closed convex nonempty subset of $\mathcal{H}$. According to item (a) there exists a unique $u_x \in \mathcal{M}$ such that $\|x-u_x\| = d(x,\mathcal{M})$. Take an arbitrary nonzero $u \in \mathcal{M}$. Since $(u_x+\alpha u) \in \mathcal{M}$ for every scalar α, it follows that

$$d(x,\mathcal{M})^2 \;\leq\; \|x-u_x-\alpha u\|^2 \;=\; \|x-u_x\|^2 + |\alpha|^2\|u\|^2 - 2\,\mathrm{Re}\,(\overline{\alpha}\langle x-u_x\,;u\rangle).$$

Set $\alpha = \|u\|^{-2}\langle x-u_x\,;u\rangle$ in the above inequality and recall that $\|x-u_x\|^2 = d(x,\mathcal{M})^2$. Thus $2|\langle x-u_x\,;u\rangle|^2 \leq |\langle x-u_x\,;u\rangle|^2$, and hence $|\langle x-u_x\,;u\rangle| = 0$. Conclusion: $x-u_x \perp u$ for every nonzero u in $\mathcal{M}$, which implies

$$x - u_x \perp \mathcal{M}.$$

Moreover, this u_x is the unique vector in $\mathcal{M}$ with the above property. Indeed, if v is a vector in $\mathcal{M}$ such that $x - v \perp \mathcal{M}$, then $\langle x - v\,; v - u\rangle = \langle x - v\,; v\rangle - \langle x - v\,; u\rangle = 0$ whenever $u \in \mathcal{M}$ so that $x - v \perp v - u$ for every $u \in \mathcal{M}$. Thus, by the Pythagorean Theorem,

$$\|x - v\|^2 \le \|x - v\|^2 + \|v - u\|^2 = \|x - v + v - u\|^2 = \|x - u\|^2$$

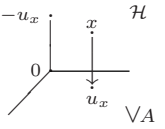
for all $u \in \mathcal{M}$. In particular, for $u = u_x$,

$$d(x, \mathcal{M}) \le \|x - v\| \le \|x - u_x\| = d(x, \mathcal{M})$$

so that $d(x, \mathcal{M}) = \|x - v\|$, and hence $v = u_x$ because u_x is the unique vector in $\mathcal{M}$ for which $d(x, \mathcal{M}) = \|x - u_x\|$. □

Let A be any nonempty subset of a Hilbert space $\mathcal{H}$. Recall that the subspace $\bigvee A = (\operatorname{span} A)^-$ of $\mathcal{H}$ is the closure in $\mathcal{H}$ of the set of all (finite) linear combinations of vectors in A. Take an arbitrary vector x in $\mathcal{H}$. The vector u_x in $\bigvee A$ that minimizes the distance of x to $\bigvee A$ is called the *best linear approximation* of x in terms of A, and the difference $x - u_x$ in $\mathcal{H}$ is called the *error* of the approximation of x in $\mathcal{H}$ by u_x in $\bigvee A$. The next result is a straightforward consequence of Theorem 5.13, which is illustrated in the figure below.

Corollary 5.14. *The best linear approximation of any vector x in a Hilbert space $\mathcal{H}$ in terms of a nonempty subset A of $\mathcal{H}$ is the vector u_x in $\bigvee A$ for which the error $x - u_x$ is orthogonal to $\bigvee A$.*



Proposition 5.15. *Let $\mathcal{M}$ be a linear manifold of a Hilbert space $\mathcal{H}$.*

$$\mathcal{M}^{\perp\perp} = \mathcal{M}^- \quad \textit{and} \quad \mathcal{M}^\perp = \{0\} \ \textit{if and only if} \ \mathcal{M}^- = \mathcal{H}.$$

In particular, if A is any subset of a Hilbert space $\mathcal{H}$, then

$$A^{\perp\perp} = \bigvee A \quad \textit{and} \quad A^\perp = \{0\} \ \textit{if and only if} \ \bigvee A = \mathcal{H}.$$

Proof. Recall that $\mathcal{M} \subseteq \mathcal{M}^{\perp\perp}$ and $\mathcal{M}^{\perp\perp}$ is closed in $\mathcal{H}$ according to Proposition 5.12. Then

$$\mathcal{M}^- \subseteq \mathcal{M}^{\perp\perp}.$$

Since $\mathcal{M}^{\perp\perp}$ is a subspace (i.e., a closed linear manifold) of a Hilbert space, it follows that it is itself a Hilbert space, and hence $\mathcal{M}^-$ is a subspace of the Hilbert space $\mathcal{M}^{\perp\perp}$. Take an arbitrary x in $\mathcal{M}^{\perp\perp}$. Theorem 5.13 ensures that

there exists $u_x \in \mathcal{M}^-$ such that $x - u_x \in (\mathcal{M}^-)^\perp = \mathcal{M}^\perp$ (Proposition 5.12). But $x - u_x \in \mathcal{M}^{\perp\perp}$ because $u_x \in \mathcal{M}^- \subseteq \mathcal{M}^{\perp\perp}$. Thus $x - u_x \in \mathcal{M}^\perp \cap \mathcal{M}^{\perp\perp}$, and hence $x = u_x \in \mathcal{M}^-$. Conclusion:

$$\mathcal{M}^{\perp\perp} \subseteq \mathcal{M}^-.$$

Therefore $\mathcal{M}^- = \mathcal{M}^{\perp\perp}$. If $\mathcal{M}^\perp = \{0\}$, then $\mathcal{M}^{\perp\perp} = \{0\}^\perp = \mathcal{H}$, and so

$$\mathcal{M}^- = \mathcal{H} \quad \text{whenever} \quad \mathcal{M}^\perp = \{0\}.$$

On the other hand, Proposition 5.12 says that $\mathcal{M}^\perp = \{0\}$ whenever $\mathcal{M}^- = \mathcal{H}$. Finally, if A is any subset of $\mathcal{H}$, then $A^\perp = (\bigvee A)^\perp$ so that $A^{\perp\perp} = (\bigvee A)^{\perp\perp}$ (Proposition 5.12 again). Since $\bigvee A$ is a subspace of $\mathcal{H}$, it follows that $A^{\perp\perp} = \bigvee A$, and $A^\perp = \{0\}$ if and only if $\bigvee A = \mathcal{H}$. □

It is worth noticing that the inner product space $\mathcal{H}$ in Theorem 5.13 was supposed to be complete only to ensure that the *closed* (convex and nonempty) subset M and the *closed* linear manifold $\mathcal{M}$ are *complete* (Theorem 3.40(b)). In fact, Theorem 5.13 can be formulated in an inner product space setting by assuming that M is *complete* (as a metric space in the inner product topology) and $\mathcal{M}$ is a *complete* linear manifold, instead of assuming that M and $\mathcal{M}$ are closed in a inner product space $\mathcal{H}$ that is itself complete. We shall see next that Theorem 5.13 does not hold without the completeness assumption.

Example 5.H. Let $\mathcal{X}$ be a proper dense linear manifold of a Hilbert space $\mathcal{H}$. ($\mathcal{X}$ is an inner product space that is not complete by Proposition 4.7.) Take $z \in \mathcal{H}\backslash\mathcal{X}$, consider the orthogonal complement $\{z\}^\perp$ of $\{z\}$ in $\mathcal{H}$, and set

$$\mathcal{M} = \{z\}^\perp \cap \mathcal{X}.$$

(a) Since $\{z\}^\perp$ is a closed linear manifold of $\mathcal{H}$ (Proposition 5.12), it follows by Problem 3.38(d) that $\{z\}^\perp \cap \mathcal{X}$ is closed in $\mathcal{X}$. Thus the intersection $\mathcal{M}$ of the linear manifolds $\{z\}^\perp$ and $\mathcal{X}$ of $\mathcal{H}$ is a linear manifold of $\mathcal{X}$ that is closed in $\mathcal{X}$ (i.e., $\mathcal{M}$ is a subspace of $\mathcal{X}$). Now, if $\mathcal{M} = \mathcal{X}$, then $\mathcal{X} \subseteq \{z\}^\perp$, which implies $\{z\}^{\perp\perp} \subseteq \mathcal{X}^\perp$. But $\mathcal{X}^\perp = \{0\}$ since $\mathcal{X}^- = \mathcal{H}$ (Proposition 5.12). Hence $\{z\} \subseteq \{z\}^{\perp\perp} = \{0\}$. Then $z = 0$, which contradicts the fact that $z \notin \mathcal{X}$. So

$$\mathcal{M} \text{ is a proper subspace of } \mathcal{X}.$$

Take $x \in \mathcal{X}\backslash\mathcal{M}$. Since $x \in \mathcal{H}$, and since $\{z\}^\perp$ is a subspace of $\mathcal{H}$, it follows by Theorem 5.13 that there is a unique $u_x \in \{z\}^\perp$ such that $x - u_x \in \{z\}^{\perp\perp} = \bigvee\{z\}$ (see Proposition 5.15), and hence $u_x = x + \alpha z$ for some nonzero scalar α (recall: $\bigvee\{z\} = \text{span}\,\{z\}$). Therefore,

$$u_x \notin \mathcal{X}$$

because $z \notin \mathcal{X}$ and $x \in \mathcal{X}$. Theorem 5.13 also says that u_x is the *unique* vector in $\{z\}^\perp$ such that

$$\|x - u_x\| \;=\; d(x, \{z\}^\perp).$$

If $\mathcal{M}$ is dense in $\{z\}^\perp$ (i.e., if $\mathcal{M}^- = \{z\}^\perp$, where $\mathcal{M}^-$ is the closure of $\mathcal{M}$ in $\mathcal{H}$), then $d(x, \mathcal{M}) = d(x, \{z\}^\perp)$ by Problem 3.43(b), and in this case there is no vector u in $\mathcal{M} = \{z\}^\perp \cap \mathcal{X}$ such that $\|x - u\| = d(x, \mathcal{M})$. This shows that Theorem 5.13 does not hold in the incomplete inner product space $\mathcal{X}$.

(b) Next we give a concrete example of the above setup by exhibiting a Hilbert space $\mathcal{H}$, a dense linear manifold $\mathcal{X}$ of $\mathcal{H}$, and a vector z in $\mathcal{H}\backslash\mathcal{X}$ for which $\mathcal{M} = \{z\}^\perp \cap \mathcal{X}$ is dense in $\{z\}^\perp$. Set

$$\mathcal{H} = \ell^2_+ \quad \text{and} \quad \mathcal{X} = \ell^0_+.$$

Recall that ℓ^0_+ is dense in ℓ^2_+ (Problem 3.44). Set $z = \{(\frac{1}{2})^k\}_{k=1}^\infty \in \ell^2_+ \backslash \ell^0_+$ and take an arbitrary $y \in \{z\}^\perp$. That is, $y = \{\upsilon_k\}_{k=1}^\infty \in \ell^2_+$ is such that

$$\sum_{k=1}^{\infty} \big(\tfrac{1}{2}\big)^k \upsilon_k \;=\; 0.$$

For each integer $n \geq 1$ consider the sequence

$$y_n \;=\; \big(\upsilon_1, \ldots, \upsilon_n, -2^{n+1}\textstyle\sum_{k=1}^n (\frac{1}{2})^k \upsilon_k,\, 0, 0, 0, \ldots\big)$$

in ℓ^0_+. It is clear that $\langle y_n \,; z\rangle = 0$, and hence $y_n \in \mathcal{M} = \{z\}^\perp \cap \ell^0_+$, for every $n \geq 1$. Moreover,

$$\|y_n - y\|^2 \;=\; \big|\upsilon_{n+1} + 2^{n+1}\textstyle\sum_{k=1}^n (\frac{1}{2})^k \upsilon_k\big|^2 + \sum_{k=n+2}^\infty |\upsilon_k|^2$$

for each integer $n \geq 1$. Since $y \in \{z\}^\perp \subset \ell^2_+$, it follows that $\sum_{k=n+2}^\infty |\upsilon_k|^2 \to 0$ as $n \to \infty$ (Problem 3.11) and $\sum_{k=1}^n (\frac{1}{2})^k \upsilon_k = -\sum_{k=n+1}^\infty (\frac{1}{2})^k \upsilon_k$ for every $n \geq 1$. Recalling that $\sum_{k=n+1}^\infty (\frac{1}{2})^k = (\frac{1}{2})^n$ for each $n \geq 0$, we get

$$\begin{aligned}
\{0\} \;\leq\; 2^{n+1}\Big|\sum_{k=1}^{n} (\tfrac{1}{2})^k \upsilon_k\Big| \;&=\; 2^{n+1}\Big|\sum_{k=n+1}^{\infty} (\tfrac{1}{2})^k \upsilon_k\Big| \\
&\leq\; \big(\sup_{k\geq n+1} |\upsilon_k|\big)\, 2^{n+1} \sum_{k=n+1}^{\infty} (\tfrac{1}{2})^k \\
&=\; \big(\sup_{k\geq n+1} |\upsilon_k|\big)\, \tfrac{2^{n+1}}{2^n} \;=\; 2 \sup_{k\geq n+1} |\upsilon_k| \;\to\; 0
\end{aligned}$$

as $n \to \infty$ (since $\lim_n \sup_{k\geq n} |\upsilon_k| = \limsup_n |\upsilon_k| = \lim_n |\upsilon_n| = 0$). Thus

$$y_n \to y \quad \text{in} \quad \ell^2_+.$$

Conclusion: For every $y \in \{z\}^\perp$ there exists an $\mathcal{M}$-valued sequence $\{y_n\}_{n=1}^\infty$ that converges in ℓ^2_+ to y, which means that $\mathcal{M}$ is dense in $\{z\}^\perp$ (Proposition 3.32(d)). That is,

$$\mathcal{M}^{-} = \{z\}^{\perp}.$$

Note that $\mathcal{M} \neq \{z\}^{\perp}$ (e.g., the sequence $(\zeta_1 - \|z\|^2(\overline{\zeta}_1)^{-1}, \zeta_2, \zeta_3, \ldots)$ lies in $\{z\}^{\perp}\backslash\ell_+^0$ for every $z = \{\zeta_k\}_{k=1}^{\infty}$ in $\ell_+^2\backslash\ell_+^0$ with $\zeta_1 \neq 0$), which implies that $\mathcal{M}$ is not closed in the Hilbert space $\mathcal{H} = \ell_+^2$, and hence $\mathcal{M}$ is not complete (Corollary 3.41).

The above example shows that completeness of $\mathcal{H}$ was crucial in the proof of Theorem 5.13. But it is not enough. Indeed, Theorem 5.13 does not necessarily hold in a Banach space that is not Hilbert (i.e., in a Banach space whose norm does not satisfy the parallelogram law). This is closely related to Lemma 4.33. In fact, what is behind that lemma is that there may exist a proper subspace $\mathcal{M}$ of a Banach space $\mathcal{X}$ for which $d(x, \mathcal{M}) < 1$ for all $x \in \mathcal{X}$ with $\|x\| = 1$ or, equivalently, $d(x, \mathcal{M}) < \|x\|$ for every nonzero x in $\mathcal{X}$. Suppose this is the case, and take an arbitrary $x \in \mathcal{X}\backslash\mathcal{M}$. Since $d(x, \mathcal{M}) = d(x - u, \mathcal{M})$ for every $x \in \mathcal{X}$ and every $u \in \mathcal{M}$, it follows that $d(x, \mathcal{M}) < \|x - u\|$ whenever u lies in $\mathcal{M}$ and x lies in $\mathcal{X}\backslash\mathcal{M}$. Therefore, if x is a vector in $\mathcal{X}\backslash\mathcal{M}$, then there is no vector u in $\mathcal{M}$ for which $\|x - u\| = d(x, \mathcal{M})$.

5.5 Orthogonal Structure

Let $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary nonempty indexed family of subspaces of a Hilbert space $\mathcal{H}$ and consider their topological sum

$$\Big(\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma\Big)^{-} = \Big[\operatorname{span}\Big(\bigcup_{\gamma\in\Gamma}\mathcal{M}_\gamma\Big)\Big]^{-} = \bigvee\Big(\bigcup_{\gamma\in\Gamma}\mathcal{M}_\gamma\Big) = \bigvee_{\gamma\in\Gamma}\mathcal{M}_\gamma.$$

If $\{\mathcal{M}_i\}_{i=1}^n$ is a *finite* (nonempty) family of pairwise *orthogonal* (i.e., $\mathcal{M}_i \perp \mathcal{M}_j$ whenever $i \neq j$) *subspaces* of a *Hilbert* space, then Corollary 5.11 says that $\sum_{i=1}^n \mathcal{M}_i = \big(\sum_{i=1}^n \mathcal{M}_i\big)^{-}$. That is, *the topological sum and the ordinary* (algebraic) *sum of a finite family of pairwise orthogonal subspaces of a Hilbert space coincide*. The next theorem is a countably infinite counterpart of the above italicized result, which emerges as an important consequence of Theorem 5.13.

Theorem 5.16. (The Orthogonal Structure Theorem). *Let $\mathcal{H}$ be a Hilbert space and suppose $\{\mathcal{M}_k\}_{k\in\mathbb{N}}$ is a countably infinite family of pairwise orthogonal subspaces of $\mathcal{H}$. If $x \in \big(\sum_{k\in\mathbb{N}}\mathcal{M}_k\big)^{-}$, then there exists a unique $\mathcal{H}$-valued sequence $\{u_k\}_{k=1}^{\infty}$ with $u_k \in \mathcal{M}_k$ for each k such that*

$$x = \sum_{k=1}^{\infty} u_k.$$

Moreover, $\|x\|^2 = \sum_{k=1}^{\infty}\|u_k\|^2$. Conversely, if $\{u_k\}_{k=1}^{\infty}$ is an $\mathcal{H}$-valued sequence such that $u_k \in \mathcal{M}_k$ for each k and $\sum_{k=1}^{\infty}\|u_k\|^2 < \infty$, then the infinite series $\sum_{k=1}^{\infty} u_k$ converges in $\mathcal{H}$ and $\sum_{k=1}^{\infty} u_k \in \big(\sum_{k\in\mathbb{N}}\mathcal{M}_k\big)^{-}$.

Proof. If $x \in \big(\sum_{k\in\mathbb{N}}\mathcal{M}_k\big)^-$, then there exists a sequence $\{x_{(n)}\}_{n=1}^{\infty}$ of vectors in $\sum_{k\in\mathbb{N}}\mathcal{M}_k$ that converges to x in $\mathcal{H}$ (Proposition 3.27). Take an arbitrary integer $n \geq 1$. Since $x_{(n)} \in \sum_{k\in\mathbb{N}}\mathcal{M}_k$, it follows that $x_{(n)}$ is a *finite* sum where each summand lies in one of the subspaces $\mathcal{M}_k$. Thus $x_{(n)}$ can be written as

$$x_{(n)} \,=\, \sum_{k=1}^{m_n} x_{(n)_k} \,\in\, \sum_{k=1}^{m_n} \mathcal{M}_k$$

with $x_{(n)_k} \in \mathcal{M}_k$ for each $k = 1, \ldots, m_n$, where m_n is an integer that depends on n. Clearly, we may take $m_{n+1} \geq m_n \geq n$. Observe that the above finite sum may contain finitely many zero summands, and $\{\mathcal{M}_k\}_{k=1}^{m_n} \subseteq \{\mathcal{M}_k\}_{k=1}^{m_{n+1}}$.

Existence. Note that $\sum_{k=1}^{m_n}\mathcal{M}_k$ is a subspace of the Hilbert space $\mathcal{H}$ (Corollary 5.11). According to Theorem 5.13 there exists a unique vector

$$u_{x(n)} \,=\, \sum_{k=1}^{m_n} u_k \,\in\, \sum_{k=1}^{m_n} \mathcal{M}_k,$$

with each u_k in $\mathcal{M}_k$, such that $\|x - u_{x(n)}\| \leq \|x - u\|$ for all vectors u in $\sum_{k=1}^{m_n}\mathcal{M}_k$ and $x - u_{x(n)} \perp \sum_{k=1}^{m_n}\mathcal{M}_k$. In particular,

$$\Big\|x - \sum_{k=1}^{m_n} u_k\Big\| \,\leq\, \|x - x_{(n)}\|.$$

Claim. The vectors u_k in the expansion of $u_{x(n)}$ do not depend on n.

Proof. Since $\sum_{k=1}^{m_{n+1}}\mathcal{M}_k = \sum_{k=1}^{m_n}\mathcal{M}_k + \sum_{k=m_n+1}^{m_{n+1}}\mathcal{M}_k$ is a subspace of $\mathcal{H}$, it follows by Theorem 5.13 that there exists a unique

$$u_{x(n+1)} \,=\, v + w \,\in\, \sum_{k=1}^{m_{n+1}} \mathcal{M}_k,$$

with $v \in \sum_{k=1}^{m_n}\mathcal{M}_k$ and $w \in \sum_{k=m_n+1}^{m_{n+1}}\mathcal{M}_k$, such that $\|x - u_{x(n+1)}\| \leq \|x - u\|$ for all vectors u in $\sum_{k=1}^{m_{n+1}}\mathcal{M}_k$ and $x - u_{x(n+1)} \perp \sum_{k=1}^{m_{n+1}}\mathcal{M}_k$. Take an arbitrary $z \in \sum_{k=1}^{m_n}\mathcal{M}_k \subseteq \sum_{k=1}^{m_{n+1}}\mathcal{M}_k$ and note that

$$0 \,=\, \langle x - u_{x(n+1)}\,;z\rangle \,=\, \langle x - v - w\,;z\rangle \,=\, \langle x - v\,;z\rangle - \langle w\,;z\rangle \,=\, \langle x - v\,;z\rangle$$

(since $\mathcal{M}_j \perp \mathcal{M}_k$ whenever $j \neq k$ so that $\sum_{k=m_n+1}^{m_{n+1}}\mathcal{M}_k \perp \sum_{k=1}^{m_n}\mathcal{M}_k$, and hence $\langle w\,;z\rangle = 0$). Thus $x - v \perp \sum_{k=1}^{m_n}\mathcal{M}_k$. But $u_{x(n)}$ is the only vector in $\sum_{k=1}^{m_n}\mathcal{M}_k$ for which $x - u_{x(n)} \perp \sum_{k=1}^{m_n}\mathcal{M}_k$. Therefore $v = u_{x(n)}$. Outcome: $u_{x(n+1)} = u_{x(n)} + w = \sum_{k=1}^{m_{n+1}} u_k$ with $u_k \in \mathcal{M}_k$. □

Set $\{u_k\}_{k=1}^{\infty} = \bigcup_{n\geq 1}\{u_k\}_{k=1}^{n}$, which is a sequence of pairwise orthogonal vectors in $\mathcal{H}$ because $\{\mathcal{M}_k\}_{k\in\mathbb{N}}$ is an orthogonal family. Since $n \leq m_n$, it follows by Proposition 5.8 that

$$\sum_{k=1}^{n} \|u_k\|^2 \le \sum_{k=1}^{m_n} \|u_k\|^2 = \Big\| \sum_{k=1}^{m_n} u_k \Big\|^2 \le (\|x - x_{(n)}\| + \|x\|)^2.$$

But $x_{(n)} \to x$ in $\mathcal{H}$ as $n \to \infty$ so that $\sum_{k=1}^{\infty} \|u_k\|^2 \le \|x\|^2$. Thus the infinite series $\sum_{k=1}^{\infty} u_k$ converges in the Hilbert space $\mathcal{H}$, and $\|\sum_{k=1}^{\infty} u_k\|^2 = \sum_{k=1}^{\infty} \|u_k\|^2$ (see Corollary 5.9). Moreover, it in fact converges to x since the sequence of partial sums $\{\sum_{k=1}^{n} u_k\}_{n=1}^{\infty}$ has a subsequence, namely $\{\sum_{k=1}^{m_n} u_k\}_{n=1}^{\infty}$, that converges to x (see Proposition 3.5), and so $\|x\|^2 = \sum_{k=1}^{\infty} \|u_k\|^2$.

Uniqueness. Suppose $x = \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} v_k$, where u_k and v_k lie in $\mathcal{M}_k$ for each k. Thus $\sum_{k=1}^{\infty} (u_k - v_k) = 0$ (see Problem 4.9(a)). But $\{(u_k - v_k)\}_{k=1}^{\infty}$ is a sequence of pairwise orthogonal vectors in the inner product space $\mathcal{H}$ (since $u_k - v_k \in \mathcal{M}_k$ for each k and $\{\mathcal{M}_k\}_{k\in\mathbb{N}}$ is an orthogonal family), and therefore $\sum_{k=1}^{\infty} \|u_k - v_k\|^2 = \|\sum_{k=1}^{\infty} (u_k - v_k)\|^2 = 0$ by Corollary 5.9(a). Hence $u_k = v_k$ for every $k \ge 1$.

Converse. If $\{u_k\}_{k=1}^{\infty}$ is an $\mathcal{H}$-valued sequence with $u_k \in \mathcal{M}_k$ for each k, then it is a sequence of pairwise orthogonal vectors in $\mathcal{H}$ (since $\mathcal{M}_j \perp \mathcal{M}_k$ whenever $j \ne k$). If $\sum_{k=1}^{\infty} \|u_k\|^2 < \infty$, then the infinite series $\sum_{k=1}^{\infty} u_k$ converges in $\mathcal{H}$ according to Corollary 5.9(b); that is, $\sum_{k=1}^{n} u_k \to \sum_{k=1}^{\infty} u_k$ in $\mathcal{H}$ as $n \to \infty$. Since $\sum_{k=1}^{n} u_k$ lies in $\sum_{k\in\mathbb{N}} \mathcal{M}_k$ for each integer $n \ge 1$, it follows by Proposition 3.27 that the limit $\sum_{k=1}^{\infty} u_k$ lies in $\big(\sum_{k\in\mathbb{N}} \mathcal{M}_k\big)^-$. □

Here are two immediate consequences of the Orthogonal Structure Theorem.

Corollary 5.17. *If $\{\mathcal{M}_k\}_{k\in\mathbb{N}}$ is a countably infinite family of pairwise orthogonal subspaces of a Hilbert space $\mathcal{H}$, then*

$$\big(\textstyle\sum_{k\in\mathbb{N}} \mathcal{M}_k\big)^- = \big\{\sum_{k=1}^{\infty} u_k \in \mathcal{H}\colon\ u_k \in \mathcal{M}_k \text{ and } \sum_{k=1}^{\infty} \|u_k\|^2 < \infty\big\}.$$

Corollary 5.18. *Let $\{\mathcal{M}_k\}_{k\in\mathbb{N}}$ be a countably infinite orthogonal family of subspaces of a Hilbert space $\mathcal{H}$. If it spans $\mathcal{H}$; that is, if*

$$\bigvee_{k\in\mathbb{N}} \mathcal{M}_k = \mathcal{H},$$

then every vector x in $\mathcal{H}$ is uniquely expressed as

$$x = \sum_{k=1}^{\infty} u_k$$

in terms of an orthogonal sequence $\{u_k\}_{k=1}^{\infty}$ with each u_k in each $\mathcal{M}_k$.

Example 5.I. Let $\{\mathcal{M}_k\}$ be a countable collection of subspaces of a Hilbert space $(\mathcal{H}, \langle\ ;\ \rangle)$ so that each $(\mathcal{M}_k, \langle\ ;\ \rangle)$ is itself a Hilbert space. If $\{\mathcal{M}_k\}$ is an infinite collection, then let it be indexed by $\mathbb{N}$. Consider the direct sum $\bigoplus_k \mathcal{M}_k$. Recall that $\bigoplus_k \mathcal{M}_k$ is itself a linear space (but it is not a subset of $\mathcal{H}$). Let $\big[\bigoplus_k \mathcal{M}_k\big]_2$ denote the linear manifold of the full direct sum $\bigoplus_{k=1}^{\infty} \mathcal{M}_k$ made up of all square-summable sequences $\{x_k\}$ in $\bigoplus_{k=1}^{\infty} \mathcal{M}_k$. That is,

$$\big[\textstyle\bigoplus_k \mathcal{M}_k\big]_2 \,=\, \big\{\{x_k\} \in \bigoplus_k \mathcal{M}_k\colon\ \textstyle\sum_k \|x_k\|^2 < \infty\big\},$$

where $\|\ \|$ is the norm induced on $\mathcal{H}$ by the inner product $\langle\ ;\ \rangle$. (It is clear that $\big[\bigoplus_k \mathcal{M}_k\big]_2$ coincides with $\bigoplus_k \mathcal{M}_k$ if the collection $\{\mathcal{M}_k\}$ is finite.) The function $\langle\ ;\ \rangle_\oplus\colon \big[\bigoplus_k \mathcal{M}_k\big]_2 \times \big[\bigoplus_k \mathcal{M}_k\big]_2 \to \mathbb{F}$, given by

$$\langle x\,; y\rangle_\oplus \,=\, \sum_k \langle x_k\,; y_k\rangle$$

for every $x = \{x_k\} \in \big[\bigoplus_k \mathcal{M}_k\big]_2$ and $y = \{y_k\} \in \big[\bigoplus_k \mathcal{M}_k\big]_2$, is an inner product on $\big[\bigoplus_k \mathcal{M}_k\big]_2$ that makes it into a Hilbert space (Examples 5.E and 5.F). If the collection $\{\mathcal{M}_k\}$ consists of pairwise *orthogonal* subspaces of $\mathcal{H}$, then the Hilbert space $\big(\big[\bigoplus_k \mathcal{M}_k\big]_2, \langle\ ;\ \rangle_\oplus\big)$ is referred to as an (internal) *orthogonal direct sum*. In this case (i.e., if $\{\mathcal{M}_k\}$ is a collection of orthogonal subspaces), then Corollary 5.17 (or Corollary 5.11, if the collection is finite) says that

$$\big(\textstyle\sum_k \mathcal{M}_k\big)^- \,=\, \big\{\sum_k x_k \in \mathcal{H}\colon\ \{x_k\} \in \big[\bigoplus_k \mathcal{M}_k\big]_2\big\},$$

and

$$\big[\textstyle\bigoplus_k \mathcal{M}_k\big]_2 \,=\, \big\{\{x_k\} \in \bigoplus_k \mathcal{M}_k\colon\ \textstyle\sum_k x_k \in \big(\sum_k \mathcal{M}_k\big)^-\big\}.$$

This establishes a natural mapping $\Phi\colon \big[\bigoplus_k \mathcal{M}_k\big]_2 \to \big(\sum_k \mathcal{M}_k\big)^-$,

$$\Phi\big(\{x_k\}\big) \,=\, \sum_k x_k$$

for every $\{x_k\} \in \big[\bigoplus_k \mathcal{M}_k\big]_2$, which is injective and surjective (by the Orthogonal Structure Theorem) and linear as well (addition and scalar multiplication of summable sequences is again a summable sequence — Problem 4.9). Thus Φ is an isomorphism of the linear space $\big[\bigoplus_k \mathcal{M}_k\big]_2$ onto the linear manifold $\big(\sum_k \mathcal{M}_k\big)^-$ of the linear space $\mathcal{H}$. Hence the orthogonal direct sum $\big[\bigoplus_k \mathcal{M}_k\big]_2$ and the topological sum $\big(\sum_k \mathcal{M}_k\big)^-$ of pairwise orthogonal subspaces of a Hilbert space are isomorphic linear spaces. They are, in fact, isometrically isomorphic Hilbert spaces: the natural mapping Φ actually is an isometric isomorphism of the Hilbert space $\big[\bigoplus_k \mathcal{M}_k\big]_2$ onto the subspace $\big(\sum_k \mathcal{M}_k\big)^-$ of the Hilbert space $\mathcal{H}$ (next section). This provides a natural identification between them. It is customary to use the same notation of the full direct sum to denote $\big[\bigoplus_k \mathcal{M}_k\big]_2$ when it is equipped with the above inner product $\langle\ ;\ \rangle_\oplus$.

Notation: $$\textstyle\bigoplus_k \mathcal{M}_k \,=\, \big(\big[\bigoplus_k \mathcal{M}_k\big]_2, \langle\ ;\ \rangle_\oplus\big).$$

We shall follow the common usage. From now on $\bigoplus_k \mathcal{M}_k$ will denote the *Hilbert space* whose linear space is the set of all *square-summable* sequences in the full direct sum of a collection $\{\mathcal{M}_k\}$ of pairwise *orthogonal subspaces* of a *Hilbert space* $\mathcal{H}$.

Two subspaces $\mathcal{M}$ and $\mathcal{N}$ of an inner product space $\mathcal{X}$ are *complementary* in $\mathcal{X}$ if they are algebraic complements of each other (i.e., $\mathcal{M} + \mathcal{N} = \mathcal{X}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$; see Problem 4.34). Recall that $\mathcal{M} \perp \mathcal{N}$ implies $\mathcal{M} \cap \mathcal{N} = \{0\}$.

Proposition 5.19. *Orthogonal complementary subspaces in an inner product space are orthogonal complements of each other.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be orthogonal complementary subspaces in an inner product space $\mathcal{X}$. Take an arbitrary x in $\mathcal{M}^\perp \subseteq \mathcal{X}$. Since $\mathcal{M} + \mathcal{N} = \mathcal{X}$, it follows that $x = u + v$ with u in $\mathcal{M}$ and v in $\mathcal{N}$, and so

$$\langle x\,;u\rangle \;=\; \langle u\,;u\rangle + \langle v\,;u\rangle.$$

But $\langle x\,;u\rangle = \langle v\,;u\rangle = 0$ (because $\mathcal{M} \perp \mathcal{M}^\perp$ and $\mathcal{M} \perp \mathcal{N}$). Hence $\|u\|^2 = 0$, which means that $u = 0$. Thus $x = v \in \mathcal{N}$. Therefore, $\mathcal{M}^\perp \subseteq \mathcal{N}$. But $\mathcal{N} \subseteq \mathcal{M}^\perp$ because $\mathcal{M} \perp \mathcal{N}$. Outcome: $\mathcal{M}^\perp = \mathcal{N}$. □

The next result is the central theorem of Hilbert space geometry.

Theorem 5.20. (Projection Theorem – First version). *Every Hilbert space $\mathcal{H}$ can be decomposed as*

$$\mathcal{H} \;=\; \mathcal{M} + \mathcal{M}^\perp$$

where $\mathcal{M}$ is any subspace of $\mathcal{H}$.

Proof. Let $\mathcal{M}$ be any subspace of a Hilbert space $\mathcal{H}$. Since $\mathcal{M}^\perp$ is a subspace of $\mathcal{H}$ (Proposition 5.12) which is orthogonal to $\mathcal{M}$ (by definition), it follows by Theorem 5.10 that $\mathcal{M} + \mathcal{M}^\perp$ is a subspace of $\mathcal{H}$. Moreover, $\mathcal{M} \subseteq \mathcal{M} + \mathcal{M}^\perp$ and $\mathcal{M}^\perp \subseteq \mathcal{M} + \mathcal{M}^\perp$. Thus $(\mathcal{M} + \mathcal{M}^\perp)^\perp \subseteq \mathcal{M}^\perp \cap \mathcal{M}^{\perp\perp} = \mathcal{M}^\perp \cap \mathcal{M}^- = \mathcal{M}^\perp \cap \mathcal{M} = 0$, and so $\mathcal{M} + \mathcal{M}^\perp = (\mathcal{M} + \mathcal{M}^\perp)^- = \mathcal{H}$ (Proposition 5.15). □

Let $\mathcal{M}$ be an arbitrary subspace of a Hilbert space $\mathcal{H}$. Since $\mathcal{M}^\perp$ is again a subspace of $\mathcal{H}$, and since $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$, what Theorem 5.20 says is: *If $\mathcal{M}$ is any subspace of a Hilbert space $\mathcal{H}$, then $\mathcal{M}$ and $\mathcal{M}^\perp$ are complementary subspaces of $\mathcal{H}$.* This in fact is the converse of Proposition 5.19 in a Hilbert space setting. Moreover, according to Theorem 2.14, *for each $x \in \mathcal{H} = \mathcal{M} + \mathcal{M}^\perp$ there exists a unique u in $\mathcal{M}$ and a unique v in $\mathcal{M}^\perp$ such that $x = u + v$* and, by the Pythagorean Theorem, $\|x\|^2 = \|u\|^2 + \|v\|^2$.

5.6 Unitary Equivalence

An *isometry* between metric spaces is a map that preserves distance, which is obviously continuous. Proposition 4.37 placed linear isometries in a normed-space setting. The next one places it in an inner-product-space setting.

Proposition 5.21. *Let $\mathcal{X}$ and $\mathcal{Y}$ be inner product spaces. A linear transformation $V \in \mathcal{L}[\mathcal{X}, \mathcal{Y}]$ is an isometry if and only if*

$$\langle Vx_1\,;Vx_2\rangle \;=\; \langle x_1\,;x_2\rangle$$

for every $x_1, x_2 \in \mathcal{X}$.

Proof. The inner product on the left-hand side is the inner product on $\mathcal{Y}$, and that on the right-hand side is the inner product on $\mathcal{X}$. If the above identity holds, then $\|Vx\|^2 = \langle Vx\,;Vx\rangle = \langle x\,;x\rangle = \|x\|^2$ for every $x \in \mathcal{X}$. Conversely, if $\|Vx\| = \|x\|$ for every $x \in \mathcal{X}$, then $\langle Vx_1\,;Vx_2\rangle = \langle x_1\,;x_2\rangle$ for every $x_1, x_2 \in \mathcal{X}$ by the polarization identity of Proposition 5.4. Thus $\langle Vx_1\,;Vx_2\rangle = \langle x_1\,;x_2\rangle$ for every $x_1, x_2 \in \mathcal{X}$ if and only if $\|Vx\| = \|x\|$ for every $x \in \mathcal{X}$ or, equivalently, if and only if $V \in \mathcal{L}[\mathcal{X},\mathcal{Y}]$ is an isometry (Proposition 4.37). □

In other words, *a linear isometry between inner product spaces is a linear transformation that preserves the inner product*. Now recall that an *isomorphism* is an invertible linear transformation between linear spaces. These concepts (isometry and isomorphism) were combined in Section 4.7 to yield the notion of an *isometric isomorphism* between normed spaces, which is precisely a *linear surjective isometry*. Between inner product spaces it has a name of its own: an isometric isomorphism between inner product spaces is called a *unitary transformation*. That is, a unitary transformation of an inner product space *onto* an inner product space is a linear surjective isometry or, equivalently, an invertible linear isometry. According to Proposition 5.21, a *unitary transformation between inner product spaces is a linear-space isomorphism that preserves the inner product*. Thus a unitary transformation preserves the algebraic structure, the topological structure, and also the geometric structure between inner product spaces. In particular, it preserves convergence and Cauchy sequences (see introduction to Section 4.6), and so it also preserves separability and completeness (cf. Problem 3.48(b) and Theorem 3.44). Two inner product spaces, say $\mathcal{X}_1$ and $\mathcal{X}_2$, are called *unitarily equivalent* if there exists a unitary transformation between them (equivalently, if they are *isometrically isomorphic* — Notation: $\mathcal{X}_1 \cong \mathcal{X}_2$). Unitarily equivalent inner product spaces are regarded as essentially the same inner product space (i.e., they are indistinguishable, except perhaps by the nature of their points).

The continuous linear extension results of Section 4.7 (viz., Theorem 4.35 and Corollaries 4.36 and 4.38) are trivially extended to inner product spaces (simply replace "normed space", "Banach space", and "isometric isomorphism" with "inner product space", "Hilbert space", and "unitary transformation", respectively). The completion results of Section 4.7 are (almost) immediately translated into the inner product space language by just recalling that in an inner-product-space setting two spaces are unitarily equivalent if and only if they are isometrically isomorphic.

Definition 5.22. If the image of a linear isometry on an inner product space $\mathcal{X}$ is a dense linear manifold of a Hilbert $\mathcal{H}$, then $\mathcal{H}$ is a *completion* of $\mathcal{X}$. Equivalently, if an inner product space $\mathcal{X}$ is unitarily equivalent to a dense linear manifold of a Hilbert space $\mathcal{H}$, then $\mathcal{H}$ is a completion of $\mathcal{X}$.

Theorem 5.23. *Every inner product space has a completion. Any two completions of an inner product space are unitarily equivalent. If $\mathcal{H}$ and $\mathcal{K}$ are com-*

pletions of inner product spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively, then every $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ has an extension $\widehat{T} \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ over the completion $\mathcal{H}$ of $\mathcal{X}$ into the completion $\mathcal{K}$ of $\mathcal{Y}$. Moreover, $\widehat{T}$ is unique up to unitary transformations, and $\|\widehat{T}\| = \|T\|$.

Proof. Every inner product space has a completion as a normed space (Theorem 4.40). The only question that has not been answered in Theorems 4.40 through 4.42 is whether this completion (which is a Banach space) is a Hilbert space; in other words, whether the norm $\| \ \|_{\widehat{X}}$ in the proof of Theorem 4.40 satisfies the parallelogram law whenever the norm $\| \ \|_X$ does. Consider the setup of the proof of Theorem 4.40 and suppose $\mathcal{X}$ is an inner product space. Recall that

$$\|[\boldsymbol{x}]\|_{\widehat{X}} = \lim \|x_n\|_X,$$

where $\boldsymbol{x} = \{x_n\}$ is an arbitrary element of an arbitrary coset $[\boldsymbol{x}]$ in the quotient space $\widehat{\mathcal{X}}$. Take any pair of cosets $[\boldsymbol{x}]$ and $[\boldsymbol{y}]$ in $\widehat{\mathcal{X}}$. Since the norm $\| \ \|_X$ on the inner product space $\mathcal{X}$ satisfies the parallelogram law,

$$\begin{aligned}
\|[\boldsymbol{x}] + [\boldsymbol{y}]\|_{\widehat{X}}^2 + \|[\boldsymbol{x}] - [\boldsymbol{y}]\|_{\widehat{X}}^2 &= \|[\boldsymbol{x} + \boldsymbol{y}]\|_{\widehat{X}}^2 + \|[\boldsymbol{x} - \boldsymbol{y}]\|_{\widehat{X}}^2 \\
&= \lim \|x_n + y_n\|_X^2 + \lim \|x_n - y_n\|_X^2 \\
&= \lim \big(\|x_n + y_n\|_X^2 + \|x_n - y_n\|_X^2\big) \\
&= \lim 2\big(\|x_n\|_X^2 + \|y_n\|_X^2\big) \\
&= 2\big(\lim \|x_n\|_X^2 + \lim \|y_n\|_X^2\big) \\
&= 2\big(\|[\boldsymbol{x}]\|_{\widehat{X}}^2 + \|[\boldsymbol{y}]\|_{\widehat{X}}^2\big)
\end{aligned}$$

by continuity (apply Corollary 3.8 recalling that squaring, addition, and scalar multiplication are continuous). Thus the norm $\| \ \|_{\widehat{X}}$ on $\widehat{\mathcal{X}}$ also satisfies the parallelogram law so that this Banach space is, in fact, a Hilbert space. □

We shall now return to orthogonal complements. Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of an inner product space $\mathcal{X}$. If $\mathcal{M}$ and $\mathcal{N}$ are algebraic complements of each other (i.e., if $\mathcal{M} + \mathcal{N} = \mathcal{X}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$), then they are said to be *complementary linear manifolds* in $\mathcal{X}$. Consider the *natural mapping* Φ of the linear space $\mathcal{M} \oplus \mathcal{N}$ into the linear space $\mathcal{M} + \mathcal{N}$, which was defined by

$$\Phi((u, v)) = u + v$$

for each $(u, v) \in \mathcal{M} \oplus \mathcal{N}$ in Section 2.8. Let $\langle \ ; \ \rangle$ be the inner product on $\mathcal{X}$ and equip the direct sum $\mathcal{M} \oplus \mathcal{N}$ with the inner product $\langle \ ; \ \rangle_{\oplus}$, viz.,

$$\langle (u_1, v_1) ; (u_2 ; v_2) \rangle_{\oplus} = \langle u_1 ; u_2 \rangle + \langle v_1 ; v_2 \rangle$$

for every (u_1, v_1) and (u_2, v_2) in $\mathcal{M} \oplus \mathcal{N}$, as in Example 5.E.

Proposition 5.24. *Let $\mathcal{M}$ and $\mathcal{N}$ be orthogonal complementary linear manifolds in an inner product space $\mathcal{X}$. The natural mapping $\Phi: \mathcal{M} \oplus \mathcal{N} \to \mathcal{M} + \mathcal{N}$*

is a unitary transformation so that $\mathcal{M} \oplus \mathcal{N}$ *and* $\mathcal{X} = \mathcal{M} + \mathcal{N}$ *are unitarily equivalent* (i.e., $\mathcal{X} \cong \mathcal{M} \oplus \mathcal{N}$).

Proof. Since $\mathcal{M}$ and $\mathcal{N}$ are complementary linear manifolds in $\mathcal{X}$, it follows by Theorem 2.14 that Φ is an isomorphism of the linear space $\mathcal{M} \oplus \mathcal{N}$ onto the linear space $\mathcal{M} + \mathcal{N} = \mathcal{X}$. If $\mathcal{M} \perp \mathcal{N}$, then

$$\begin{aligned} \big\langle \Phi((u_1, v_1)) \,; \Phi((u_2, v_2)) \big\rangle &= \langle u_1 + v_1 \,; u_2 + v_2 \rangle \\ &= \langle u_1 \,; u_2 \rangle + \langle v_1 \,; v_2 \rangle \\ &= \langle (u_1, v_1) \,; (u_2 \,; v_2) \rangle_{\oplus} \end{aligned}$$

for every (u_1, v_1) and (u_2, v_2) in $\mathcal{M} \oplus \mathcal{N}$. Thus the natural mapping Φ is a linear-space isomorphism that preserves the inner product; that is, Φ is a unitary transformation. □

In light of Proposition 5.24, we may identify the inner product space $\mathcal{X}$ with the orthogonal direct sum $\mathcal{M} \oplus \mathcal{N}$ (equipped with the above inner product $\langle\ ;\ \rangle_{\oplus}$) through the natural mapping Φ, whenever $\mathcal{M}$ and $\mathcal{N}$ are orthogonal complementary linear manifolds in $\mathcal{X}$. Next suppose $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal complementary subspaces* in a *Hilbert space* $\mathcal{H}$. That is, suppose $\mathcal{M}$ and $\mathcal{N}$ are closed linear manifolds of a Hilbert space $\mathcal{H}$ such that $\mathcal{M} + \mathcal{N} = \mathcal{H}$ and $\mathcal{M} \perp \mathcal{N}$ (thus $\mathcal{M} \cap \mathcal{N} = \{0\}$). According to Proposition 5.19, $\mathcal{N} = \mathcal{M}^\perp$ so that $\mathcal{M} + \mathcal{M}^\perp = \mathcal{H}$, and hence $\mathcal{M} \oplus \mathcal{M}^\perp \cong \mathcal{H}$ by Proposition 5.24. In this case it is usual to identify the orthogonal direct sum $\mathcal{M} \oplus \mathcal{M}^\perp$ with its unitarily equivalent image $\Phi(\mathcal{M} \oplus \mathcal{M}^\perp) = \mathcal{M} + \mathcal{M}^\perp = \mathcal{H}$, and write $\mathcal{M} \oplus \mathcal{M}^\perp = \mathcal{H}$. Under this identification the central Theorem 5.20 can be restated as follows.

Theorem 5.25. (Projection Theorem – Second version). *Every Hilbert space* $\mathcal{H}$ *has an orthogonal direct sum decomposition*

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$$

where $\mathcal{M}$ *is any subspace of* $\mathcal{H}$.

Observe that both $\mathcal{M}$ and $\mathcal{M}^\perp$ are themselves Hilbert spaces, so that each of them can be further decomposed as direct sums of orthogonal complementary subspaces. The decomposition of Theorem 5.25 leads to a useful notation for the orthogonal complement of a *subspace* $\mathcal{M}$ of a *Hilbert space* $\mathcal{H}$, namely,

$$\mathcal{M}^\perp = \mathcal{H} \ominus \mathcal{M}.$$

Example 5.J. Let $\{\mathcal{M}_k\}$ be a countable collection of orthogonal subspaces of a Hilbert space $(\mathcal{H}, \langle\ ;\ \rangle)$. If $\{\mathcal{M}_k\}$ is countably infinite, then assume that it is indexed by $\mathbb{N}$. Consider the natural isomorphism Φ of $\bigoplus_k \mathcal{M}_k$ onto $\big(\sum_k \mathcal{M}_k\big)^-$ defined in Example 5.I,

$$\Phi\big(\{x_k\}\big) = \sum_k x_k \quad \text{for every } \{x_k\} \in \textstyle\bigoplus_k \mathcal{M}_k,$$

where $\bigoplus_k \mathcal{M}_k$ denotes the Hilbert space of all square-summable sequences in the full direct sum of $\{\mathcal{M}_k\}$, and the inner product on $\bigoplus_k \mathcal{M}_k$ is given by

$$\big\langle \{x_k\}\,;\{y_k\}\big\rangle_{\oplus} \;=\; \sum_k \langle x_k\,;y_k\rangle$$

for every $\{x_k\}$ and $\{y_k\}$ in $\bigoplus_k \mathcal{M}_k$ (see Example 5.I). Since $\mathcal{M}_j \perp \mathcal{M}_k$ whenever $j \neq k$, and since the inner product is continuous (in both arguments — Problem 5.6), it follows that

$$\begin{aligned}\big\langle \Phi\big(\{x_k\}\big)\,;\Phi\big(\{y_k\}\big)\big\rangle \;&=\; \Big\langle \sum_j x_j\,;\sum_k y_k\Big\rangle \;=\; \sum_j\sum_k \langle x_j\,;y_k\rangle \\ &=\; \sum_k \langle x_k\,;y_k\rangle \;=\; \big\langle \{x_k\}\,;\{y_k\}\big\rangle_{\oplus}\end{aligned}$$

for every pair of square-summable sequences $\{x_k\}$ and $\{y_k\}$ in $\bigoplus_k \mathcal{M}_k$. Thus the linear-space isomorphism Φ preserves the inner product, which means that it is a unitary transformation. Conclusion: The orthogonal direct sum $\bigoplus_k \mathcal{M}_k$ and the topological sum $\big(\sum_k \mathcal{M}_k\big)^-$ of pairwise orthogonal subspaces of a Hilbert space are unitarily equivalent Hilbert spaces. That is,

$$\bigoplus_k \mathcal{M}_k \;\cong\; \Big(\sum_k \mathcal{M}_k\Big)^-$$

if $\{\mathcal{M}_k\}$ is a collection of pairwise orthogonal subspaces of a Hilbert space. This, in fact, is a restatement of the Orthogonal Structure Theorem (Theorem 5.16). (Recall: For a finite collection, $\big(\sum_k \mathcal{M}_k\big)^- = \sum_k \mathcal{M}_k$ by Corollary 5.11, so that $\bigoplus_k \mathcal{M}_k \cong \sum_k \mathcal{M}_k$.) If, in addition, the orthogonal collection $\{\mathcal{M}_k\}$ spans $\mathcal{H}$ (i.e., if $\bigvee_k \mathcal{M}_k = \mathcal{H}$), then it is usual to identify the orthogonal direct sum $\bigoplus_k \mathcal{M}_k$ with its unitarily equivalent image $\Phi\big(\bigoplus_k \mathcal{M}_k\big) = \big(\sum_k \mathcal{M}_k\big)^- = \bigvee_k \mathcal{M}_k = \mathcal{H}$, and write $\bigoplus_k \mathcal{M}_k = \mathcal{H}$. Under this identification the result in Corollary 5.18 can be restated as follows.

If the orthogonal collection $\{\mathcal{M}_k\}$ spans $\mathcal{H}$, then $\mathcal{H} = \bigoplus_k \mathcal{M}_k$.

Remark: If $\{\mathcal{M}_i\}_{i=1}^n$ is a *finite* orthogonal collection of *linear manifolds* of a Hilbert space, then $\big(\bigoplus_{i=1}^n \mathcal{M}_i\big)^- = \bigoplus_{i=1}^n \mathcal{M}_i^-$ (by induction and Problem 5.7).

5.7 Summability

The first half of this section pertains to Banach spaces and, as such, could have been introduced in Chapter 4. The final part, however, is a genuine Hilbert space subject.

Definition 5.26. Let Γ be any index set and let $\{x_\gamma\}_{\gamma\in\Gamma}$ be an indexed family of vectors in a normed space $\mathcal{X}$. $\{x_\gamma\}_{\gamma\in\Gamma}$ is a *summable family* with *sum* $x \in \mathcal{X}$ (notation: $x = \sum_{\gamma\in\Gamma} x_\gamma$) if for each $\varepsilon > 0$ there exists a finite set of indices $N_\varepsilon \subseteq \Gamma$ such that

$$\Big\| \sum_{k\in N} x_k - \sum_{\gamma\in\Gamma} x_\gamma \Big\| < \varepsilon$$

for every finite subset N of Γ that includes N_ε (i.e., for every finite N such that $N_\varepsilon \subseteq N \subseteq \Gamma$). It is called a *p-summable family* for some $p \geq 1$ if $\{\|x_\gamma\|^p\}_{\gamma\in\Gamma}$ is a summable family of positive numbers. In particular, $\{x_\gamma\}_{\gamma\in\Gamma}$ is an *absolutely summable family* if $\{\|x_\gamma\|\}_{\gamma\in\Gamma}$ is a summable family, and a *square-summable family* if $\{\|x_\gamma\|^2\}_{\gamma\in\Gamma}$ is a summable family.

Remark: It is readily verified from Definition 5.26 that if $\{x_\gamma\}_{\gamma\in\Gamma}$ and $\{y_\gamma\}_{\gamma\in\Gamma}$ are two (similarly indexed) summable families of vectors in a normed space $\mathcal{X}$ with sums $\sum_{\gamma\in\Gamma} x_\gamma$ and $\sum_{\gamma\in\Gamma} y_\gamma$ in $\mathcal{X}$, respectively, then $\{\alpha x_\gamma + \beta y_\gamma\}_{\gamma\in\Gamma} = \alpha\{x_\gamma\}_{\gamma\in\Gamma} + \beta\{y_\gamma\}_{\gamma\in\Gamma}$ is again a summable family of vectors in $\mathcal{X}$ with sum

$$\sum_{\gamma\in\Gamma} (\alpha x_\gamma + \beta y_\gamma) = \alpha \sum_{\gamma\in\Gamma} x_\gamma + \beta \sum_{\gamma\in\Gamma} y_\gamma$$

for every pair of scalars α and β. This shows that the collection of all summable families of vectors in $\mathcal{X}$ is a linear manifold of the linear space $\mathcal{X}^\Gamma$ (cf. Example 2.F), and so is a linear space itself. It is also easy to verify that if $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family of vectors in $\mathcal{X}$ with sum $\sum_{\gamma\in\Gamma} x_\gamma$ in $\mathcal{X}$, and if $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is a bounded linear transformation of $\mathcal{X}$ into some normed space $\mathcal{Y}$, then $\{Tx_\gamma\}_{\gamma\in\Gamma}$ is a summable family of vectors in $\mathcal{Y}$ with sum

$$\sum_{\gamma\in\Gamma} Tx_\gamma = T \sum_{\gamma\in\Gamma} x_\gamma.$$

Theorem 5.27. (Cauchy Criterion). *Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be an indexed family of vectors in a normed space $\mathcal{X}$ and consider the following assertions.*

(a) *$\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family.*

(b) *For each $\varepsilon > 0$ there exists a finite set of indices $N_\varepsilon \subseteq \Gamma$ such that*

$$\Big\| \sum_{k\in N} x_k \Big\| < \varepsilon$$

for every finite subset N of Γ that is disjoint with N_ε (i.e., *whenever $N \subseteq \Gamma$ is finite and $N \cap N_\varepsilon = \varnothing$*).

Claim: (a) *implies* (b), *and* (b) *implies* (a) *if $\mathcal{X}$ is a Banach space.*

Proof. Suppose (a) holds true and take an arbitrary $\varepsilon > 0$. According to Definition 5.26 there exists a finite $N_\varepsilon \subseteq \Gamma$ such that

$$\Big\| \sum_{k\in N'} x_k - \sum_{\gamma\in\Gamma} x_\gamma \Big\| < \tfrac{\varepsilon}{2}$$

whenever N' is finite and $N_\varepsilon \subseteq N' \subseteq \Gamma$. If N is a finite subset of Γ such that $N \cap N_\varepsilon = \varnothing$, then $\sum_{k\in N} x_k = \sum_{k\in N\cup N_\varepsilon} x_k - \sum_{k\in N_\varepsilon} x_k$. Since $N \cup N_\varepsilon$ is finite, and since $N_\varepsilon \subseteq N \cup N_\varepsilon \subseteq \Gamma$, it follows that

$$\Big\|\sum_{k\in N} x_k\Big\| \le \Big\|\sum_{k\in N\cup N_\varepsilon} x_k - \sum_{\gamma\in\Gamma} x_\gamma\Big\| + \Big\|\sum_{k\in N_\varepsilon} x_k - \sum_{\gamma\in\Gamma} x_\gamma\Big\| < \varepsilon.$$

Thus (a)⇒(b). Conversely, if (b) holds true, then there exists a sequence $\{N_i'\}_{i=1}^\infty$ of finite subsets of Γ such that, for each $i\ge 1$, $\|\sum_{k\in N'} x_k\| < \frac{1}{i}$ whenever $N'\subseteq\Gamma$ is finite and $N'\cap N_i' = \varnothing$. Set $N_i = \bigcup_{j=1}^i N_j'$, which is a finite subset of Γ such that $N_i\subseteq N_{i+1}$ for each $i\ge 1$. Take any $i\ge 1$ and an arbitrary finite $N\subseteq\Gamma$ such that $N\cap N_i = \varnothing$. Since $N\cap N_i' = \varnothing$, it follows that $\|\sum_{k\in N} x_k\| < \frac{1}{i}$. Conclusion: $\{N_i\}_{i=1}^\infty$ is an *increasing* sequence of finite subsets of Γ such that, for each $i\ge 1$,

$$\Big\|\sum_{k\in N} x_k\Big\| < \tfrac{1}{i}$$

whenever $N\subseteq\Gamma$ is finite and $N\cap N_i = \varnothing$. Now set $y_i = \sum_{k\in N_i} x_k$ in $\mathcal{X}$ for each $i\ge 1$. Since $\{N_i\}_{i=0}^\infty$ is an increasing sequence, we get

$$y_{i+j} - y_i = \sum_{k\in N_{i+j}} x_k - \sum_{k\in N_i} x_k = \sum_{k\in N_{i+j}\setminus N_i} x_k$$

for every $i,j\ge 1$, which implies

$$\|y_{i+j} - y_i\| \le \Big\|\sum_{k\in N_{i+j}\setminus N_i} x_k\Big\| < \tfrac{1}{i}$$

for every $i\ge 1$ and all $j\ge 1$ (reason: $N_{i+j}\setminus N_i$ is a finite subset of Γ such that $(N_{i+j}\setminus N_i)\cap N_i = \varnothing$ for every $i,j\ge 1$). Hence (see Problem 3.51) $\{y_i\}_{i=1}^\infty$ is a Cauchy sequence in $\mathcal{X}$. If $\mathcal{X}$ is a Banach space, then $\{y_i\}_{i=1}^\infty$ converges in $\mathcal{X}$ to, say, $x\in\mathcal{X}$. Therefore, for each $\varepsilon>0$ there exists

(i) an integer $i_\varepsilon\ge 1$ such that $\|\sum_{k\in N_i} x_k - x\| < \frac{\varepsilon}{2}$ whenever $i\ge i_\varepsilon$

and, as (b) holds true,

(ii) a finite $N_\varepsilon\subseteq\Gamma$ such that $\|\sum_{k\in N'} x_k\| < \frac{\varepsilon}{2}$ whenever $N'\subseteq\Gamma$ is finite and $N'\cap N_\varepsilon = \varnothing$.

Since $N_\varepsilon\cup N_{i_\varepsilon}$ is finite and since $\{N_i\}_{i=1}^\infty$ is increasing, it follows that there exists an integer $i_\varepsilon^*\ge i_\varepsilon$, and consequently a finite subset $N_{i_\varepsilon^*}$ of Γ, such that $N_\varepsilon\cup N_{i_\varepsilon}\subseteq N_{i_\varepsilon^*}$. If N is finite and $N_{i_\varepsilon^*}\subseteq N\subseteq\Gamma$, then $N\setminus N_{i_\varepsilon^*}\subseteq\Gamma$ is finite and $(N\setminus N_{i_\varepsilon^*})\cap N_\varepsilon = \varnothing$. Hence

$$\begin{aligned}\Big\|\sum_{k\in N} x_k - x\Big\| &= \Big\|\sum_{k\in N\setminus N_{i_\varepsilon^*}} x_k + \sum_{k\in N_{i_\varepsilon^*}} x_k - x\Big\| \\ &\le \Big\|\sum_{k\in N\setminus N_{i_\varepsilon^*}} x_k\Big\| + \Big\|\sum_{k\in N_{i_\varepsilon^*}} x_k - x\Big\| < \varepsilon,\end{aligned}$$

and therefore (b)⇒(a). □

Corollary 5.28. *If $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family of vectors in a normed space $\mathcal{X}$, then the set $\{\gamma\in\Gamma\colon x_\gamma\neq 0\}$ is countable.*

Proof. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a summable family of vectors in a normed space $\mathcal{X}$. The previous theorem ensures that for every integer $n\geq 1$ there is a finite subset N_n of Γ such that $\|\sum_{k\in N}x_k\|<\frac{1}{n}$ whenever $N\subseteq\Gamma$ is finite and $N\cap N_n=\varnothing$. Set $S=\bigcup_{n=1}^{\infty}N_n\subseteq\Gamma$ and recall from Corollary 1.11 that S is a countable set. If $\gamma\in\Gamma\backslash S$, then $\{\gamma\}\cap N_n=\varnothing$, and hence $\|x_\gamma\|<\frac{1}{n}$, for every $n\geq 1$, which implies that $x_\gamma=0$. Thus x_γ is a nonzero vector in $\mathcal{X}$ only if γ lies in the countable set S. □

What Corollary 5.28 says is that an uncountable indexed family of vectors in a normed space may be summable but, in this case, it has only a countable number of nonzero vectors.

Corollary 5.29. *Every absolutely summable family of vectors in a Banach space is a summable family.*

Proof. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be an absolutely summable family of vectors in a normed space $\mathcal{X}$ so that $\{\|x_\gamma\|\}_{\gamma\in\Gamma}$ is a summable family of nonnegative numbers. Thus (cf. Theorem 5.27) for each $\varepsilon>0$ there exists a finite $N_\varepsilon\subseteq\Gamma$ such that

$$\Big\|\sum_{k\in N}x_k\Big\|\leq\sum_{k\in N}\|x_k\|<\varepsilon$$

for every finite $N\subseteq\Gamma$ such that $N\cap N_\varepsilon=\varnothing$. Another application of Theorem 5.27 ensures that $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family if $\mathcal{X}$ is a Banach space. □

The converse of Corollary 5.29 holds for finite-dimensional Banach spaces. That is, *if $\mathcal{X}$ is a finite-dimensional normed space, then every summable family of vectors in $\mathcal{X}$ is absolutely summable*. But it fails in general. In fact, Dvoretzky and Rogers proved in 1950 that there are summable families of vectors in infinite-dimensional Banach spaces that are not absolutely summable.

Proposition 5.30. *If $\mathcal{X}$ is a finite-dimensional normed space, then $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family of vectors in $\mathcal{X}$ if and only if it is absolutely summable.*

Proof. Recall that a finite-dimensional normed space is a Banach space (Corollary 4.28). Then, according to Corollary 5.29, it remains to show that every summable family of vectors in a finite-dimensional normed space is absolutely summable. Consider a normed space $(\mathcal{X},\|\ \|)$ with $\dim\mathcal{X}=n$ for some positive integer n and let $B=\{e_j\}_{j=1}^n$ be a Hamel basis for $\mathcal{X}$. Take an arbitrary $x\in\mathcal{X}$ and consider its *unique* expansion on B,

$$x=\sum_{j=1}^{n}\xi_j e_j,$$

where $\{\xi_j\}_{j=1}^n$ is a family of scalars; the coordinates of x with respect to the basis B. It is readily verified that the function $\|\ \|_1\colon\mathcal{X}\to\mathbb{R}$, defined by

$$\|x\|_1 = \sum_{j=1}^{n} |\xi_j|$$

for every $x \in \mathcal{X}$, is a norm on $\mathcal{X}$. Since any two norms on $\mathcal{X}$ are equivalent, it follows that there exist real constants $\alpha > 0$ and $\beta > 0$ such that

$$\|x\| \le \alpha\|x\|_1 \quad \text{and} \quad \|x\|_1 \le \beta\|x\|$$

for every $x \in \mathcal{X}$ (Proposition 4.26 and Theorem 4.27). Now suppose $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family of vectors in $\mathcal{X}$ and take an arbitrary $\varepsilon > 0$. Theorem 5.27 ensures the existence of a finite $N_\varepsilon \subseteq \Gamma$ such that

$$\sum_{j=1}^{n} \Big| \sum_{k\in N} \xi_{j(k)} \Big| = \Big\| \sum_{k\in N} x_k \Big\|_1 \le \beta \Big\| \sum_{k\in N} x_k \Big\| < \beta\varepsilon,$$

and hence

$$\Big| \sum_{k\in N} \xi_{j(k)} \Big| < \beta\varepsilon \quad \text{for every} \quad j = 1, \ldots, n,$$

whenever $N \subseteq \Gamma$ is finite and $N \cap N_\varepsilon = \varnothing$, where $\{\xi_{j(k)}\}_{j=1}^n$ are the coordinates of each $x_k \in \mathcal{X}$ with respect to the basis B. That is, $x_k = \sum_{j=1}^n \xi_{j(k)} e_j$ with $\|x_k\|_1 = \sum_{j=1}^n |\xi_{j(k)}|$, and therefore $\sum_{k\in N} x_k = \sum_{k\in N} \sum_{j=1}^n \xi_{j(k)} e_j = \sum_{j=1}^n \sum_{k\in N} \xi_{j(k)} e_j$ is the unique expansion of $\sum_{k\in N} x_k$ on B, which implies that $\|\sum_{k\in N} x_k\|_1 = \sum_{j=1}^n |\sum_{k\in N} \xi_{j(k)}|$. Take any finite subset N of Γ for which $N \cap N_\varepsilon = \varnothing$ and observe that

$$\begin{aligned} \sum_{k\in N} \|x_k\| &\le \alpha \sum_{k\in N} \|x_k\|_1 = \alpha \sum_{k\in N} \sum_{j=1}^{n} |\xi_{j(k)}| \\ &\le \alpha \sum_{j=1}^{n} \sum_{k\in N} |\operatorname{Re} \xi_{j(k)}| + \alpha \sum_{j=1}^{n} \sum_{k\in N} |\operatorname{Im} \xi_{j(k)}|. \end{aligned}$$

Set $N_j^+ = \{k \in N\colon \operatorname{Re}\xi_{j(k)} \ge 0\}$ and $N_j^- = \{k \in N\colon \operatorname{Re}\xi_{j(k)} < 0\}$ for each $j = 1, \ldots, n$, which (since they are subsets of N) are finite subsets of Γ such that $N_j^+ \cap N_\varepsilon = \varnothing$ and $N_j^- \cap N_\varepsilon = \varnothing$ for every $j = 1, \ldots, n$. Therefore, $|\sum_{k\in N_j^+} \xi_{j(k)}| < \beta\varepsilon$ and $|\sum_{k\in N_j^-} \xi_{j(k)}| < \beta\varepsilon$ for all $j = 1, \ldots, n$, and hence

$$\begin{aligned} \sum_{j=1}^{n} \sum_{k\in N} |\operatorname{Re} \xi_{j(k)}| &= \sum_{j=1}^{n} \Big| \sum_{k\in N_j^+} \operatorname{Re} \xi_{j(k)} \Big| + \sum_{j=1}^{n} \Big| \sum_{k\in N_j^-} \operatorname{Re} \xi_{j(k)} \Big| \\ &= \sum_{j=1}^{n} \Big| \operatorname{Re} \Big(\sum_{k\in N_j^+} \xi_{j(k)} \Big) \Big| + \sum_{j=1}^{n} \Big| \operatorname{Re} \Big(\sum_{k\in N_j^-} \xi_{j(k)} \Big) \Big| \\ &\le \sum_{j=1}^{n} \Big| \sum_{k\in N_j^+} \xi_{j(k)} \Big| + \sum_{j=1}^{n} \Big| \sum_{k\in N_j^-} \xi_{j(k)} \Big| < 2n\beta\varepsilon. \end{aligned}$$

Similarly, $\sum_{j=1}^{n}\sum_{k\in N}|\mathrm{Im}\,\xi_{j(k)}| < 2n\beta\varepsilon$, and so

$$\sum_{k\in N}\|x_k\| \;<\; 4n\alpha\beta\varepsilon.$$

Conclusion: $\{x_\gamma\}_{\gamma\in\Gamma}$ is an absolutely summable family. □

Remark: If a family of vectors in a normed space $\mathcal{X}$ is indexed by $\mathbb{N}$ (or by $\mathbb{N}_0$), then it can be viewed as an $\mathcal{X}$-valued sequence (the very indexing process establishes a function from $\mathbb{N}$ to $\mathcal{X}$). If $\{x_k\}_{k\in\mathbb{N}}$ is a summable family, then $\{x_k\}_{k=1}^{\infty}$ is a summable sequence (or, equivalently, the infinite series $\sum_{k=1}^{\infty}x_k$ converges). Indeed, if for each $\varepsilon>0$ there exists a finite $N_\varepsilon\subset\mathbb{N}$ such that $\|\sum_{k\in N}x_k - x\|<\varepsilon$ for some $x\in\mathcal{X}$ whenever N is finite and $N_\varepsilon\subseteq N\subset\mathbb{N}$, then by setting $n_\varepsilon=\#N_\varepsilon$ it follows that $\|\sum_{k=1}^{n}x_k - x\|<\varepsilon$ whenever $n\geq n_\varepsilon$. However, the converse fails even for scalar-valued sequences. For instance, consider the sequence $\{x_k\}_{k=1}^{\infty}$ with $x_{2k-1}=-x_{2k}=\frac{1}{k}$. The infinite series $\sum_{k=1}^{\infty}x_k$ converges (to zero, since $|\sum_{k=1}^{n}x_k|\leq\frac{2}{n+1}$ for each $n\geq1$), but it is not an *absolutely convergent series* (since $\sum_{k=1}^{2n}|x_k| = 2\sum_{k=1}^{n}\frac{1}{k}$ for each $n\geq1$). Thus $\{x_k\}_{k=1}^{\infty}$ is a summable sequence but not an absolutely summable sequence, and so $\{x_k\}_{k\in\mathbb{N}}$ is not an absolutely summable family of vectors in the one-dimensional normed space $\mathbb{R}$. Hence $\{x_k\}_{k\in\mathbb{N}}$ is not a summable family by Proposition 5.30. An $\mathcal{X}$-valued sequence $\{x_k\}_{k=1}^{\infty}$ for which $\{x_k\}_{k\in\mathbb{N}}$ is a summable family of vectors in $\mathcal{X}$ is referred to as an *unconditionally summable sequence*. In this case it is also common to say that the infinite series $\sum_{k=1}^{\infty}x_k$ is an *unconditionally convergent series*.

Proposition 5.31. *Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a family of vectors in a normed space $\mathcal{X}$ and take any $p\geq1$. The following assertions are equivalent.*

(a) *$\{x_\gamma\}_{\gamma\in\Gamma}$ is a p-summable family.*

(b) *$\sup_N\sum_{k\in N}\|x_k\|^p<\infty$, where the supremum is taken over all finite subsets of Γ, which is expressed by writing $\sum_{\gamma\in\Gamma}\|x_\gamma\|^p<\infty$.*

Moreover, if any of the above equivalent assertions holds, then

$$\sum_{\gamma\in\Gamma}\|x_\gamma\|^p \;=\; \sup_N\sum_{k\in N}\|x_k\|^p.$$

Proof. Suppose (a) holds true. Theorem 5.27 ensures that for each $\varepsilon>0$ there is a finite subset N_ε of Γ such that $\sum_{k\in N}\|x_k\|^p<\varepsilon$ for every finite subset N of Γ that is disjoint with N_ε (i.e., for every finite $N\subseteq\Gamma$ such that $N\cap N_\varepsilon=\varnothing$). If (b) fails, then for every integer $n\geq1$ there is a finite subset N_n of Γ such that $\sum_{k\in N_n}\|x_k\|^p>n$. Therefore, since $(N_n\backslash N_\varepsilon)\cap N_\varepsilon=\varnothing$, it follows that

$$n \;<\; \sum_{k\in N_n}\|x_k\|^p \;=\; \sum_{k\in N_n\cap N_\varepsilon}\|x_k\|^p \;+\; \sum_{k\in N_n\backslash N_\varepsilon}\|x_k\|^p \;<\; \sum_{k\in N_\varepsilon}\|x_k\|^p+\varepsilon$$

for every integer $n \geq 1$, which is a contradiction. Hence (a) implies (b). Conversely, suppose (b) holds and set $\alpha = \sup_N \sum_{k\in N} \|x_k\|^p$. Thus for every $\varepsilon > 0$ there exists a finite subset N_ε of Γ for which $\sum_{k\in N_\varepsilon} \|x_k\|^p > \alpha - \varepsilon$. If N is any finite subset of Γ such that $N \cap N_\varepsilon = \varnothing$, then

$$\sum_{k\in N} \|x_k\|^p \;=\; \sum_{k\in N\cup N_\varepsilon} \|x_k\|^p - \sum_{k\in N_\varepsilon} \|x_k\|^p \;<\; \alpha - (\alpha - \varepsilon) \;=\; \varepsilon.$$

Hence (b) implies (a) by Theorem 5.27 (in the Banach space $\mathbb{R}$). Finally, if any of the above equivalent assertions holds, then for every $\varepsilon > 0$ there exists a finite subset N_ε of Γ such that

$$\Big| \sum_{k\in N} \|x_k\|^p - \sum_{\gamma\in\Gamma} \|x_\gamma\|^p \Big| \;<\; \varepsilon$$

or, equivalently, such that

$$\sum_{k\in N} \|x_k\|^p < \sum_{\gamma\in\Gamma} \|x_\gamma\|^p + \varepsilon \quad \text{and} \quad \sum_{\gamma\in\Gamma} \|x_\gamma\|^p < \sum_{k\in N} \|x_k\|^p + \varepsilon,$$

whenever N is finite and $N_\varepsilon \subseteq N \subseteq \Gamma$ (Definition 5.26). Hence

$$\sup_N \sum_{k\in N} \|x_k\|^p \;\leq\; \sum_{\gamma\in\Gamma} \|x_\gamma\|^p + \varepsilon \;\leq\; \sup_N \sum_{k\in N} \|x_k\|^p + 2\varepsilon$$

for every $\varepsilon > 0$, with the supremum taken over all finite subsets of Γ (recall: $\sum_{k\in N_1} \|x_k\|^p \leq \sum_{k\in N_2} \|x_k\|^p$ whenever N_1 and N_2 are finite subsets of Γ such that $N_1 \subseteq N_2$). By taking the infimum over all $\varepsilon > 0$ in the above inequalities,

$$\sum_{\gamma\in\Gamma} \|x_\gamma\|^p \;=\; \sup_N \sum_{k\in N} \|x_k\|^p. \qquad \square$$

Remark: Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a family of vectors in a normed space $\mathcal{X}$. It is obvious that $\sum_{k\in N} \|x_k\|^p < \infty$ for every finite subset N of Γ. By convention, the *empty sum* is null (i.e., $\sum_{k\in\varnothing} \|x_k\|^p = 0$ — in general, $\sum_{k\in\varnothing} x_k = 0 \in \mathcal{X}$). If $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family, then it has only a countable set of nonzero vectors (cf. Corollary 5.28). If this set is infinite (countably infinite, that is), then the family of all nonzero vectors from $\{x_\gamma\}_{\gamma\in\Gamma}$ can be indexed by $\mathbb{N}$, say $\{x_k\}_{k\in\mathbb{N}}$ (or by any other countably infinite index set). Clearly, $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable (a p-summable) family if and only if $\{x_k\}_{k\in\mathbb{N}}$ is. If $\{x_k\}_{k=1}^\infty$ is *any* sequence containing all vectors from $\{x_k\}_{k\in\mathbb{N}}$, then it follows by the remark that precedes Proposition 5.31 that $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family if and only if $\{x_k\}_{k=1}^\infty$ is an unconditionally summable sequence. In particular, $\{x_\gamma\}_{\gamma\in\Gamma}$ is a p-summable family if and only if $\{\|x_k\|^p\}_{k=1}^\infty$ is an unconditionally summable sequence of positive numbers — but a sequence of *positive* numbers is unconditionally summable if and only if it is (plain) summable, isn't it? (Indeed, Proposition 5.30 ensures that *a series of real numbers is unconditionally convergent if and only if it is absolutely convergent*.)

Example 5.K. Let Γ be any index set and let ℓ^2_Γ denote the collection of all square-summable families $\{\xi_\gamma\}_{\gamma\in\Gamma}$ of scalars in $\mathbb{F}$ (as usual, $\mathbb{F}$ stands either for the real field $\mathbb{R}$ or for the complex field $\mathbb{C}$). It is easy to check that ℓ^2_Γ is a linear space over $\mathbb{F}$. Observe that $\{\xi_\gamma\overline{\upsilon}_\gamma\}_{\gamma\in\Gamma}$ is a summable family whenever $x=\{\xi_\gamma\}_{\gamma\in\Gamma}$ and $y=\{\upsilon_\gamma\}_{\gamma\in\Gamma}$ are square-summable families. Indeed, by the Hölder inequality for finite sums,

$$\sum_{k\in N}|\xi_k\overline{\upsilon}_k| \le \Big(\sum_{k\in N}|\xi_k|^2\Big)^{\frac{1}{2}}\Big(\sum_{k\in N}|\upsilon_k|^2\Big)^{\frac{1}{2}}$$

for every finite subset N of Γ. Therefore,

$$\sum_{\gamma\in\Gamma}|\xi_\gamma\overline{\upsilon}_\gamma| \le \Big(\sum_{\gamma\in\Gamma}|\xi_\gamma|^2\Big)^{\frac{1}{2}}\Big(\sum_{\gamma\in\Gamma}|\upsilon_\gamma|^2\Big)^{\frac{1}{2}}$$

by Proposition 5.31, which implies that $\{\xi_\gamma\overline{\upsilon}_\gamma\}_{\gamma\in\Gamma}$ is an absolutely summable family of scalars, and hence a summable family in the Banach space $\mathbb{F}$ (Corollary 5.29). Thus the function $\langle\ ;\ \rangle\colon \ell^2_\Gamma\times\ell^2_\Gamma\to\mathbb{F}$ given by

$$\langle x\,;y\rangle = \sum_{\gamma\in\Gamma}\xi_\gamma\overline{\upsilon}_\gamma$$

for every $x=\{\xi_\gamma\}_{\gamma\in\Gamma}$ and $y=\{\upsilon_\gamma\}_{\gamma\in\Gamma}$ in ℓ^2_Γ is well defined. Moreover, it is readily verified that it is an inner product on ℓ^2_Γ. In fact, it is not difficult to show that

$$(\ell^2_\Gamma,\langle\ ;\ \rangle) \text{ is a Hilbert space.}$$

In particular, if $\Gamma=\mathbb{N}$, then $(\ell^2_\Gamma,\langle\ ;\ \rangle)$ becomes the Hilbert space $(\ell^2_+,\langle\ ;\ \rangle)$ of Example 5.B. The natural generalization is easy. Let $\ell^2_\Gamma(\mathcal{H})$ denote the linear space of all square-summable families $\{x_\gamma\}_{\gamma\in\Gamma}$ of vectors in a Hilbert space $(\mathcal{H},\langle\ ;\ \rangle_H)$. Proceeding as above, it can also be shown that the function $\langle\ ;\ \rangle\colon\ell^2_\Gamma(\mathcal{H})\times\ell^2_\Gamma(\mathcal{H})\to\mathbb{F}$ given by

$$\big\langle\{x_\gamma\}_{\gamma\in\Gamma}\,;\{y_\gamma\}_{\gamma\in\Gamma}\big\rangle = \sum_{\gamma\in\Gamma}\langle x_\gamma\,;y_\gamma\rangle_H$$

for every $\{x_\gamma\}_{\gamma\in\Gamma}$ and $\{y_\gamma\}_{\gamma\in\Gamma}$ in $\ell^2_\Gamma(\mathcal{H})$ is an inner product on $\ell^2_\Gamma(\mathcal{H})$, and

$$(\ell^2_\Gamma(\mathcal{H}),\langle\ ;\ \rangle) \text{ is a Hilbert space.}$$

Again, if $\Gamma=\mathbb{N}$, then $(\ell^2_\Gamma(\mathcal{H}),\langle\ ;\ \rangle)$ becomes the Hilbert space $(\ell^2_+(\mathcal{H}),\langle\ ;\ \rangle)$ of Example 5.F.

From now on suppose $\mathcal{X}$ is an inner product space. It is easy to show using Definition 5.26 that if $\{x_\gamma\}_{\gamma\in\Gamma}$ and $\{y_\gamma\}_{\gamma\in\Gamma}$ are (similarly indexed) summable families of vectors in $\mathcal{X}$ with sums $\sum_{\gamma\in\Gamma}x_\gamma$ and $\sum_{\gamma\in\Gamma}y_\gamma$ in $\mathcal{X}$, respectively,

and if x and y are arbitrary vectors in $\mathcal{X}$, then $\{\langle x_\gamma\,;y\rangle\}_{\gamma\in\Gamma}$ and $\{\langle x\,;y_\gamma\rangle\}_{\gamma\in\Gamma}$ are summable families of scalars with sums

$$\sum_{\gamma\in\Gamma}\langle x_\gamma\,;y\rangle=\Big\langle\sum_{\gamma\in\Gamma}x_\gamma\,;y\Big\rangle\quad\text{and}\quad\sum_{\gamma\in\Gamma}\langle x\,;y_\gamma\rangle=\Big\langle x\,;\sum_{\gamma\in\Gamma}y_\gamma\Big\rangle.$$

The next result is the general version of the Pythagorean Theorem. It extends Corollary 5.9 to any infinite (not necessarily countable) orthogonal family of vectors in an inner product space.

Theorem 5.32. *Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a family of pairwise orthogonal vectors in a inner product space $\mathcal{X}$.*

(a) *If $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family, then it is a square-summable family and $\|\sum_{\gamma\in\Gamma}x_\gamma\|^2=\sum_{\gamma\in\Gamma}\|x_\gamma\|^2$.*

(b) *If $\mathcal{X}$ is a Hilbert space and $\{x_\gamma\}_{\gamma\in\Gamma}$ is a square-summable family, then $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family.*

Proof. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be an orthogonal family of vectors in $\mathcal{X}$.

(a) If $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family, then for each $\varepsilon>0$ there exists a finite $N_\varepsilon\subseteq\Gamma$ such that

$$\sum_{k\in N}\|x_k\|^2=\Big\|\sum_{k\in N}x_k\Big\|^2<\varepsilon^2$$

for every finite $N\subseteq\Gamma$ such that $N\cap N_\varepsilon=\varnothing$ (by Proposition 5.8 and Theorem 5.27). Another application of Theorem 5.27 ensures that $\{\|x_\gamma\|^2\}_{\gamma\in\Gamma}$ is a summable family in the Banach space $\mathbb{R}$. Moreover, since $x_\alpha\perp x_\beta$ whenever x_α and x_β are distinct vectors from $\{x_\gamma\}_{\gamma\in\Gamma}$ (i.e., $\langle x_\alpha\,;x_\beta\rangle=0$ for every $\alpha,\beta\in\Gamma$ such that $\alpha\neq\beta$), it follows that $\{\langle x_\gamma\,;x_\gamma\rangle\}_{\gamma\in\Gamma}$ includes the family of all nonzero scalars from the family $\{\langle x_\alpha\,;x_\beta\rangle\}_{(\alpha,\beta)\in\Gamma\times\Gamma}$. Therefore,

$$\begin{aligned}\Big\|\sum\nolimits_{\gamma\in\Gamma}x_\gamma\Big\|^2&=\Big\langle\sum_{\gamma\in\Gamma}x_\gamma\,;\sum_{\gamma\in\Gamma}x_\gamma\Big\rangle=\sum_{\alpha\in\Gamma}\Big\langle x_\alpha\,;\sum_{\beta\in\Gamma}x_\beta\Big\rangle\\&=\sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma}\langle x_\alpha\,;x_\beta\rangle=\sum_{\gamma\in\Gamma}\langle x_\gamma\,;x_\gamma\rangle=\sum_{\gamma\in\Gamma}\|x_\gamma\|^2.\end{aligned}$$

(b) If $\{\|x_\gamma\|^2\}_{\gamma\in\Gamma}$ is a summable family of nonnegative numbers, then for every $\varepsilon>0$ there exists a finite $N_\varepsilon\subseteq\Gamma$ such that

$$\Big\|\sum_{k\in N}x_k\Big\|=\Big(\sum_{k\in N}\|x_k\|^2\Big)^{\frac{1}{2}}<\varepsilon^{\frac{1}{2}}$$

whenever $N\subseteq\Gamma$ is finite and $N\cap N_\varepsilon=\varnothing$ (by Proposition 5.8 and Theorem 5.27 again). Another application of Theorem 5.27 ensures that $\{x_\gamma\}_{\gamma\in\Gamma}$ is a summable family in the Hilbert space $\mathcal{X}$. □

5.8 Orthonormal Basis

An orthogonal set in an inner product space $\mathcal{X}$ may contain the origin of $\mathcal{X}$, which is orthogonal to every vector in $\mathcal{X}$. The next proposition presents the main algebraic property of orthogonal sets that *do not* contain the origin.

Proposition 5.33. *An orthogonal set consisting of nonzero vectors in an inner product space is linearly independent.*

Proof. Let A be an orthogonal set consisting of nonzero vectors (i.e., set of pairwise orthogonal nonzero vectors) in an inner product space $\mathcal{X}$. Suppose there exists a finite subset of A containing more than one vector that is not linearly independent; for instance, $\{x_i\}_{i=0}^n \subseteq A$ such that $x_0 = \sum_{i=1}^n \alpha_i x_i$ for some integer $n \geq 1$ and some set of scalars $\{\alpha_i\}_{i=1}^n$. Since $x_0 \perp x_i$ for every $i = 1, \ldots, n$ we get $\|x_0\|^2 = \langle x_0 ; x_0 \rangle = \sum_{i=1}^n \alpha_i \langle x_i ; x_0 \rangle = 0$, which is a contradiction (because A does not contain the origin). Conclusion: Every finite subset of A containing more than one vector is linearly independent. Recall that every singleton $\{x\}$ in a linear space $\mathcal{X}$ such that $x \neq 0$ is linearly independent. Therefore, every finite subset of A is linearly independent, which means that A is itself linearly independent (Proposition 2.3). □

A *unit vector* in a normed space is a vector with norm equal to 1. An *orthonormal set* in an inner product space $\mathcal{X}$ is an orthogonal set consisting of unit vectors. That is, a subset A of $\mathcal{X}$ is an orthonormal set if $x \perp y$ for every pair $\{x, y\}$ of distinct vectors in A and $\|x\| = 1$ for every $x \in A$. Equivalently, $\{e_\gamma\}_{\gamma \in \Gamma}$ is an orthonormal family of vectors in $\mathcal{X}$ if $\langle e_\alpha ; e_\beta \rangle = \delta_{\alpha\beta}$ for every $\alpha, \beta \in \Gamma$, where $\delta_{\alpha\beta}$ is the Kronecker delta (i.e., $\langle e_\alpha ; e_\beta \rangle = 0$ for every $\alpha, \beta \in \Gamma$ such that $\alpha \neq \beta$ and $\|e_\gamma\|^2 = \langle e_\gamma ; e_\gamma \rangle = 1$ for every $\gamma \in \Gamma$). Each orthogonal set of nonzero vectors can be normalized. In fact, if A is an orthogonal set of nonzero vectors in $\mathcal{X}$, then the set $\{\|x\|^{-1} x \in \mathcal{X} \colon\ x \in A\} \subseteq \mathcal{X}$ is orthonormal. The next two results are immediate consequences of the definition of orthonormal sets (Proposition 5.34 below is a particular case of Proposition 5.33).

Proposition 5.34. *Every orthonormal set in any inner product space is linearly independent.*

Proposition 5.35. *If A is an orthonormal set in an inner product space $\mathcal{X}$, and if there exists $x \in \mathcal{X}$ such that $x \perp A$ and $\|x\| = 1$, then $A \cup \{x\}$ is an orthonormal set in $\mathcal{X}$ (that properly includes A).*

Let $\mathcal{O}$ be the collection of all orthonormal sets in an inner product space $\mathcal{X}$. We may argue by contradiction that *any singleton $\{x\}$ with $\|x\| = 1$ is an orthonormal set in $\mathcal{X}$* (although the expression "pairwise orthonormal" is meaningless in this case). As a subcollection of the power set $\wp(\mathcal{X})$, $\mathcal{O}$ is partially ordered in the inclusion ordering of $\wp(\mathcal{X})$. Recall from Section 1.5 that a set $A \in \mathcal{O}$ (i.e., an orthonormal set A in an inner product space $\mathcal{X}$) is maximal in $\mathcal{O}$ if there is no set $A' \in \mathcal{O}$ such that $A \subset A'$ (i.e., if there is no

orthonormal set A' in $\mathcal{X}$ that properly includes A). If A is maximal in $\mathcal{O}$, then we say that A is a *maximal orthonormal set* in the inner product space $\mathcal{X}$.

Proposition 5.36. *Let $\mathcal{X}$ be an inner product space and let A be an orthonormal set in $\mathcal{X}$. The following assertions are pairwise equivalent.*

(a) *A is a maximal orthonormal set in $\mathcal{X}$.*

(b) *There is no unit vector x for which $A \cup \{x\}$ is an orthonormal set.*

(c) *If $x \perp A$, then $x = 0$* (i.e., $A^\perp = \{0\}$).

Proof. Let A be an orthonormal set in an inner product space $\mathcal{X}$.

Proof of (a)⇒(b). If there exists a unit vector x in $\mathcal{X}$ for which $A \cup \{x\}$ is an orthonormal set, then $A \cup \{x\}$ is an orthonormal set that properly includes A (since $x \perp A$). Thus A is not a maximal orthonormal set in $\mathcal{X}$.

Proof of (b)⇒(c). If there exists a nonzero vector x in $\mathcal{X}$ such that $x \perp A$, then there exists a unit vector $x' = \|x\|^{-1}x$ in $\mathcal{X}$ such that $A \cup \{x'\}$ is an orthonormal set.

Proof of (c)⇒(a). If (a) fails, then there exists an orthonormal set A' in $\mathcal{X}$ that properly includes A so that $A' \backslash A \neq \varnothing$. Take any x in $A' \backslash A$, which is a nonzero vector (actually, x is a unit vector) orthogonal to A (because $x \in A'$, $A \subset A'$, and A' is an orthonormal set). Thus (c) fails. □

Proposition 5.37. *If A is an orthonormal set in an inner product space $\mathcal{X}$, then there exists a maximal orthonormal set B in $\mathcal{X}$ such that $A \subseteq B$.*

Proof. Let A be an orthonormal set in an inner product space $\mathcal{X}$. Set

$$\mathcal{O}_A = \big\{S \in \wp(\mathcal{X}):\ S \text{ is an orthonormal set in } \mathcal{X} \text{ and } A \subseteq S\big\},$$

the collection of all orthonormal sets in $\mathcal{X}$ that include A. As a nonempty subcollection (e.g., $A \in \mathcal{O}_A$) of the power set $\wp(\mathcal{X})$, $\mathcal{O}_A$ is partially ordered in the inclusion ordering. Take an arbitrary chain $\mathcal{C}$ in $\mathcal{O}_A$ and consider the union $\bigcup \mathcal{C}$ of all sets in $\mathcal{C}$. If x and y are distinct vectors in $\bigcup \mathcal{C}$, then $x \in C$ and $y \in D$, where $C, D \in \mathcal{C} \subseteq \mathcal{O}_A$. Since $\mathcal{C}$ is a chain, it follows that either $C \subseteq D$ or $D \subseteq C$. Suppose (without loss of generality) that $C \subseteq D$. Thus $x, y \in D$, and so $\bigcup \mathcal{C}$ is an orthonormal set (since $D \in \mathcal{O}_A$). Moreover, $A \subseteq \bigcup \mathcal{C}$ (reason: every set in $\mathcal{C}$ includes A). Outcome: $\bigcup \mathcal{C} \in \mathcal{O}_A$. Since $\bigcup \mathcal{C}$ is an upper bound for $\mathcal{C}$, we may conclude that every chain in $\mathcal{O}_A$ has an upper bound in $\mathcal{O}_A$. Hence $\mathcal{O}_A$ has a maximal element by Zorn's Lemma. Thus let B be a maximal element of $\mathcal{O}_A$, which clearly is an orthonormal set in $\mathcal{X}$ that includes A. If there is a unit vector x in $\mathcal{X}$ such that $B \cup \{x\}$ is an orthonormal set, then $B \cup \{x\}$ lies in $\mathcal{O}_A$ and properly includes B, which contradicts the fact that B is a maximal element of $\mathcal{O}_A$. Therefore, there is no unit vector x in $\mathcal{X}$ such that $B \cup \{x\}$ is an orthonormal set, and hence B is a maximal orthonormal set in $\mathcal{X}$ (Proposition 5.36). □

Proposition 5.37 says that there are plenty of maximal orthonormal sets in any inner product space of dimension greater than 1. The next proposition says that the maximal orthonormal sets in a Hilbert space $\mathcal{H}$ are precisely those orthonormal sets that span $\mathcal{H}$.

Proposition 5.38. *Let A be an orthonormal set in an inner product space $\mathcal{X}$.*

(a) *If $\bigvee A = \mathcal{X}$, then A is a maximal orthonormal set.*

(b) *If $\mathcal{X}$ is a Hilbert space and A is a maximal orthonormal set, then $\bigvee A = \mathcal{X}$.*

Proof. Take any orthonormal set A in $\mathcal{X}$. According to Proposition 5.36, A is a maximal orthonormal set if and only if $A^\perp = \{0\}$. If $\bigvee A = \mathcal{X}$, then $\big(\bigvee A\big)^\perp = \mathcal{X}^\perp = \{0\}$. But Proposition 5.12 says that $A^\perp = \big(\bigvee A\big)^\perp$, and hence $A^\perp = \{0\}$. The converse is an immediate consequence of Proposition 5.15. If $\mathcal{X}$ is a Hilbert space and $A^\perp = \{0\}$, then $\bigvee A = \mathcal{X}$. □

An orthonormal set in an inner product space $\mathcal{X}$ that spans $\mathcal{X}$ is called an *orthonormal basis* (or a *Hilbert basis*) for $\mathcal{X}$. In other words, a subset B of an inner product space $\mathcal{X}$ is an orthonormal basis if

$$\begin{aligned} &\text{(i)} \quad B \text{ is an orthonormal set, and} \\ &\text{(ii)} \quad \textstyle\bigvee B = \mathcal{X}. \end{aligned}$$

This is a combined topological and algebraic concept, while the Hamel basis of Section 2.4 is a purely algebraic concept. However, *every orthonormal basis for a given inner product space $\mathcal{X}$ is included in a Hamel basis for the linear space $\mathcal{X}$* (Proposition 5.34 and Theorem 2.5). Recall from Proposition 5.37 that *every nonzero inner product space has a maximal orthonormal set*. Using the above terminology, Proposition 5.38(a) says that *if B is an orthonormal basis for an inner product space $\mathcal{X}$, then B is a maximal orthonormal set in $\mathcal{X}$*. Note that this does not ensure the existence of orthonormal bases in incomplete inner product spaces, but in nonzero Hilbert spaces they do exist. In fact, in a Hilbert-space setting the concepts of maximal orthonormal set and orthonormal basis coincide (Proposition 5.38). That is, *B is an orthonormal basis for a Hilbert space $\mathcal{H}$ if and only if B is a maximal orthonormal set in $\mathcal{H}$*. Therefore (cf. Proposition 5.37 again), *every nonzero Hilbert space has an orthonormal basis*. As we could expect (suggested perhaps by Section 2.4), *the cardinality of all maximal orthonormal sets in an inner product space $\mathcal{X}$ is an invariant for $\mathcal{X}$*. First we prove this statement for finite-dimensional spaces.

Proposition 5.39. *Let $\mathcal{X}$ be an inner product space.*

(a) *If $\mathcal{X}$ is finite dimensional, then every orthonormal basis for $\mathcal{X}$ is a Hamel basis for $\mathcal{X}$.*

(b) *If there exists an orthonormal basis for $\mathcal{X}$ with a finite number of vectors, then every orthonormal basis for $\mathcal{X}$ is a Hamel basis for $\mathcal{X}$.*

Consequently, in both cases, every orthonormal basis for $\mathcal{X}$ has the same finite cardinality, which coincides with the linear dimension of $\mathcal{X}$.

Proof. Let $\mathcal{X}$ be an inner product space.

(a) If $\mathcal{X}$ is finite dimensional, then $\mathcal{M} = \mathcal{M}^-$ for every linear manifold $\mathcal{M}$ of $\mathcal{X}$. In particular, $\operatorname{span} B = \bigvee B$ for every orthonormal basis B for $\mathcal{X}$. Recall that every orthonormal basis for $\mathcal{X}$ is linearly independent (Proposition 5.34). Therefore, if B is an orthonormal basis for $\mathcal{X}$, then B is a linearly independent subset of $\mathcal{X}$ such that $\operatorname{span} B = \mathcal{X}$; that is, B is a Hamel basis for $\mathcal{X}$.

(b) If $B = \{e_i\}_{i=1}^n$ is an orthonormal basis for $\mathcal{X}$, then $\operatorname{span}\{e_i\}_{i=1}^n$ is an n-dimensional linear manifold of $\mathcal{X}$, which in fact is a subspace of $\mathcal{X}$ (Corollary 4.29). Hence $\operatorname{span} B = \bigvee B$. But $\bigvee B = \mathcal{X}$ so that B is a linearly independent subset of $\mathcal{X}$ (Proposition 5.34) such that $\operatorname{span} B = \mathcal{X}$. In other words, B is a Hamel basis for $\mathcal{X}$.

Finally, recall that the cardinality of any Hamel basis for $\mathcal{X}$ is an invariant for $\mathcal{X}$: the linear dimension of $\mathcal{X}$ (Theorem 2.7). Thus, in both cases, $\# B = \dim \mathcal{X}$ in $\mathbb{N}$ for every orthonormal basis B for $\mathcal{X}$. □

To verify such an invariance for infinite-dimensional spaces, we shall use the following fundamental inequality.

Lemma 5.40. (Bessel Inequality). *Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be a family of vectors in an inner product space $\mathcal{X}$ and let x be any vector in $\mathcal{X}$. If $\{e_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal family, then $\{\langle x\,;e_\gamma\rangle\}_{\gamma\in\Gamma}$ is a square-summable family of scalars and*

$$\sum_{\gamma\in\Gamma} |\langle x\,;e_\gamma\rangle|^2 \le \|x\|^2.$$

Proof. Take an arbitrary $x \in \mathcal{X}$ and an arbitrary finite set of indices $N \subseteq \Gamma$. Since $\{e_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal family, it follows by the Pythagorean Theorem (Proposition 5.8) that

$$\begin{aligned} 0 &\le \Big\| \sum_{k\in N} \langle x\,;e_k\rangle e_k - x \Big\|^2 \\ &= \Big\| \sum_{k\in N} \langle x\,;e_k\rangle e_k \Big\|^2 - 2\operatorname{Re}\sum_{k\in N} \langle x\,;e_k\rangle\langle e_k\,;x\rangle + \|x\|^2 \\ &= \sum_{k\in N} |\langle x\,;e_k\rangle|^2 - 2\sum_{k\in N} |\langle x\,;e_k\rangle|^2 + \|x\|^2, \end{aligned}$$

and hence

$$\sum_{k\in N} |\langle x\,;e_k\rangle|^2 \le \|x\|^2.$$

Therefore, as N is an arbitrary finite subset of Γ,

$$\sum_{\gamma\in\Gamma} |\langle x\,;e_\gamma\rangle|^2 \;=\; \sup_N \sum_{k\in N} |\langle x\,;e_k\rangle|^2 \;\le\; \|x\|^2,$$

where the supremum is taken over all finite subsets of Γ. The family of scalars $\{\langle x\,;e_\gamma\rangle\}_{\gamma\in\Gamma}$ is then square-summable by Proposition 5.31. □

Corollary 5.41. *Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be any orthonormal family of vectors in an inner product space $\mathcal{X}$. For each $x\in\mathcal{X}$ the set $\{\gamma\in\Gamma\colon \langle x\,;e_\gamma\rangle\neq 0\}$ is countable.*

Proof. According to Lemma 5.40 $\{|\langle x\,;e_\gamma\rangle|^2\}_{\gamma\in\Gamma}$ is a summable family of nonnegative numbers. Thus Corollary 5.28 ensures that $\{\gamma\in\Gamma\colon |\langle x\,;e_\gamma\rangle|^2\neq 0\}$ is a countable set. Equivalently, $\{\gamma\in\Gamma\colon \langle x\,;e_\gamma\rangle\neq 0\}$ is a countable set. □

Theorem 5.42. *Every orthonormal basis for a given Hilbert space has the same cardinality.*

Proof. Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{H}$ is finite dimensional, then the above result follows by Proposition 5.39. Suppose $\mathcal{H}$ is infinite dimensional and let B and C be orthonormal bases for $\mathcal{H}$. Proposition 5.39(b) ensures that B and C are infinite, and so $\#\mathbb{N}\le \#B$. For each $b\in B$ set $C_b=\{c\in C\colon \langle c\,;b\rangle\neq 0\}$. According to Corollary 5.41, $\#C_b\le\#\mathbb{N}$, and hence $\#C_b\le\#B$ for all $b\in B$. Then, since B is an infinite set (cf. Theorems 1.10 and 1.9),

$$\#\Big(\bigcup_{b\in B} C_b\Big) \;\le\; \#\big(B\times B\big) \;=\; \#B.$$

If $c\in C$, then $c\in C_b$ for some $b\in B$ (reason: since B is a maximal orthonormal set, it follows that if $c\in C$ and $c\perp B$, then $c=0$, which contradicts the fact that $\|c\|=1$; hence $\langle c\,;b\rangle\neq 0$ for some $b\in B$). Therefore, $C\subseteq\bigcup_{b\in B}C_b$. But $\bigcup_{b\in B}C_b\subseteq C$ (each C_b is a subset of C). Thus $C=\bigcup_{b\in B}C_b$ and, consequently, $\#C\le\#B$. Since C also is infinite, we may swap B and C, apply the same argument, and get $\#B\le\#C$. Hence $\#C=\#B$ by the Cantor–Bernstein Theorem (Theorem 1.6). □

The preceding theorem might be restated as "*every maximal orthonormal set in a given inner product space $\mathcal{X}$ has the same cardinality*". The proof remains essentially the same. Such an invariant (i.e., the cardinality of every orthonormal basis for a given Hilbert space) is called the *orthogonal dimension* of the Hilbert space $\mathcal{H}$. According to Proposition 5.39, the orthogonal dimension of $\mathcal{H}$ is finite if and only if its linear dimension is finite and, in this case, these two dimensions coincide. Thus the concepts of "infinite-dimensional" and 'finite-dimensional" Hilbert spaces are unambiguously defined.

Proposition 5.43. *If $\mathcal{X}$ is a separable inner product space, then every orthonormal set in $\mathcal{X}$ is countable.*

Proof. Let $B=\{e_\gamma\}_{\gamma\in\Gamma}$ be any family of orthonormal vectors in the inner product space $\mathcal{X}$. If $\mathcal{X}$ is separable (as a metric space), then there exists a

countable dense subset $A = \{a_k\}_{k\in\mathbb{N}}$ of $\mathcal{X}$. Since A is dense in $\mathcal{X}$, it follows by Proposition 3.32 that every nonempty open ball centered at any point of $\mathcal{X}$ meets A. In particular, every open ball of radius, say $\frac{1}{2}$, centered at any point of B meets A. Then for each $\gamma \in \Gamma$ there exists a least integer $k_\gamma \in \mathbb{N}$ such that $\|e_\gamma - a_{k_\gamma}\| < \frac{1}{2}$. This establishes a function $F\colon \Gamma \to \mathbb{N}$ that assigns to each $\gamma \in \Gamma$ the integer $k_\gamma \in \mathbb{N}$. We show that this function is injective. In fact, consider the family $\{a_{k_\gamma}\}_{\gamma\in\Gamma}$. Recall that B is an orthonormal set. Thus

$$\begin{aligned}\sqrt{2} = \|e_\alpha - e_\beta\| &= \|e_\alpha - a_{k_\alpha} - e_\beta + a_{k_\beta} + a_{k_\alpha} - a_{k_\beta}\| \\ &\le \|e_\alpha - a_{k_\alpha}\| + \|e_\beta - a_{k_\beta}\| + \|a_{k_\alpha} - a_{k_\beta}\| \\ &\le 1 + \|a_{k_\alpha} - a_{k_\beta}\|,\end{aligned}$$

and so $\|a_{k_\alpha} - a_{k_\beta}\| \ge \sqrt{2} - 1 > 0$, for every pair of distinct indices $\alpha, \beta \in \Gamma$. This implies that $k_\alpha \ne k_\beta$ whenever $\alpha \ne \beta$, which means that $F\colon \Gamma \to \mathbb{N}$ is injective. Therefore $\#\Gamma \le \#\mathbb{N}$, and so B is countable. □

Recall that every (nonzero) Hilbert space has an orthonormal basis. The next theorem characterizes the (nonzero) separable Hilbert spaces in terms of their orthogonal dimension.

Theorem 5.44. *A nonzero Hilbert space is separable if and only if it has a countable orthonormal basis.*

Proof. Propositions 4.9(b) and 5.43. □

We close this section with a useful result for constructing countable orthonormal sets, which is known as the *Gram-Schmidt orthogonalization process*.

Proposition 5.45. *If $A = \{a_k\}$ is a countable linearly independent nonempty set in an inner product space $\mathcal{X}$, then there exists a countable orthonormal set $B = \{e_k\}$ in $\mathcal{X}$ with the following property:* $\operatorname{span}\{e_k\}_{k=1}^n = \operatorname{span}\{a_k\}_{k=1}^n$ *for every integer n such that $1 \le n \le \#A$, and hence* $\operatorname{span} B = \operatorname{span} A$.

Proof. Let $A = \{a_k\}$ be a countable (either a finite and nonempty or a countably infinite) linearly independent subset of an inner product space $\mathcal{X}$.

Claim. For every integer $n \ge 1$ ($n \le \#A$) there exists an orthonormal subset $\{e_k\}_{k=1}^n$ of $\mathcal{X}$ such that $\operatorname{span}\{e_k\}_{k=1}^n = \operatorname{span}\{a_k\}_{k=1}^n$.

Proof. $a_1 \ne 0$ (because $A = \{a_k\}$ is a linearly independent subset of $\mathcal{X}$). Set $e_1 = \|a_1\|^{-1} a_1$ in $\mathcal{X}$ so that the result holds for $n = 1$. (Recall that any singleton $\{x\}$ such that $\|x\| = 1$ is an orthonormal set in $\mathcal{X}$.) If $\#A = 1$ the proposition is proved. Thus assume that $1 < \#A \le \aleph_0$. Suppose the result holds for some integer $n \ge 1$ such that $n \le \#A$. Observe that

$$a_{n+1} - \sum_{k=1}^{n} \langle a_{n+1}; e_k\rangle e_k \ \ne\ 0$$

(otherwise $a_{n+1} \in \operatorname{span}\{e_k\}_{k=1}^n = \operatorname{span}\{a_k\}_{k=1}^n$, which contradicts the fact that A is linearly independent). Set

$$e_{n+1} \;=\; \beta_{n+1}\Big(a_{n+1} - \sum_{k=1}^{n} \langle a_{n+1}\,;e_k\rangle e_k\Big),$$

where $\beta_{n+1} = \|a_{n+1} - \sum_{k=1}^n \langle a_{n+1}\,;e_k\rangle e_k\|^{-1}$ so that $\|e_{n+1}\| = 1$. Take an arbitrary $j = 1, \ldots, n$. Since $\{e_k\}_{k=1}^n$ is an orthonormal set, we get

$$\begin{aligned}\langle e_{n+1}\,;e_j\rangle \;&=\; \beta_{n+1}\Big(\langle a_{n+1}\,;e_j\rangle - \sum_{k=1}^{n}\langle a_{n+1}\,;e_k\rangle\langle e_k\,;e_j\rangle\Big)\\ &=\; \beta_{n+1}\big(\langle a_{n+1}\,;e_j\rangle - \langle a_{n+1}\,;e_j\rangle\big) \;=\; 0,\end{aligned}$$

and hence $e_{n+1} \perp \{e_k\}_{k=1}^n$. Therefore, $\{e_k\}_{k=1}^{n+1}$ is an orthonormal set. But $e_{n+1} \in \operatorname{span}(\{e_k\}_{k=1}^n \cup \{a_{n+1}\}) = \operatorname{span}\{a_k\}_{k=1}^{n+1}$. Thus

$$\begin{aligned}\operatorname{span}\{e_k\}_{k=1}^{n+1} \;&=\; \operatorname{span}(\{e_k\}_{k=1}^n \cup \{e_{n+1}\})\\ &=\; \operatorname{span}(\{a_k\}_{k=1}^n \cup \{e_{n+1}\}) \;=\; \operatorname{span}\{a_k\}_{k=1}^{n+1},\end{aligned}$$

which completes the proof by induction. □

Finally, set $B = \bigcup_{n=1}^{\#A}\{e_k\}_{k=1}^n$. Since $\{e_k\}_{k=1}^n \subseteq \{e_k\}_{k=1}^{n+1}$ for each integer $n \geq 1$ such that $n \leq \#A$, it follows that if e and e' are distinct vectors in B, then they belong to $\{e_k\}_{k=1}^m$ for some integer m such that $1 \leq m \leq \#A$. Since $\{e_k\}_{k=1}^m$ is an orthonormal set, $e \perp e'$ and $\|e\| = \|e'\| = 1$. Thus B is an orthonormal set. Moreover, since $\operatorname{span}\{e_k\}_{k=1}^n = \operatorname{span}\{a_k\}_{k=1}^n$ for every integer $n \geq 1$ such that $n \leq \#A$, then $\operatorname{span} B = \operatorname{span} A$. □

Corollary 5.46. *There is no Hilbert space with a countably infinite Hamel basis. In other words, a Hamel basis for a Hilbert space is either finite or uncountable.*

Proof. Let $\mathcal{H}$ be a Hilbert space and suppose there exists a countably infinite Hamel basis for the linear space $\mathcal{H}$, say $\{f_k\}_{k=1}^\infty$. The Gram-Schmidt orthogonalization process ensures the existence of a countably infinite orthonormal set, say $\{e_k\}_{k=1}^\infty$, such that $\operatorname{span}\{e_k\}_{k=1}^n = \operatorname{span}\{f_k\}_{k=1}^n$ for every $n \geq 1$. If $\{\alpha_k\}_{k=1}^\infty$ is a square-summable sequence of scalars, then $\{\alpha_k e_k\}_{k=1}^\infty$ is a square-summable sequence of pairwise orthogonal vectors in $\mathcal{H}$, and so the infinite series $\sum_{k=1}^\infty \alpha_k e_k$ converges in the Hilbert space $\mathcal{H}$ by Corollary 5.9(b). Take any square-summable sequence $\{\alpha_k\}_{k=1}^\infty$ of nonzero scalars (e.g., $\alpha_k = \frac{1}{k}$ for each $k \geq 1$) and put $x = \sum_{k=1}^\infty \alpha_k e_k$ in $\mathcal{H}$. Since $x \notin \operatorname{span}\{e_k\}_{k=1}^n$, and since $\operatorname{span}\{e_k\}_{k=1}^n = \operatorname{span}\{f_k\}_{k=1}^n$, for every $n \geq 1$, it follows that $x \notin \operatorname{span}\{f_k\}_{k=1}^\infty$ (i.e., x is not a finite linear combination of vectors from $\{f_k\}_{k=1}^\infty$), which contradicts the assumption that $\{f_k\}_{k=1}^\infty$ is a Hamel basis for $\mathcal{H}$. Conclusion: There is no countably infinite Hamel basis for $\mathcal{H}$. □

This result is the main reason why the concept of linear dimension is of no interest in Hilbert space theory. Here the useful notion is orthogonal dimension and, from now on, $\dim \mathcal{H}$ will always mean the orthogonal dimension of the Hilbert space $\mathcal{H}$. Observe that the notions of finite-dimensional and infinite-dimensional Hilbert spaces remain unchanged.

5.9 The Fourier Series Theorem

The Fourier Series Theorem states the fundamental properties of an orthonormal basis in a Hilbert space. Precisely, it exhibits a collection of necessary and sufficient conditions for an orthonormal set to be an orthonormal basis in a Hilbert space and, in particular, it ensures existence and uniqueness of the expansion of any vector in a Hilbert space in terms of each orthonormal basis. Before stating and proving it, we need the following auxiliary result.

Proposition 5.47. *Let $\{e_k\}_{k\in N}$ be a finite orthonormal set in a Hilbert space $\mathcal{H}$ and let x be an arbitrary vector in $\mathcal{H}$.*

$$u_x \;=\; \sum_{k\in N} \langle x\,;e_k\rangle e_k$$

is the unique vector in $\operatorname{span}\{e_k\}_{k\in N}$ *such that*

$$\|x-u_x\| \;\leq\; \|x-u\| \quad \textit{for every} \quad u\in \operatorname{span}\{e_k\}_{k\in N}.$$

Proof. Since $\operatorname{span}\{e_k\}_{k\in N}$ is a finite-dimensional linear manifold of $\mathcal{H}$ (Theorem 2.6), it follows by Corollary 4.29 that it is a subspace of $\mathcal{H}$ so that $\operatorname{span}\{e_k\}_{k\in N} = \big(\operatorname{span}\{e_k\}_{k\in N}\big)^{-} = \bigvee\{e_k\}_{k\in N}$. Theorem 5.13 says that there exists a unique vector u_x in $\bigvee\{e_k\}_{k\in N}$ such that

$$\|x-u_x\| \;\leq\; \|x-u\| \quad \text{for every} \quad u\in \bigvee\{e_k\}_{k\in N}.$$

Moreover, Theorem 5.13 also says that this u_x is the unique vector in $\bigvee\{e_k\}_{k\in N}$ such that

$$x-u_x \;\perp\; \bigvee\{e_k\}_{k\in N}.$$

Since $u_x\in \operatorname{span}\{e_k\}_{k\in N}$, it follows that $u_x=\sum_{k\in N}\alpha_k e_k$ for some finite family of scalars $\{\alpha_k\}_{k\in N}$. Since $x-u_x\perp \bigvee\{e_k\}_{k\in N}$, it follows that $x-u_x\perp e_j$ for every $j\in N$. Thus, recalling that $\{e_k\}_{k\in N}$ is an orthonormal set, we get

$$0 \;=\; \langle x-u_x\,;e_j\rangle \;=\; \langle x\,;e_j\rangle - \sum_{k\in N}\alpha_k\langle e_k\,;e_j\rangle \;=\; \langle x\,;e_j\rangle - \alpha_j$$

for every $j\in N$, and hence $u_x \;=\; \sum_{k\in N}\langle x\,;e_k\rangle e_k$. □

Theorem 5.48. (The Fourier Series Theorem). *Let $B=\{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal set in a Hilbert space $\mathcal{H}$. The following assertions are pairwise equivalent.*

(a) *B is an orthonormal basis for $\mathcal{H}$.*

(b) *Every $x \in \mathcal{H}$ has a unique expansion on B, namely*

$$x = \sum_{\gamma \in \Gamma} \langle x\,;e_\gamma\rangle e_\gamma.$$

This is referred to as the Fourier series expansion *of x. The elements of the family of scalars $\{\langle x\,;e_\gamma\rangle\}_{\gamma\in\Gamma}$ are called the* Fourier coefficients *of x with respect to B.*

(c) *For every pair of vectors x, y in $\mathcal{H}$,*

$$\langle x\,;y\rangle = \sum_{\gamma \in \Gamma} \langle x\,;e_\gamma\rangle\overline{\langle y\,;e_\gamma\rangle}.$$

(d) *For every $x \in \mathcal{H}$,*

$$\|x\|^2 = \sum_{\gamma \in \Gamma} |\langle x\,;e_\gamma\rangle|^2.$$

This is the Parseval identity.

(e) *Every linear manifold of $\mathcal{H}$ that includes B is dense in $\mathcal{H}$.*

Proof. We shall verify that (e)⇔(a)⇔(d) and (b)⇒(c)⇒(d)⇒(b).

Proof of (a)⇔(d). Take any x in $\mathcal{H}$. If $(\operatorname{span} B)^- = \mathcal{H}$, then every nonempty open ball centered at x meets $\operatorname{span} B$ (Proposition 3.32). That is, for every $\varepsilon > 0$ there exists a finite set $N_\varepsilon \subseteq \Gamma$ and a vector $u \in \operatorname{span}\{e_k\}_{k\in N_\varepsilon}$ such that $\|x - u\| < \varepsilon$. Set $u_x = \sum_{k\in N_\varepsilon}\langle x\,;e_k\rangle e_k$ in $\operatorname{span}\{e_k\}_{k\in N_\varepsilon}$. Proposition 5.4 ensures that $\|x - u_x\| \le \|x - u\|$, and hence

$$\Big\|x - \sum_{k\in N_\varepsilon} \langle x\,;e_k\rangle e_k\Big\|^2 < \varepsilon^2.$$

Since $\{e_k\}_{k\in N_\varepsilon}$ is an orthonormal set, it follows by the Pythagorean Theorem (Proposition 5.8) that

$$\begin{aligned}\Big\|\sum_{k\in N_\varepsilon} \langle x\,;e_k\rangle e_k - x\Big\|^2 &= \Big\|\sum_{k\in N_\varepsilon} \langle x\,;e_k\rangle e_k\Big\|^2 - 2\operatorname{Re}\sum_{k\in N_\varepsilon}\langle x\,;e_k\rangle\langle e_k\,;x\rangle + \|x\|^2\\ &= \sum_{k\in N_\varepsilon}|\langle x\,;e_k\rangle|^2 - 2\sum_{k\in N_\varepsilon}|\langle x\,;e_k\rangle|^2 + \|x\|^2\\ &= \|x\|^2 - \sum_{k\in N_\varepsilon}|\langle x\,;e_k\rangle|^2.\end{aligned}$$

Therefore, by using the Bessel inequality (Lemma 5.40) and recalling that $\sum_{k\in N_\varepsilon}|\langle x\,;e_k\rangle|^2 \le \sum_{\gamma\in\Gamma}|\langle x\,;e_\gamma\rangle|^2$ (Proposition 5.31), we get

$$\Big|\|x\|^2 - \sum_{\gamma\in\Gamma} |\langle x\,;e_\gamma\rangle|^2\Big| \;=\; \|x\|^2 - \sum_{\gamma\in\Gamma} |\langle x\,;e_\gamma\rangle|^2 \;\le\; \|x\|^2 - \sum_{k\in N_\varepsilon} |\langle x\,;e_k\rangle|^2$$
$$= \Big\|x - \sum_{k\in N_\varepsilon} \langle x\,;e_k\rangle e_k\Big\|^2 \;<\; \varepsilon^2$$

for all $\varepsilon > 0$. Hence $\|x\|^2 = \sum_{\gamma\in\Gamma}|\langle x\,;e_\gamma\rangle|^2$. Outcome: (a) implies (d). Conversely, if the orthonormal set B is not an orthonormal basis for $\mathcal{H}$, then it is not maximal in the Hilbert space $\mathcal{H}$ (Proposition 5.38(b)). Thus there exists a unit vector e in $\mathcal{H}$ such that $B\cup\{e\}$ is an orthonormal set (Proposition 5.36). Therefore $\langle e\,;e_\gamma\rangle = 0$ for all $\gamma\in\Gamma$, and hence $1 = \|e\|^2 \neq \sum_{\gamma\in\Gamma}|\langle e\,;e_\gamma\rangle|^2 = 0$. Conclusion: If (a) fails then (d) fails. Equivalently, (d) implies (a).

Proposition 4.9(a) ensures that (a)⇔(e). It is readily verified that (b)⇒(c). Indeed, if (b) holds, then

$$\langle x\,;y\rangle \;=\; \Big\langle \sum_{\alpha\in\Gamma}\langle x\,;e_\alpha\rangle e_\alpha\,;\sum_{\beta\in\Gamma}\langle y\,;e_\beta\rangle e_\beta\Big\rangle \;=\; \sum_{\alpha\in\Gamma}\langle x\,;e_\alpha\rangle\Big\langle e_\alpha\,;\sum_{\beta\in\Gamma}\langle y\,;e_\beta\rangle e_\beta\Big\rangle$$
$$= \sum_{\alpha\in\Gamma}\langle x\,;e_\alpha\rangle\sum_{\beta\in\Gamma}\overline{\langle y\,;e_\beta\rangle}\langle e_\alpha\,;e_\beta\rangle \;=\; \sum_{\alpha\in\Gamma}\langle x\,;e_\alpha\rangle\overline{\langle y\,;e_\alpha\rangle}$$

for every $x, y\in\mathcal{H}$ (because $\{e_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal set). Hence (b) implies (c). Moreover, (c)⇒(d) trivially.

Proof of (d)⇒(b). Take any $x\in\mathcal{H}$. Assertion (d) implies that $\{\langle x\,;e_\gamma\rangle e_\gamma\}_{\gamma\in\Gamma}$ is a square-summable family of orthogonal vectors in $\mathcal{H}$. Thus $\{\langle x\,;e_\gamma\rangle e_\gamma\}_{\gamma\in\Gamma}$ is a summable family of orthogonal vectors in the Hilbert space $\mathcal{H}$ by Theorem 5.32(b). Let x' be the sum of $\{\langle x\,;e_\gamma\rangle e_\gamma\}_{\gamma\in\Gamma}$. That is, $x' = \sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma\in\mathcal{H}$. Since $\{e_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal set, it follows by Theorem 5.32(a) that

$$\|x'-x\|^2 \;=\; \Big\|\sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma - x\Big\|^2$$
$$= \Big\|\sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma\Big\|^2 - 2\,\mathrm{Re}\sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle\langle e_\gamma\,;x\rangle + \|x\|^2$$
$$= \sum_{\gamma\in\Gamma}|\langle x\,;e_\gamma\rangle|^2 - 2\sum_{\gamma\in\Gamma}|\langle x\,;e_\gamma\rangle|^2 + \|x\|^2 \;=\; 0$$

because assertion (d) holds true (i.e., because $\|x\|^2 = \sum_{\gamma\in\Gamma}|\langle x\,;e_\gamma\rangle|^2$). Therefore, $x' = x$ so that

$$x \;=\; \sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma.$$

Finally, if $x = \sum_{\gamma\in\Gamma}\alpha_\gamma(x)e_\gamma$ for some family of scalars $\{\alpha_\gamma(x)\}_{\gamma\in\Gamma}$, then $\sum_{\gamma\in\Gamma}\big(\alpha_\gamma(x) - \langle x\,;e_\gamma\rangle\big)e_\gamma = 0$ so that $\sum_{\gamma\in\Gamma}|\alpha_\gamma(x) - \langle x\,;e_\gamma\rangle|^2 = 0$ by Theorem 5.32(a). Thus $\alpha_\gamma(x) = \langle x\,;e_\gamma\rangle$ for every $\gamma\in\Gamma$, which proves that the Fourier series expansion of x is unique. □

Remark: Consider the sums in (b), (c), and (d) of Theorem 5.48. If the Hilbert space $\mathcal{H}$ is finite dimensional, then any orthonormal basis for $\mathcal{H}$ is finite (Proposition 5.39), and so these are finite sums. If $\mathcal{H}$ is infinite dimensional and separable, then any orthonormal basis B for $\mathcal{H}$ is countably infinite (Proposition 5.43) so that B can be indexed by $\mathbb{N}$ (or $\mathbb{N}_0$, or $\mathbb{Z}$). For instance, if $B = \{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for an infinite-dimensional separable Hilbert space $\mathcal{H}$, then we have a countable summable family of vectors $\{\langle x\,;e_k\rangle e_k\}_{k\in\mathbb{N}}$ in (b), a countable summable family of scalars $\{\langle x\,;e_k\rangle\overline{\langle y\,;e_k\rangle}\}_{k\in\mathbb{N}}$ in (c), and a countable summable family of nonnegative numbers $\{|\langle x\,;e_k\rangle|^2\}_{k\in\mathbb{N}}$ in (d). If $\{e_k\}_{k=1}^{\infty}$ is *any* $\mathcal{H}$-valued orthonormal sequence containing all vectors from the orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ for $\mathcal{H}$ (which is itself an orthonormal basis for $\mathcal{H}$), then the Fourier series expansion for any $x \in \mathcal{H}$ can be written as

$$x = \sum_{k=1}^{\infty} \langle x\,;e_k\rangle e_k.$$

Similarly,

$$\langle x\,;y\rangle = \sum_{k=1}^{\infty} \langle x\,;e_k\rangle\overline{\langle y\,;e_k\rangle}$$

for every x, y in $\mathcal{H}$, and

$$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x\,;e_k\rangle|^2$$

for every $x \in \mathcal{H}$, where all the above infinite series are *unconditionally convergent*. In particular, *the Fourier series expansion is unconditionally convergent*. If $\mathcal{H}$ is a nonseparable Hilbert space, then any orthonormal basis for $\mathcal{H}$ is uncountable (Theorem 5.44). However, Corollary 5.41 says that, even in this case, the sums in (b), (c), and (d) have only a countable number of nonzero summands for each $x, y \in \mathcal{H}$.

Example 5.L. In this example we shall exhibit orthonormal bases for some classical separable Hilbert spaces.

(a) First recall that, for any integer $n \geq 1$, $\mathbb{F}^n$ is a Hilbert space (see Example 5.A). Moreover, the finite set $\{e_k\}_{k=1}^n$, where each $e_k = (\delta_{k1}, \ldots, \delta_{kn}) \in \mathbb{F}^n$ has 1 at the kth position and zeros elsewhere, clearly is an orthonormal set in $\mathbb{F}^n$ and also a Hamel basis for the linear space $\mathbb{F}^n$ (Example 2.I). Then $\{e_k\}_{k=1}^n$ is an orthonormal basis for the finite-dimensional Hilbert space $\mathbb{F}^n$, which is called the *canonical orthonormal basis* for $\mathbb{F}^n$.

(b) Now let ℓ_+^2 be the Hilbert space of Example 5.B. Consider the ℓ_+^2-valued sequence $\{e_k\}_{k\in\mathbb{N}}$, where each e_k is a scalar-valued sequence in ℓ_+^2 with just one nonzero entry (equal to 1) at the kth position. That is, $e_k = \{\delta_{kj}\}_{j\in\mathbb{N}} \in \ell_+^2$ for every $k \in \mathbb{N}$ with $\delta_{kj} = 1$ if $j = k$ and $\delta_{kj} = 0$ if $j \neq k$. Again, it is clear that $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal sequence of vectors in ℓ_+^2. Moreover, recall that $x =$

$\{\xi_j\}_{j\in\mathbb{N}}$ lies in ℓ^2_+ if and only if $\sum_{j=1}^{\infty}|\xi_j|^2<\infty$. Therefore, if $x=\{\xi_j\}_{j\in\mathbb{N}}\in\ell^2_+$, then $x=\lim x_n$, where $x_n=(\xi_1,\dots,\xi_n,0,0,0,\dots)\in\ell^2_+$ for every $n\in\mathbb{N}$. Since each $x_n\in\operatorname{span}\{e_k\}_{k\in\mathbb{N}}$ (for $x_n=\sum_{k=1}^{n}\xi_k e_k$), it follows by the Closed Set Theorem that $x\in\big(\operatorname{span}\{e_k\}_{k\in\mathbb{N}}\big)^-=\bigvee\{e_k\}_{k\in\mathbb{N}}$. Hence $\ell^2_+\subseteq\bigvee\{e_k\}_{k\in\mathbb{N}}\subseteq\ell^2_+$, which means that $\bigvee\{e_k\}_{k\in\mathbb{N}}=\ell^2_+$. Conclusion: The orthonormal sequence $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for ℓ^2_+. It can be similarly shown that $\{e_k\}_{k\in\mathbb{Z}}$, where each e_k is a scalar-valued net in ℓ^2 with just one nonzero entry (equal to 1) at the kth position (i.e., $e_k=\{\delta_{kj}\}_{j\in\mathbb{Z}}\in\ell^2$ for every $k\in\mathbb{Z}$ with $\delta_{kj}=1$ if $j=k$ and $\delta_{kj}=0$ if $j\ne k$) is an orthonormal basis for the Hilbert space ℓ^2 of Example 5.B. These are referred to as the *canonical orthonormal bases* for ℓ^2_+ and ℓ^2, respectively.

Next consider the complex Hilbert space $L^2(S)$ for some nondegenerate interval S of the real line (see Example 5.D). Recall that $L^2(S)$ is the completion of $R^2(S)$, which in turn is the quotient space $r^2(S)/\!\sim$ of Example 3.E. Also recall that, according to the usual convention, we write $x\in L^2(S)$ instead of $[x]\in L^2(S)$, where x is any representative of the equivalence class $[x]$.

(c) Suppose $S=[a,b]$ for some pair of real numbers $a<b$ so that $L^2(S)=L^2[a,b]$. It is not difficult to verify that the countable set $\{e_k\}_{k\in\mathbb{Z}}$, with each $e_k\in L^2[a,b]$ given by

$$e_k(t)\;=\;(b-a)^{-\frac{1}{2}}\exp\Big(2\pi i k\,\tfrac{t-a}{b-a}\Big)\quad\text{ for every }\quad t\in[a,b],$$

is a maximal orthonormal set in $L^2[a,b]$, and hence an orthonormal basis for the Hilbert space $L^2[a,b]$. In particular, for $S=[0,2\pi]$, the countable set $\{e_k\}_{k\in\mathbb{Z}}$, with each $e_k\in L^2[0,2\pi]$ given by

$$e_k(t)\;=\;\tfrac{1}{\sqrt{2\pi}}\,e^{ikt}\;=\;\tfrac{1}{\sqrt{2\pi}}\big(\cos kt+i\sin kt\big)\quad\text{ for every }\quad t\in[0,2\pi],$$

is an orthonormal basis for the Hilbert space $L^2[0,2\pi]$. Similarly (and equivalently), let $\mathbb{D}$ be the open unit disk (about the origin) in the complex plane $\mathbb{C}$, and let $\mathbb{T}=\partial\mathbb{D}$ denote the unit circle in $\mathbb{C}$. Suppose the length of $\mathbb{T}$ is normalized (i.e., suppose $L^2(\mathbb{T})$ is the Hilbert space of all equivalence classes of square-integrable complex functions defined on $\mathbb{T}$ with respect to normalized Lebesgue measure μ on the unit circle so that $\mu(\mathbb{T})=1$). The countable set $\{e_k\}_{k\in\mathbb{Z}}$, with each $e_k\in L^2(\mathbb{T})$ given by

$$e_k(z)\;=\;z^k\quad\text{ for every }\quad z\;=\;e^{i\theta}\in\mathbb{T}\qquad(\text{with }\;0\le\theta<2\pi),$$

is an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$.

(d) If $S=[0,\infty)$, then it can be shown that the sequence $\{e_n\}_{n=0}^{\infty}$ with each $e_n\in L^2[0,\infty)$ given by

$$e_n(t)\;=\;\tfrac{1}{n!}\,e^{-\frac{t}{2}}L_n(t)\quad\text{ for every }\quad t\ge0,$$

where each $L_n\in L^2[0,\infty)$ is defined by

$$L_n(t) \,=\, e^t \tfrac{d^n}{dt^n}\big(t^n e^{-t}\big) \,=\, \textstyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{n!}{k!}\,t^k \quad \text{for every} \quad t \ge 0,$$

is an orthonormal basis for the Hilbert space $L^2[0,\infty)$. (Note: The elements of $\{e_n\}_{n=0}^{\infty}$ and $\{L_n\}_{n=0}^{\infty}$ are referred to as *Chebyshev–Laguerre functions* and *Chebyshev–Laguerre polynomials*, respectively.) If $S = \mathbb{R}$, then it can also be shown that the sequence $\{e_n\}_{n=0}^{\infty}$ with each $e_n \in L^2(-\infty,\infty)$ given by

$$e_n(t) \,=\, \big(2^n n!\sqrt{\pi}\big)^{-\frac{1}{2}} e^{-\frac{t^2}{2}} H_n(t) \quad \text{for every} \quad t \in \mathbb{R},$$

where each $H_n \in L^2(-\infty,\infty)$ is defined by

$$H_n(t) \,=\, (-1)^n e^{t^2}\tfrac{d^n}{dt^n}\, e^{-t^2} \quad \text{for every} \quad t \in \mathbb{R},$$

is an orthonormal basis for the Hilbert space $L^2(-\infty,\infty)$. (Note: The elements of $\{e_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$ are called *Chebyshev–Hermite functions* and *Chebyshev–Hermite polynomials*, respectively.)

(e) Let S and U be bilateral shifts of infinite multiplicity acting on a separable Hilbert space $\mathcal{H}$ as defined in Problem 5.30 (which are unitary operators, i.e., invertible isometries). Suppose the bilateral shifts S and U in $\mathcal{B}[\mathcal{H}]$ do not commute but S intertwines U^2 to U; that is, suppose

$$SU \ne US \quad \text{and} \quad SU^2 = US.$$

If a nonzero w in $\mathcal{H}$ makes the double-indexed family $\{S^m U^n w\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ an orthonormal basis for $\mathcal{H}$, then w is called a *wavelet* (or an *orthogonal wavelet*), and the vectors $w_{m,n} = S^m U^n w$ in $\mathcal{H}$ are called the *wavelet vectors* generated by the wavelet w. The Fourier series expansion $x = \sum_{m,n}\langle x\,; w_{m,n}\rangle w_{m,n}$, in terms of the double-indexed orthonormal basis $\{w_{m,n}\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ for $\mathcal{H}$ consisting of wavelet vectors, is referred to as a *wavelet expansion* of $x \in \mathcal{H}$. If $\mathcal{H}$ represents some concrete Hilbert space of functions, then the term "wavelet functions" is used instead of "wavelet vectors". For instance, consider a pair of operators S and U on $\mathcal{H} = L^2(\mathbb{R})$ defined by

$$y = Sx \in L^2(\mathbb{R}) \quad \text{with} \quad y(t) = \sqrt{2}\,x(2t) \;\text{ for }\; t \in \mathbb{R},$$

and

$$y = Ux \in L^2(\mathbb{R}) \quad \text{with} \quad y(t) = x(t-1) \;\text{ for }\; t \in \mathbb{R},$$

for every $x \in L^2(\mathbb{R})$. It can be verified that S and U are bilateral shifts on $L^2(\mathbb{R})$ satisfying the preceding assumptions. A well-known wavelet associated with them is the *Haar wavelet*, which is the function w in $L^2(\mathbb{R})$ defined by

$$w(t) = \begin{cases} 1, & 0 < t \le \frac{1}{2}, \\ -1, & \frac{1}{2} < t \le 1, \\ 0, & t \in \mathbb{R}\backslash(0,1]. \end{cases}$$

Example 5.M. All the orthonormal bases in the previous example are countable, and so those Hilbert spaces are all separable Hilbert spaces. However, there exist nonseparable Hilbert spaces. Actually, *there exist Hilbert spaces of arbitrary orthogonal dimension*. Indeed, let Γ be any index set and let ℓ^2_Γ be the Hilbert space of Example 5.K. Consider the family $\{e_\gamma\}_{\gamma\in\Gamma}$ of vectors in ℓ^2_Γ, where each $e_\gamma = \{\delta_{\gamma\alpha}\}_{\alpha\in\Gamma}$ is a family of scalars such that $\delta_{\gamma\alpha} = 1$ if $\alpha = \gamma$ and $\delta_{\gamma\alpha} = 0$ if $\alpha \neq \gamma$. Note that each e_γ is a square-summable family of scalars (with just one nonzero element equal to 1), and so each e_γ in fact lies in ℓ^2_Γ. It is also clear that $\langle e_\alpha\,;e_\beta\rangle = \delta_{\alpha\beta}$ for every $\alpha,\beta\in\Gamma$, and hence $\{e_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal family of vectors in ℓ^2_Γ. Let x be any unit vector in ℓ^2_Γ. That is, $x = \{\xi_\gamma\}_{\gamma\in\Gamma}$ is a square-summable family of scalars such that $\|x\|^2 = \sum_{\gamma\in\Gamma}|\xi_\gamma|^2 = 1$. If x is orthogonal to every e_γ (i.e., if $x \perp e_\gamma$ for all $\gamma\in\Gamma$), then $0 = \langle x\,;e_\gamma\rangle = \sum_{\alpha\in\Gamma}\xi_\alpha\delta_{\gamma\alpha} = \xi_\gamma$ for all $\gamma\in\Gamma$, and hence $x = 0$. But this contradicts the fact that $\|x\| = 1$. Therefore, there is no unit vector x in ℓ^2_Γ for which $\{e_\gamma\}_{\gamma\in\Gamma}\cup\{x\}$ is an orthonormal set. That is (Proposition 5.36), $\{e_\gamma\}_{\gamma\in\Gamma}$ is a maximal orthonormal set in the Hilbert space ℓ^2_Γ, which means that $\{e_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal basis for ℓ^2_Γ (Proposition 5.38). Hence

$$\dim \ell^2_\Gamma \;=\; \#\Gamma.$$

If the index set Γ is uncountable, then $\{e_\gamma\}_{\gamma\in\Gamma}$ is an uncountable orthonormal basis for ℓ^2_Γ. In this case the Hilbert space ℓ^2_Γ is not separable (Theorem 5.44).

Theorem 5.49. *Two Hilbert spaces are unitarily equivalent if and only if they have the same orthogonal dimension.*

Proof. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces (over the same field $\mathbb{F}$) and let $B = \{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal basis for $\mathcal{H}$, indexed by an index set Γ.

(a) If $\mathcal{H}\cong\mathcal{K}$ (i.e., if $\mathcal{H}$ and $\mathcal{K}$ are unitarily equivalent), then there is a unitary transformation $U\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ such that $U(B)$ is an orthonormal basis for $\mathcal{K}$. In fact, since B is an orthonormal set in $\mathcal{H}$, and U preserves the inner product, it follows that $U(B)$ is an orthonormal set in $\mathcal{K}$. Moreover, as U is surjective, for every $y\in\mathcal{K}$ there is an $x\in\mathcal{H}$ such that $y = Ux$, and hence (Theorem 5.48)

$$y \;=\; U\sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma \;=\; \sum_{\gamma\in\Gamma}\langle Ux\,;Ue_\gamma\rangle Ue_\gamma \;=\; \sum_{\gamma\in\Gamma}\langle y\,;Ue_\gamma\rangle Ue_\gamma$$

because U is a bounded linear transformation that preserves the inner product. Thus the orthogonal set $U(B) = \{Ue_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal basis for $\mathcal{K}$ by Theorem 5.48. Since U is invertible, it establishes a one-to-one correspondence between B and $U(B)$, and so $\#B = \#U(B)$. Therefore, $\dim\mathcal{H} = \dim\mathcal{K}$.

(b) Conversely, suppose $\dim\mathcal{H} = \dim\mathcal{K}$. Then $\#B = \#C$, where C is an arbitrary orthonormal basis for $\mathcal{K}$, so that C and B can be similarly indexed, say $C = \{f_\gamma\}_{\gamma\in\Gamma}$. Consider the Hilbert space ℓ^2_Γ of Example 5.K (over the field $\mathbb{F}$). First we show that $\mathcal{H}$ and ℓ^2_Γ are unitarily equivalent, that is,

$$\mathcal{H} \cong \ell^2_\Gamma.$$

In fact, take an arbitrary $x \in \mathcal{H}$. The Parseval identity (Theorem 5.48) says that $\|x\|^2 = \sum_{\gamma\in\Gamma}|\langle x\,;e_\gamma\rangle|^2$, and so $\{\langle x\,;e_\gamma\rangle\}_{\gamma\in\Gamma}$ lies in ℓ^2_Γ. Consider the mapping $U\colon\mathcal{H}\to\ell^2_\Gamma$ defined by $Ux = \{\langle x\,;e_\gamma\rangle\}_{\gamma\in\Gamma}$ for every $x\in\mathcal{H}$. It is readily verified that U is a linear transformation (by the linearity of the inner product in the first argument), and

$$\|Ux\|^2 \;=\; \sum_{\gamma\in\Gamma}|\langle x\,;e_\gamma\rangle|^2 \;=\; \|x\|^2$$

for every $x\in\mathcal{H}$ (Parseval identity again). Thus U is a linear isometry. Next we verify that U is surjective. If $\{\alpha_\gamma\}_{\gamma\in\Gamma}$ is any family of scalars in ℓ^2_Γ (i.e., if $\sum_{\gamma\in\Gamma}|\alpha_\gamma|^2<\infty$; cf. Proposition 5.31), then $\{\alpha_\gamma e_\gamma\}_{\gamma\in\Gamma}$ is a summable family of vectors in $\mathcal{H}$. Indeed, as $\{e_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal set, $\sum_{\gamma\in\Gamma}\|\alpha_\gamma e_\gamma\|^2 = \sum_{\gamma\in\Gamma}|\alpha_\gamma|^2<\infty$ so that $\{\alpha_\gamma e_\gamma\}_{\gamma\in\Gamma}$ is a square-summable family, and hence a summable family of vectors in the Hilbert space $\mathcal{H}$ (Proposition 5.31 and Theorem 5.32). Therefore, for every $\{\alpha_\gamma\}_{\gamma\in\Gamma}\in\ell^2_\Gamma$ there is an x in $\mathcal{H}$ such that $x=\sum_{\gamma\in\Gamma}\alpha_\gamma e_\gamma$. But the Fourier series expansion of x is unique, which implies that $\alpha_\gamma = \langle x\,;e_\gamma\rangle$ for every $\gamma\in\Gamma$. Then $Ux = \{\langle x\,;e_\gamma\rangle\}_{\gamma\in\Gamma} = \{\alpha_\gamma\}_{\gamma\in\Gamma}$ so that $\{\alpha_\gamma\}_{\gamma\in\Gamma}\in\mathcal{R}(U)$. That is, $\ell^2_\Gamma\subseteq\mathcal{R}(U)$. Since $\mathcal{R}(U)\subseteq\ell^2_\Gamma$ trivially, it follows that $\mathcal{R}(U)=\ell^2_\Gamma$. Thus the linear isometry U is surjective, which means that U is a unitary transformation. Hence $\mathcal{H}$ and ℓ^2_Γ are unitarily equivalent. Similarly, since B and C are indexed by a common index set Γ, the same argument shows that $\mathcal{K}$ and ℓ^2_Γ are unitarily equivalent as well:

$$\ell^2_\Gamma \cong \mathcal{K}.$$

Therefore, $\mathcal{H}$ and $\mathcal{K}$ are unitarily equivalent, that is,

$$\mathcal{H}\cong\mathcal{K},$$

because unitary equivalence is transitive: composition of isometric isomorphism is again an isometric isomorphism. □

According to Theorem 5.44 and Examples 5.L and 5.M, the next result is an immediate consequence of Theorem 5.49.

Corollary 5.50. *Let Γ be an arbitrary index set. Every Hilbert space $\mathcal{H}$ with $\dim\mathcal{H} = \#\Gamma$ is unitarily equivalent to ℓ^2_Γ. In particular, every n-dimensional Hilbert space (for any integer $n\in\mathbb{N}$) is unitarily equivalent to $\mathbb{F}^n$, and every infinite-dimensional separable Hilbert space is unitarily equivalent to ℓ^2_+.*

Our first example of an unbounded linear transformation defined on a Banach space appeared in the proof of Corollary 4.30 part (b). It is easy to exhibit unbounded linear transformations defined on linear manifolds of a Hilbert space (see Problem 4.33(b)). Now we apply the Fourier Series Theorem

to exhibit an unbounded linear transformation defined on a whole Hilbert space; precisely, defined on an infinite-dimensional separable Hilbert space.

Example 5.N. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Let $B = \{f_\gamma\}_{\gamma\in\Gamma}$ be a Hamel basis for $\mathcal{H}$ that properly includes $\{e_k\}_{k=1}^{\infty}$ (see Theorem 2.5 and Corollary 5.46). Take $f_0 \in B\backslash\{e_k\}_{k=1}^{\infty}$ (obviously $f_0 \neq 0$) and consider the mapping $F\colon B \to \mathcal{H}$ such that $Ff_0 = f_0$ and $Ff = 0$ for all $f \neq f_0$ in B. Now consider the transformation $L\colon \mathcal{H} \to \mathcal{H}$ defined as follows. For each $x \in \mathcal{H}$ take its unique representation as a (finite) linear combination of vectors in the Hamel basis B,

$$x \;=\; \sum_{k=1}^{n(x)} \alpha_k(x)\, f_k.$$

Here $n(x)$ is a positive integer and $\{\alpha_k(x)\}_{k=1}^{n(x)}$ is a finite family of scalars containing all nonzero coordinates of x with respect to the Hamel basis B. Set

$$Lx \;=\; \sum_{k=1}^{n(x)} \alpha_k(x)\, F f_k.$$

It is not difficult to verify that L is linear (i.e., $L \in \mathcal{L}[\mathcal{H}]$). Moreover, $Lf = 0$ for every $f \in B$ such that $f \neq f_0$, and $Lf_0 = f_0$ (so that $L \neq O$). In particular, $Le_k = 0$ for all $k \geq 1$. Take an arbitrary $x \in \mathcal{H}$ and consider its Fourier series expansion, viz., $x = \sum_{k=1}^{\infty}\langle x\,; e_k\rangle e_k$. If L is bounded (i.e., if L is continuous), then it follows by Corollary 3.8 that $Lx = \sum_{k=1}^{\infty}\langle x\,; e_k\rangle Le_k = 0$. But this implies that $Lx = 0$ for all $x \in \mathcal{H}$ (i.e., $L = O$), which is a contradiction. Thus L is unbounded; that is, $L \in \mathcal{L}[\mathcal{H}]\backslash\mathcal{B}[\mathcal{H}]$.

5.10 Orthogonal Projection

A projection is an idempotent linear transformation of a linear space into itself (Section 2.9). An *orthogonal projection* on an inner product space $\mathcal{X}$ is a projection $P\colon \mathcal{X} \to \mathcal{X}$ such that $\mathcal{R}(P) \perp \mathcal{N}(P)$. Since orthogonal projections are projections, all the results of Section 2.9 apply to orthogonal projections in particular. If P is an orthogonal projection on $\mathcal{X}$, then so is the complementary projection $E = (I - P)\colon \mathcal{X} \to \mathcal{X}$ (reason: $E = (I - P)$ is a projection with $\mathcal{R}(E) = \mathcal{N}(P)$ and $\mathcal{N}(E) = \mathcal{R}(P)$).

Proposition 5.51. *An orthogonal projection $P\colon \mathcal{X} \to \mathcal{X}$ on an inner product space $\mathcal{X}$ has the following basic properties.*

(a) $P \in \mathcal{B}[\mathcal{X}]$ (i.e., *P is bounded*) *and* $\|P\| = 1$ *whenever* $P \neq O$,

(b) $\mathcal{N}(P)$ *and* $\mathcal{R}(P)$ *are subspaces of* $\mathcal{X}$,

(c) $\mathcal{N}(P) = \mathcal{R}(P)^\perp$ *and* $\mathcal{R}(P) = \mathcal{N}(P)^\perp$,

(d) $\mathcal{R}(P)$ *and* $\mathcal{N}(P)$ *are orthogonal complementary subspaces in* $\mathcal{X}$*. That is, besides being orthogonal subspaces of* $\mathcal{X}$*,* $\mathcal{R}(P)$ *and* $\mathcal{N}(P)$ *are such that*

$$\mathcal{X} \;=\; \mathcal{R}(P) + \mathcal{N}(P).$$

Thus $\mathcal{X}$ *can be decomposed as an orthogonal direct sum*

$$\mathcal{X} \;=\; \mathcal{R}(P) \oplus \mathcal{N}(P).$$

Proof. Let P be an orthogonal projection on an inner product space $\mathcal{X}$.

(a) Take an arbitrary $x \in \mathcal{X}$. Since $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are algebraic complements of each other (Theorem 2.19), we can write $x = u + v$ with $u \in \mathcal{R}(P)$ and $v \in \mathcal{N}(P)$. Moreover, $u \perp v$ because $\mathcal{R}(P) \perp \mathcal{N}(P)$. Then $\|x\|^2 = \|u\|^2 + \|v\|^2$ by the Pythagorean Theorem. Recalling that $\mathcal{R}(P) = \{u \in \mathcal{X}\colon\, Pu = u\}$, we get $Px = Pu + Pv = u$ so that $\|Px\|^2 = \|u\|^2 \le \|x\|^2$. Hence $\|P\| \le 1$. That is, P is a contraction. If $P \neq O$, then $\mathcal{R}(P) \neq \{0\}$, and so $\|Pu\| = \|u\| \neq 0$ (because $Pu = u$) for all nonzero u in $\mathcal{R}(P)$. Thus $\|P\| \ge 1$. Outcome: $\|P\| = 1$.

(b) According to item (a), $P \in \mathcal{B}[\mathcal{X}]$. Thus Proposition 4.13 says that $\mathcal{N}(P)$ is a subspace of $\mathcal{X}$. Since $E = I - P$ is an orthogonal projection on $\mathcal{X}$, it follows that $E \in \mathcal{B}[\mathcal{X}]$ and $\mathcal{R}(P) = \mathcal{N}(E)$ is a subspace of $\mathcal{X}$ (Proposition 4.13 again).

(c) Recall from the definition of orthogonal complement that if A and B are subsets of $\mathcal{X}$ such that $A \perp B$, then $A \subseteq B^\perp$. Therefore, as $\mathcal{N}(P) \perp \mathcal{R}(P)$, we get $\mathcal{N}(P) \subseteq \mathcal{R}(P)^\perp$. Now take an arbitrary $x \in \mathcal{R}(P)^\perp$ so that $x \perp \mathcal{R}(P)$. Theorem 2.19 says that $x = u + v$ with $u \in \mathcal{R}(P)$ and $v \in \mathcal{N}(P)$. Hence $0 = \langle x\,;u\rangle = \langle u\,;u\rangle + \langle v\,;u\rangle = \|u\|^2$ (since $\mathcal{R}(P) \perp \mathcal{N}(P)$) and so $u = 0$, which implies that $x = v \in \mathcal{N}(P)$. Therefore $\mathcal{R}(P)^\perp \subseteq \mathcal{N}(P)$. Outcome: $\mathcal{N}(P) = \mathcal{R}(P)^\perp$. Considering the complementary orthogonal projection $E = I - P$, we conclude that $\mathcal{R}(P) = \mathcal{N}(E) = \mathcal{R}(E)^\perp = \mathcal{N}(P)^\perp$.

(d) Theorem 2.19 says that $\mathcal{N}(P)$ and $\mathcal{R}(P)$ are algebraic complements of each other so that $\mathcal{X} = \mathcal{R}(P) + \mathcal{N}(P)$. Since $\mathcal{R}(P) \perp \mathcal{N}(P)$, it follows by Proposition 5.24 that $\mathcal{X} \cong \mathcal{R}(P) \oplus \mathcal{N}(P)$. As usual, we identify the orthogonal direct sum $\mathcal{R}(P) \oplus \mathcal{N}(P)$ with its unitarily equivalent image $\mathcal{R}(P) + \mathcal{N}(P) = \mathcal{X}$, and write $\mathcal{X} = \mathcal{R}(P) \oplus \mathcal{N}(P)$. □

Theorem 5.52. (Projection Theorem – Third version). *For every subspace* $\mathcal{M}$ *of a Hilbert space* $\mathcal{H}$ *there exists a unique orthogonal projection* $P \in \mathcal{B}[\mathcal{H}]$ *with* $\mathcal{R}(P) = \mathcal{M}$.

Proof. Existence. Theorem 5.20 says that $\mathcal{H}$ can be decomposed as

$$\mathcal{H} \;=\; \mathcal{M} + \mathcal{M}^\perp.$$

Since $\mathcal{M}$ and $\mathcal{M}^\perp$ are algebraic complements of each other, for every $x \in \mathcal{H}$ there exists a unique $u \in \mathcal{M}$ and a unique $v \in \mathcal{M}^\perp$ such that $x = u + v$ (Theorem 2.14). For each $x \in \mathcal{H}$ set $Px = P(u + v) = u$ in $\mathcal{M} \subseteq \mathcal{H}$. It is easy to check that this defines a linear transformation $P: \mathcal{H} \to \mathcal{H}$ with $\mathcal{R}(P) = \mathcal{M}$. Moreover, $P^2x = Pu = u = Px$ (because $u = u + 0$ is the unique decomposition of $u \in \mathcal{M}$ in $\mathcal{M} + \mathcal{M}^\perp$) so that P is idempotent. Furthermore, $Px = 0$ if and only if $x = 0 + v$ for some $v \in \mathcal{M}^\perp$, and hence $\mathcal{N}(P) = \mathcal{M}^\perp$. Conclusion: P is an orthogonal projection on $\mathcal{H}$ with $\mathcal{R}(P) = \mathcal{M}$.

Uniqueness. Take an arbitrary vector $x \in \mathcal{H}$ and consider its unique decomposition $x = u + v \in \mathcal{M} + \mathcal{M}^\perp = \mathcal{H}$ with $u \in \mathcal{M}$ and $v \in \mathcal{M}^\perp$. Suppose P' is an orthogonal projection on $\mathcal{H}$ such that $\mathcal{R}(P') = \mathcal{M}$. Thus $P'u = u = Pu$ for every $u \in \mathcal{M}$ (since the range of any projection is the set of all its fixed points — see proof of Theorem 2.19), and $P'v = 0 = Pv$ for every $v \in \mathcal{M}^\perp$. (Reason: If $\mathcal{M} = \mathcal{R}(P') = \mathcal{R}(P)$, then $\mathcal{M}^\perp = \mathcal{R}(P')^\perp = \mathcal{R}(P)^\perp = \mathcal{N}(P') = \mathcal{N}(P)$.) Therefore, $P'x = P'(u + v) = P(u + v) = Px$, and hence $P' = P$. □

Let $\mathcal{M}$ be any subspace of a Hilbert space $\mathcal{H}$. The unique orthogonal projection $P: \mathcal{H} \to \mathcal{H}$ for which $\mathcal{R}(P) = \mathcal{M}$ is called *the orthogonal projection onto* $\mathcal{M}$. The above proof shows that Theorem 5.20 implies Theorem 5.52. In fact, all versions of the Projection Theorem, viz., Theorems 5.20, 5.25, and 5.52, are pairwise equivalent. That Theorems 5.20 and 5.25 are equivalent is an immediate consequence of Proposition 5.24. We shall verify that Theorem 5.52 implies Theorem 5.20. Indeed, if $P: \mathcal{H} \to \mathcal{H}$ is any orthogonal projection on a Hilbert space $\mathcal{H}$, then $\mathcal{H} = \mathcal{R}(P) + \mathcal{N}(P)$ by Theorem 2.19, and hence $\mathcal{H} = \mathcal{R}(P) + \mathcal{R}(P)^\perp$, where $\mathcal{R}(P)$ is a subspace of $\mathcal{H}$ (Proposition 5.51). But Theorem 5.52 says that for every subspace $\mathcal{M}$ of $\mathcal{H}$ there exists an orthogonal projection $P: \mathcal{H} \to \mathcal{H}$ such that $\mathcal{R}(P) = \mathcal{M}$. Therefore, for every subspace $\mathcal{M}$ of $\mathcal{H}$ we get $\mathcal{H} = \mathcal{M} + \mathcal{M}^\perp$. Outcome: Theorem 5.52 implies Theorem 5.20. Such an equivalence justifies the terminology "Projection Theorem" for Theorems 5.20 and 5.25. The pivotal result of Theorem 5.13 can also be translated into the orthogonal projection language. Actually, the unique vector $u_x \in \mathcal{M}$ of Theorem 5.13 is given by $u_x = Px$, where $P: \mathcal{H} \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{M}$ (i.e., with $\mathcal{R}(P) = \mathcal{M}$).

Theorem 5.53. *Let $\mathcal{M}$ be a subspace of a Hilbert space $\mathcal{H}$ and let $P \in \mathcal{B}[\mathcal{H}]$ be the orthogonal projection onto $\mathcal{M}$. Take any x in $\mathcal{H}$. Px is the unique vector in $\mathcal{M}$ such that*

$$\|x - Px\| = d(x, \mathcal{M}).$$

Moreover, Px also is the unique vector in $\mathcal{M}$ such that

$$x - Px \perp \mathcal{M}.$$

Proof. Let P be *the* orthogonal projection onto $\mathcal{M}$. $P(x - Px) = Px - P^2x = 0$ so that $x - Px \in \mathcal{N}(P) = \mathcal{R}(P)^\perp = \mathcal{M}^\perp$. Thus Px is the unique vector in

$\mathcal{M}$ such that $x - Px \in \mathcal{M}^\perp$, which in turn is the unique vector in $\mathcal{M}$ such that $\|x - Px\| = d(x, \mathcal{M})$ (Theorem 5.13). □

The next two results connect the notion of orthogonal projection with the Fourier Series Theorem.

Theorem 5.54. *Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal family of vectors in a Hilbert space $\mathcal{H}$ and set $\mathcal{M} = \bigvee\{e_\gamma\}_{\gamma\in\Gamma}$. For every $x \in \mathcal{H}$, $\{\langle x\,;e_\gamma\rangle e_\gamma\}_{\gamma\in\Gamma}$ is a summable family of vectors in $\mathcal{M}$, and the mapping $P\colon\mathcal{H}\to\mathcal{H}$, defined by*

$$Px \;=\; \sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma,$$

is the orthogonal projection onto $\mathcal{M}$.

Proof. Take an arbitrary $x \in \mathcal{H}$. The Bessel inequality (Lemma 5.40) says that $\sum_{\gamma\in\Gamma}|\langle x\,;e_\gamma\rangle|^2 \le \|x\|^2$, and hence $\{\langle x\,;e_\gamma\rangle e_\gamma\}_{\gamma\in\Gamma}$ is a square-summable family (cf. Proposition 5.31) of vectors in the Hilbert space $\mathcal{M}$ (see Proposition 4.7). Then $\{\langle x\,;e_\gamma\rangle e_\gamma\}_{\gamma\in\Gamma}$ is a summable family of vectors in $\mathcal{M}$ by Theorem 5.32. Set $Px = \sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma \in \mathcal{M}$. It is readily verified that this defines a linear transformation $P\colon\mathcal{H}\to\mathcal{H}$ such that $\mathcal{R}(P) \subseteq \mathcal{M}$. Moreover, $\|Px\|^2 = \sum_{\gamma\in\Gamma}|\langle x\,;e_\gamma\rangle|^2$ (Theorem 5.32 again), and hence $\|Px\|^2 \le \|x\|^2$. Thus P lies in $\mathcal{B}[\mathcal{H}]$ (i.e., P is continuous — a contraction, actually), which implies that

$$P^2x \;=\; P\Big(\sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma\Big) \;=\; \sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle Pe_\gamma.$$

But the very definition of P ensures that $Pe_\alpha = \sum_{\gamma\in\Gamma}\langle e_\alpha\,;e_\gamma\rangle e_\gamma = e_\alpha$ for every $\alpha \in \Gamma$. Therefore,

$$P^2x \;=\; \sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma \;=\; Px,$$

and so P is a projection. Since $\{e_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal basis for the Hilbert space $\mathcal{M}$, the Fourier Series Theorem ensures that every $u \in \mathcal{M}$ has a unique Fourier series expansion

$$u \;=\; \sum_{\gamma\in\Gamma}\langle u\,;e_\gamma\rangle e_\gamma.$$

Hence $Pu = \sum_{\gamma\in\Gamma}\langle u\,;e_\gamma\rangle Pe_\gamma = \sum_{\gamma\in\Gamma}\langle u\,;e_\gamma\rangle e_\gamma = u$ so that $u \in \mathcal{R}(P)$. Thus $\mathcal{M} \subseteq \mathcal{R}(P)$ and so $\mathcal{R}(P) = \mathcal{M}$. Note that $v \in \mathcal{N}(P)$ (i.e., $Pv = 0$) if and only if $\|Pv\|^2 = \sum_{\gamma\in\Gamma}|\langle v\,;e_\gamma\rangle|^2 = 0$. Equivalently, if and only if $\langle v\,;e_\gamma\rangle = 0$ for all $\gamma \in \Gamma$, which means that $v \in \big(\{e_\gamma\}_{\gamma\in\Gamma}\big)^\perp = \big(\bigvee\{e_\gamma\}_{\gamma\in\Gamma}\big)^\perp = \mathcal{M}^\perp = \mathcal{R}(P)^\perp$. Thus $\mathcal{N}(P) = \mathcal{R}(P)^\perp$, which implies that $\mathcal{R}(P) \perp \mathcal{N}(P)$. Conclusion: $P\colon\mathcal{H}\to\mathcal{H}$ is an orthogonal projection with $\mathcal{R}(P) = \mathcal{M}$; that is, *the* projection onto $\mathcal{M}$ (Theorem 5.52). □

Corollary 5.55. *Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be any orthonormal basis for a subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$. The orthogonal projection $P \in \mathcal{B}[\mathcal{H}]$ onto $\mathcal{M}$ is given by*

$$Px \;=\; \sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma \quad \textit{for every} \quad x\in\mathcal{H}.$$

Note that if $\mathcal{M}=\mathcal{H}$ in Theorem 5.54 (and Corollary 5.55), then $P=I$, thus pointing back to the Fourier Series Theorem (Theorem 5.48). Now we consider "orthogonal families of orthogonal projections". This is a rather important notion upon which the Spectral Theorem of the next chapter will be built.

Proposition 5.56. *Let $P_1\in\mathcal{B}[\mathcal{X}]$ and $P_2\in\mathcal{B}[\mathcal{X}]$ be orthogonal projections on an inner product space $\mathcal{X}$. The following assertions are pairwise equivalent.*

(a) $\mathcal{R}(P_1)\perp\mathcal{R}(P_2)$.

(b) $P_1P_2=O$.

(c) $P_2P_1=O$.

(d) $\mathcal{R}(P_2)\subseteq\mathcal{N}(P_1)$.

(e) $\mathcal{R}(P_1)\subseteq\mathcal{N}(P_2)$.

Proof. Clearly, $\mathcal{R}(P_1)\perp\mathcal{R}(P_2)$ implies $\mathcal{R}(P_2)\subseteq\mathcal{R}(P_1)^\perp=\mathcal{N}(P_1)$. Conversely, if $\mathcal{R}(P_2)\subseteq\mathcal{N}(P_1)=\mathcal{R}(P_1)^\perp$, then $\mathcal{R}(P_2)\perp\mathcal{R}(P_1)$. Hence (a)⇔(d). Similarly (recall: orthogonality is symmetric so that $\mathcal{R}(P_1)\perp\mathcal{R}(P_2)$ if and only if $\mathcal{R}(P_2)\perp\mathcal{R}(P_1)$) we get (a)⇔(e). Since $\mathcal{R}(P_2)\subseteq\mathcal{N}(P_1)$ if and only if $P_1P_2=O$, it follows that (d)⇔(b). Swap P_1 and P_2 to get (e)⇔(c). □

If two orthogonal projections P_1 and P_2 on an inner product space $\mathcal{X}$ satisfy any of the equivalent assertions of Proposition 5.56, then they are said to be *orthogonal to each other* (or *mutually orthogonal*). If $\{P_\gamma\}_{\gamma\in\Gamma}$ is a family of orthogonal projections on an inner product space $\mathcal{X}$ which are orthogonal to each other (i.e., $\mathcal{R}(P_\alpha)\perp\mathcal{R}(P_\beta)$ whenever $\alpha\neq\beta$), then we say that $\{P_\gamma\}_{\gamma\in\Gamma}$ is an *orthogonal family of orthogonal projections* on $\mathcal{X}$. An *orthogonal sequence of orthogonal projections* $\{P_k\}_{k=0}^\infty$ is similarly defined. If $\{P_\gamma\}_{\gamma\in\Gamma}$ is an orthogonal family of orthogonal projections and

$$\sum_{\gamma\in\Gamma}P_\gamma x \;=\; x \quad \text{for every} \quad x\in\mathcal{X},$$

then $\{P_\gamma\}_{\gamma\in\Gamma}$ is called a *resolution of the identity* on $\mathcal{X}$. (For each $x\in\mathcal{X}$ the sum $x=\sum_{\gamma\in\Gamma}P_\gamma x$ has only a countable number of nonzero vectors — why?) If $\{P_k\}_{k=0}^\infty$ is an infinite sequence, then the above identity in $\mathcal{X}$ means convergence in the strong operator topology:

$$\sum_{k=0}^{n}P_k \;\xrightarrow{\;s\;}\; I.$$

If $\{P_i\}_{i=0}^n$ is a finite family, then the above identity in $\mathcal{X}$ obviously coincides with the identity $\sum_{i=0}^n P_i=I$ in $\mathcal{B}[\mathcal{X}]$. For instance, if P is an orthogonal

projection on $\mathcal{X}$ and $E = I - P$ is the complementary projection on $\mathcal{X}$, then P and E are orthogonal projections orthogonal to each other (since $PE = P - P^2 = O$). Moreover, $\{P, E\}$ clearly is a resolution of the identity on $\mathcal{X}$ since $P + E = I$.

Proposition 5.57. *Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. For each $\gamma \in \Gamma$ define the mapping $P_\gamma\colon \mathcal{H} \to \mathcal{H}$ by*

$$P_\gamma x \;=\; \langle x\,; e_\gamma\rangle e_\gamma \quad \textit{for every} \quad x \in \mathcal{H}.$$

Claim: *$\{P_\gamma\}_{\gamma\in\Gamma}$ is a resolution of the identity on $\mathcal{H}$.*

Proof. Each P_γ is the orthogonal projection onto the one-dimensional space $\mathcal{M} = \operatorname{span}\{e_\gamma\}$. This is a particular case of Theorem 5.54. It is plain that $P_\alpha P_\beta x \;=\; \langle x\,; e_\beta\rangle\langle e_\beta\,; e_\alpha\rangle e_\alpha$, and so $P_\alpha P_\beta x = 0$ whenever $\alpha \neq \beta$, for every $x \in \mathcal{H}$. Thus $P_\alpha P_\beta = O$ for every $\alpha, \beta \in \Gamma$ such that $\alpha \neq \beta$. Hence $\{P_\gamma\}_{\gamma\in\Gamma}$ is an orthogonal family of orthogonal projections on $\mathcal{H}$. The Fourier Series Theorem ensures that $\{\langle x\,; e_\gamma\rangle e_\gamma\}_{\gamma\in\Gamma}$ is a summable family of vectors in $\mathcal{H}$ and, for every $x \in \mathcal{H}$,

$$x \;=\; \sum_{\gamma\in\Gamma}\langle x\,; e_\gamma\rangle e_\gamma \;=\; \sum_{\gamma\in\Gamma} P_\gamma x.$$

Conclusion: $\{P_\gamma\}_{\gamma\in\Gamma}$ is a resolution of the identity on $\mathcal{H}$. □

Example 5.O. Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for an infinite-dimensional separable Hilbert space $\mathcal{H}$. According to Theorem 5.54 and Proposition 5.57, for each $k \geq 1$ the mapping $P_k\colon \mathcal{H} \to \mathcal{H}$ defined by

$$P_k x \;=\; \langle x\,; e_k\rangle e_k \quad \text{for every} \quad x \in \mathcal{H}$$

is an orthogonal projection and $\{P_k\}_{k=1}^\infty$ is a resolution of the identity on $\mathcal{H}$. Thus the sequence of operators $\{\sum_{k=1}^n P_k\}_{n=1}^\infty$ converges strongly to the identity I on $\mathcal{H}$. In fact, take an arbitrary $x \in \mathcal{H}$. By the Fourier Series Theorem,

$$\sum_{k=1}^n P_k x - x \;=\; \sum_{k=1}^n \langle x\,; e_k\rangle e_k - \sum_{k=1}^\infty \langle x\,; e_k\rangle e_k \;=\; -\sum_{k=n+1}^\infty \langle x\,; e_k\rangle e_k \;\to\; 0$$

as $n \to \infty$ because the infinite series $\sum_{k=1}^\infty \langle x\,; e_k\rangle e_k$ converges in $\mathcal{H}$ (see Problem 4.9(b)). This confirms that

$$\sum_{k=1}^n P_k \;\xrightarrow{\;s\;}\; I.$$

We shall see now that the sequence of operators $\{\sum_{k=1}^n P_k\}_{n=1}^\infty$, which converges strongly to the identity I on $\mathcal{H}$, does not converge uniformly. Indeed,

if $\{\sum_{k=1}^{n} P_k\}_{n=1}^{\infty}$ converges uniformly, then the identity I must be its uniform limit. (Reason: $\sum_{k=1}^{n} P_k \xrightarrow{s} I$ and uniform convergence implies strong convergence to the same limit.) However, for each $n \geq 1$,

$$\Big\|\Big(I - \sum_{k=1}^{n} P_k\Big)e_{n+1}\Big\| \;=\; \Big\| \sum_{k=n+1}^{\infty} \langle e_{n+1}\,; e_k\rangle e_k \Big\| \;=\; \|e_{n+1}\| \;=\; 1.$$

Thus $\|I - \sum_{k=1}^{n} P_k\| = \sup_{\|x\|=1}\|(I - \sum_{k=1}^{n} P_k)x\| \geq 1$ for every $n \geq 1$, and hence $\sum_{k=1}^{n} P_k \xrightarrow{u} I$. Conclusion: $\{\sum_{k=1}^{n} P_k\}_{n=1}^{\infty}$ does not converge uniformly.

Proposition 5.58. *If $\{P_k\}_{k\in\mathbb{N}}$ is an orthogonal sequence of orthogonal projections on a Hilbert space $\mathcal{H}$, then $\{\sum_{k=1}^{n} P_k\}_{n\in\mathbb{N}}$ converges strongly to the orthogonal projection $P\colon \mathcal{H} \to \mathcal{H}$ onto $\big(\sum_{k\in\mathbb{N}} \mathcal{R}(P_k)\big)^{-}$. That is, $\sum_{k=1}^{n} P_k \xrightarrow{s} P$, where $P \in \mathcal{B}[\mathcal{H}]$ is the orthogonal projection with $\mathcal{R}(P) = \big(\sum_{k\in\mathbb{N}} \mathcal{R}(P_k)\big)^{-}$.*

Proof. Set $\mathcal{M} = \big(\sum_{k\in\mathbb{N}} \mathcal{R}(P_k)\big)^{-}$, which is a subspace of the Hilbert space $\mathcal{H}$, and consider the decomposition

$$\mathcal{H} \;= \mathcal{M} + \mathcal{M}^{\perp}$$

of Theorem 5.20. Take any $x \in \mathcal{H}$ so that $x = u + v$ with $u \in \mathcal{M}$ and $v \in \mathcal{M}^{\perp}$. Observe that $v \in \big(\sum_{k\in\mathbb{N}} \mathcal{R}(P_k)\big)^{\perp}$ by Proposition 5.12 so that $v \perp \mathcal{R}(P_k)$ for every $k \in \mathbb{N}$. Thus $v \in \mathcal{R}(P_k)^{\perp} = \mathcal{N}(P_k)$, and hence $P_k v = 0$, for every $k \in \mathbb{N}$, which implies that $\sum_{k=1}^{\infty} P_k v = 0$. Since $u \in \big(\sum_{k\in\mathbb{N}} \mathcal{R}(P_k)\big)^{-}$ and $\{\mathcal{R}(P_k)\}_{k\in\mathbb{N}}$ is a countably infinite family of pairwise orthogonal subspaces of $\mathcal{H}$, it follows by the Orthogonal Structure Theorem (Theorem 5.16) that $u = \sum_{k=1}^{\infty} u_k$ with $u_k \in \mathcal{R}(P_k)$ for each k. Since P_j is continuous, we get $P_j u = \sum_{k=1}^{\infty} P_j u_k = u_j$ (reason: $u_k \in \mathcal{R}(P_k)$, $P_j P_k = O$ whenever $j \neq k$, and $P_j u_j = u_j$) for each $j \in \mathbb{N}$. Hence $u = \sum_{k=1}^{\infty} P_k u$. Therefore, for every $x \in \mathcal{H}$,

$$\sum_{k=1}^{\infty} P_k x \;= \sum_{k=1}^{\infty} P_k u + \sum_{k=1}^{\infty} P_k v \;= \; u,$$

so that the $\mathcal{B}[\mathcal{H}]$-valued sequence $\{\sum_{k=1}^{n} P_k\}_{n\in\mathbb{N}}$ is strongly convergent (Proposition 4.44). Let $P \in \mathcal{B}[\mathcal{H}]$ be the strong limit of $\{\sum_{k=1}^{n} P_k\}_{n\in\mathbb{N}}$ so that

$$Px \;= \sum_{k=1}^{\infty} P_k x \quad \text{ for every } \quad x \in \mathcal{H}.$$

Then $P^2 x = \sum_{k=1}^{\infty} P P_k x = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} P_j P_k x = \sum_{k=1}^{\infty} P_k x = Px$ for every x in $\mathcal{H}$ (since P is continuous and $P_j P_k = O$ whenever $j \neq k$), and so P is idempotent. Moreover, $\mathcal{R}(P) = \mathcal{M}$ and $\mathcal{N}(P) = \mathcal{M}^{\perp}$. Indeed, $Px = P(u+v) = u$ for every x in $\mathcal{H} = \mathcal{M} + \mathcal{M}^{\perp}$, where u is the unique vector in $\mathcal{M}$ and v is the unique vector in $\mathcal{M}^{\perp}$ such that $x = u + v$. Thus $\mathcal{R}(P) \perp \mathcal{N}(P)$. Outcome: $P \in \mathcal{B}[\mathcal{H}]$ is the unique orthogonal projection onto $\mathcal{M}$ (cf. Theorem 5.52). □

The preceding proposition is a consequence of the Projection Theorem (i.e., Theorems 5.20 and 5.52) and the Orthogonal Structure Theorem (Theorem 5.16). Here is the full version of the Orthogonal Structure Theorem in terms of orthogonal projections.

Theorem 5.59. *Let $\mathcal{H}$ be a Hilbert space. If $\{P_k\}_{k\in\mathbb{N}}$ is a resolution of the identity on $\mathcal{H}$, then*

$$\mathcal{H} = \Big(\sum_{k\in\mathbb{N}} \mathcal{R}(P_k)\Big)^{-}.$$

Conversely, if $\{\mathcal{M}_k\}_{k\in\mathbb{N}}$ is a sequence of pairwise orthogonal subspaces of $\mathcal{H}$ such that $\mathcal{H} = \big(\sum_{k\in\mathbb{N}} \mathcal{M}_k\big)^{-}$, then the sequence $\{P_k\}_{k\in\mathbb{N}}$ of the orthogonal projections $P_k \in \mathcal{B}[\mathcal{H}]$ onto each $\mathcal{M}_k$ is a resolution of the identity on $\mathcal{H}$.

Proof. Set $\mathcal{M} = \big(\sum_{k\in\mathbb{N}} \mathcal{R}(P_k)\big)^{-}$, which is a subspace of the Hilbert space $\mathcal{H}$. If $x \in \mathcal{M}^{\perp} = \big(\sum_{k\in\mathbb{N}} \mathcal{R}(P_k)\big)^{\perp}$, then $x \perp \mathcal{R}(P_k)$ for every $k \in \mathbb{N}$. Therefore, $x \in \mathcal{R}(P_k)^{\perp} = \mathcal{N}(P_k)$, so that $P_k x = 0$, for every $k \in \mathbb{N}$. Then $\sum_{k\in\mathbb{N}} P_k x = 0$. But $\sum_{k\in\mathbb{N}} P_k x = x$ for every $x \in \mathcal{H}$ because $\{P_k\}_{k\in\mathbb{N}}$ is a resolution of the identity on $\mathcal{H}$. Thus $x \in \mathcal{M}^{\perp}$ implies $x = 0$ so that $\mathcal{M}^{\perp} = \{0\}$, and hence

$$\mathcal{M} = \mathcal{H}$$

by Proposition 5.15. Conversely, let $\{\mathcal{M}_k\}_{k\in\mathbb{N}}$ be a sequence of pairwise orthogonal subspaces of $\mathcal{H}$ such that $\mathcal{H} = \big(\sum_{k\in\mathbb{N}} \mathcal{M}_k\big)^{-}$. For each $k \in \mathbb{N}$ let $P_k \in \mathcal{B}[\mathcal{H}]$ be the orthogonal projection onto $\mathcal{M}_k$ (Theorem 5.52), and note that $\mathcal{R}(P_j) = \mathcal{M}_j \perp \mathcal{M}_k = \mathcal{R}(P_k)$ if $j \neq k$. Thus $\{P_k\}_{k\in\mathbb{N}}$ is an orthogonal sequence of orthogonal projections on $\mathcal{H}$. Therefore, by Proposition 5.58,

$$\sum_{k=1}^{n} P_k \xrightarrow{s} I,$$

where the identity $I \in \mathcal{B}[\mathcal{H}]$ is the unique orthogonal projection on $\mathcal{H}$ with $\mathcal{R}(I) = \mathcal{H} = \big(\sum_{k\in\mathbb{N}} \mathcal{M}_k\big)^{-} = \big(\sum_{k\in\mathbb{N}} \mathcal{R}(P_k)\big)^{-}$. Outcome: $\{P_k\}_{k\in\mathbb{N}}$ is a resolution of the identity on $\mathcal{H}$. □

It is worth noticing that, as $\{\mathcal{R}(P_k)\}_{k\in\mathbb{N}}$ is a sequence of pairwise orthogonal subspaces of a Hilbert space $\mathcal{H}$, then $\bigoplus_{k\in\mathbb{N}} \mathcal{R}(P_k) \cong \big(\sum_{k\in\mathbb{N}} \mathcal{R}(P_k)\big)^{-}$ (see Example 5.J). Therefore, under the usual identification, Proposition 5.58 says that $\sum_{k=1}^{n} P_k \xrightarrow{s} P$, *where* $P \in \mathcal{B}[\mathcal{H}]$ *is the orthogonal projection with* $\mathcal{R}(P) = \bigoplus_{k\in\mathbb{N}} \mathcal{R}(P_k)$, and Theorem 5.59 says that $\mathcal{H} = \bigoplus_{k\in\mathbb{N}} \mathcal{R}(P_k)$ *whenever* $\{P_k\}_{k\in\mathbb{N}}$ *is a resolution of the identity on* $\mathcal{H}$.

Definition 5.60. Let $\{P_\gamma\}_{\gamma\in\Gamma}$ be a resolution of the identity on a Hilbert space $\mathcal{H} \neq \{0\}$ with $P_\gamma \neq O$ for every $\gamma \in \Gamma$. Let $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ be a similarly indexed family of scalars. Set

$$\mathcal{D}(T) = \big\{x \in \mathcal{H}\colon\ \{\lambda_\gamma P_\gamma x\}_{\gamma\in\Gamma} \text{ is a summable family of vectors in } \mathcal{H}\big\}.$$

The mapping $T\colon \mathcal{D}(T) \to \mathcal{H}$, defined by

$$Tx = \sum_{\gamma\in\Gamma} \lambda_\gamma P_\gamma x \quad \text{for every} \quad x \in \mathcal{D}(T),$$

is said to be a *weighted sum of projections*.

Proposition 5.61. *Every weighted sum of projections is a linear transformation. Moreover, if $T \in \mathcal{L}[\mathcal{D}(T), \mathcal{H}]$ is a weighted sum of projections, then the following assertions are pairwise equivalent.*

(a) $\mathcal{D}(T) = \mathcal{H}$.

(b) $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ *is a bounded family of scalars.*

(c) T *is bounded.*

If any of the above equivalent assertions holds true, then $T \in \mathcal{B}[\mathcal{H}]$ is such that $\|T\| = \sup_{\gamma\in\Gamma}|\lambda_\gamma|$.

Proof. It is readily verified that the domain $\mathcal{D}(T)$ of a weighted sum of projections is a linear manifold of $\mathcal{H}$, and also that the transformation $T\colon \mathcal{D}(T) \to \mathcal{H}$ is linear. (See the remark that follows Definition 5.26.)

Proof of (a)⇒(b). If $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ is not bounded, then for each integer $n \geq 1$ there exists a $\gamma_n \in \Gamma$ such that $|\lambda_{\gamma_n}| \geq n$. Consider the scalar-valued sequence $\{\lambda_{\gamma_n}\}_{n=1}^\infty$, which is clearly unbounded. Consider the sequence $\{P_{\gamma_n}\}_{n=1}^\infty$ from $\{P_\gamma\}_{\gamma\in\Gamma}$ and, for each $n \geq 1$, take $e_{\gamma_n} \in \mathcal{R}(P_{\gamma_n})$ such that $\|e_{\gamma_n}\| = 1$ (recall: $\mathcal{R}(P_{\gamma_n}) \neq \{0\}$ because $P_\gamma \neq O$ for every $\gamma \in \Gamma$). Observe that the orthogonal sequence $\{\lambda_{\gamma_n}^{-1} e_{\gamma_n}\}_{n=1}^\infty$ is square-summable (because $\sum_{n=1}^\infty \|\lambda_{\gamma_n}^{-1} e_{\gamma_n}\|^2 = \sum_{n=1}^\infty |\lambda_{\gamma_n}|^{-2} \leq \sum_{n=1}^\infty \frac{1}{n^2} < \infty$), and hence it is a summable sequence in the Hilbert space $\mathcal{H}$ (Corollary 5.9(b)). Set $x = \sum_{k=1}^\infty \lambda_{\gamma_k}^{-1} e_{\gamma_k}$ in $\mathcal{H}$ and note that (since each P_{γ_n} is continuous)

$$\lambda_{\gamma_n} P_{\gamma_n} x = \sum_{k=1}^\infty \lambda_{\gamma_n} \lambda_{\gamma_k}^{-1} P_{\gamma_n} e_{\gamma_k} = e_{\gamma_n} \quad \text{for each} \quad n \geq 1$$

(because $P_{\gamma_n} e_{\gamma_k} = 0$ if $k \neq n$ and $P_{\gamma_n} e_{\gamma_n} = e_{\gamma_n}$, once $e_{\gamma_k} \in \mathcal{R}(P_{\gamma_k})$ for each $k \geq 1$ and $P_{\gamma_n} P_{\gamma_k} = O$ whenever $k \neq n$). But $\{e_{\gamma_n}\}_{n=1}^\infty$ is not a square-summable sequence (since it is an orthogonal sequence of unit vectors). Thus $\{\lambda_{\gamma_n} P_{\gamma_n} x\}_{n=1}^\infty$ is not a square-summable sequence of vectors in $\mathcal{H}$, which means that $\sup_{n\geq 1} \sum_{k=1}^n \|\lambda_{\gamma_k} P_{\gamma_k} x\|^2 = \infty$, and hence the orthogonal family $\{\lambda_\gamma P_\gamma x\}_{\gamma\in\Gamma}$ of vectors in $\mathcal{H}$ is not square-summable (cf. Proposition 5.31). Therefore, $\{\lambda_\gamma P_\gamma x\}_{\gamma\in\Gamma}$ is not a summable family of vectors in $\mathcal{H}$ by Theorem 5.32, which means that $x \notin \mathcal{D}(T)$. Conclusion: If $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ is not bounded, then $\mathcal{D}(T) \neq \mathcal{H}$. Equivalently, (a) implies (b).

Proof of (b)⇒(a,c). Let x be an arbitrary vector in $\mathcal{H}$. Since $\{P_\gamma\}_{\gamma\in\Gamma}$ is a resolution of the identity on $\mathcal{H}$, it follows that $\{P_\gamma x\}_{\gamma\in\Gamma}$ is a summable family (for

$\sum_{\gamma\in\Gamma} P_\gamma x = x$) of orthogonal vectors in $\mathcal{H}$, and so a square-summable family by Theorem 5.32(a). Suppose $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ is bounded and set $\beta = \sup_{\gamma\in\Gamma}|\lambda_\gamma|$ (which is a nonnegative real number). Then, for any finite $N \subseteq \Gamma$,

$$\sum_{k\in N} \|\lambda_k P_k x\|^2 \le \beta^2 \sum_{k\in N} \|P_k x\|^2 \le \beta^2 \sum_{\gamma\in\Gamma} \|P_\gamma x\|^2 < \infty,$$

so that $\{\lambda_\gamma P_\gamma x\}_{\gamma\in\Gamma}$ is a square-summable family of orthogonal vectors in $\mathcal{H}$ (Proposition 5.31), and hence a summable family of vectors in the Hilbert space $\mathcal{H}$ by Theorem 5.32(b). Thus $x \in \mathcal{D}(T)$, and therefore (b) implies (a). Moreover, since $\{\lambda_\gamma P_\gamma x\}_{\gamma\in\Gamma}$ is an orthogonal family of vectors in $\mathcal{H}$, and $\{P_\gamma\}_{\gamma\in\Gamma}$ is a resolution of the identity on $\mathcal{H}$, we get

$$\begin{aligned}\|Tx\|^2 &= \Big\|\sum_{\gamma\in\Gamma} \lambda_\gamma P_\gamma x\Big\|^2 = \sum_{\gamma\in\Gamma} |\lambda_\gamma|^2 \|P_\gamma x\|^2 \\ &\le \beta^2 \sum_{\gamma\in\Gamma} \|P_\gamma x\|^2 = \beta^2 \Big\|\sum_{\gamma\in\Gamma} P_\gamma x\Big\|^2 = \beta^2 \|x\|^2\end{aligned}$$

(see Theorem 5.32 again). Hence (b) implies (c).

Proof of (c)⇒(b). For each $\gamma \in \Gamma$ take a unit vector e_γ in $\mathcal{R}(P_\gamma)$. Thus

$$Te_\gamma = \sum_{\gamma\in\Gamma} \lambda_\gamma P_\gamma e_\gamma = \lambda_\gamma e_\gamma.$$

If T is bounded, then

$$|\lambda_\gamma| = \|\lambda_\gamma e_\gamma\| = \|Te_\gamma\| \le \|T\|\|e_\gamma\| = \|T\|$$

for all $\gamma \in \Gamma$ so that (c) implies (b).

Finally, by the above inequality, $\sup_{\gamma\in\Gamma}|\lambda_\gamma| \le \|T\|$. On the other hand, we have already seen that $\|Tx\| \le \beta\|x\|$ for every $x \in \mathcal{H}$, where $\beta = \sup_{\gamma\in\Gamma}|\lambda_\gamma|$, which implies that $\|T\| \le \sup_{\gamma\in\Gamma}|\lambda_\gamma|$. Hence $\|T\| = \sup_{\gamma\in\Gamma}|\lambda_\gamma|$. □

Let the infinite sequence $\{P_k\}_{k=1}^\infty$ be a resolution of the identity on $\mathcal{H}$, where $P_k \ne O$ for every $k \ge 1$, and let $\{\lambda_k\}_{k=1}^\infty$ be a bounded sequence of scalars. Observe that the identity in $\mathcal{H}$

$$Tx = \sum_{k=1}^{\infty} \lambda_k P_k x \quad \text{for every} \quad x \in \mathcal{H}$$

that defines the weighted sum of projections $T \in \mathcal{B}[\mathcal{H}]$ actually means convergence in the strong topology; that is,

$$\sum_{k=1}^{n} \lambda_k P_k \xrightarrow{s} T.$$

5.11 The Riesz Representation Theorem and Weak Convergence

Associated with an arbitrary vector y in an inner product space $\mathcal{X}$, consider the functional $f\colon \mathcal{X} \to \mathbb{F}$ defined by

$$f(x) \;=\; \langle x\,;y\rangle \quad \text{ for every } \quad x \in \mathcal{X}.$$

This is a linear and bounded functional. Indeed, f is linear because the inner product is linear in the first argument, and bounded by the Schwarz inequality: $|f(x)| = |\langle x\,;y\rangle| \le \|y\|\|x\|$ for every $x \in \mathcal{X}$. That is, $f \in \mathcal{B}[\mathcal{X},\mathbb{F}]$ and $\|f\| = \sup_{\|x\|=1}|f(x)| \le \|y\|$. On the other hand, observe that $\|y\|\|y\| = |\langle y\,;y\rangle| = |f(y)| \le \|f\|\|y\|$ so that $\|y\| \le \|f\|$. Therefore,

$$\|f\| \;=\; \|y\|.$$

Outcome: A bounded (i.e., continuous) linear functional f is naturally associated with each vector y in an inner product space $\mathcal{X}$. The remarkable fact is that the converse holds true in a Hilbert space.

Theorem 5.62. (The Riesz Representation Theorem). *For every bounded linear functional f on a Hilbert space $\mathcal{H}$, there exists a unique vector $y \in \mathcal{H}$ such that*

$$f(x) \;=\; \langle x\,;y\rangle \quad \textit{for every} \quad x \in \mathcal{H}.$$

Moreover, $\|f\| = \|y\|$. Such a unique vector y in $\mathcal{H}$ is called the Riesz representation *of the functional f in $\mathcal{B}[\mathcal{H},\mathbb{F}]$.*

Proof. If $f = 0$, then it is clear that $y = 0$ is the unique vector in $\mathcal{H}$ for which $f(x) = \langle x\,;y\rangle$ for every $x \in \mathcal{H}$. Thus suppose $f \neq 0$.

Existence. Consider the null space $\mathcal{N}(f)$ of $f \in \mathcal{B}[\mathcal{H},\mathbb{F}]$, which is a proper subspace of $\mathcal{H}$ (i.e., $\mathcal{N}(f)$ is a subspace of $\mathcal{H}$ by Proposition 4.13 and $\mathcal{N}(f) \neq \mathcal{H}$ since $f \neq 0$). Thus, as $\mathcal{H}$ is a Hilbert space, $\mathcal{N}(f)^\perp \neq \{0\}$ by Proposition 5.15. Let z be any nonzero vector in $\mathcal{N}(f)^\perp$. Since $z \notin \mathcal{N}(f)$ (for $\mathcal{N}(f) \cap \mathcal{N}(f)^\perp = \{0\}$), it follows that $f(z) \neq 0$. Now take an arbitrary x in $\mathcal{H}$ and note that

$$f\Big(x - \tfrac{f(x)}{f(z)}z\Big) \;=\; f(x) - f(x)\tfrac{f(z)}{f(z)} \;=\; 0.$$

Thus $x - \frac{f(x)}{f(z)}z \in \mathcal{N}(f)$. Since $z \in \mathcal{N}(f)^\perp$ we get

$$0 \;=\; \Big\langle x - \tfrac{f(x)}{f(z)}z\,;z\Big\rangle \;=\; \langle x\,;z\rangle - \tfrac{f(x)}{f(z)}\|z\|^2,$$

and hence $f(x) = \big\langle x\,;\frac{\overline{f(z)}}{\|z\|^2}z\big\rangle$. Then there exists a vector $y = \frac{\overline{f(z)}}{\|z\|^2}z$ in $\mathcal{H}$, which does not depend on x, such that $f(x) = \langle x\,;y\rangle$.

Uniqueness. If $y' \in \mathcal{H}$ is such that $f(x) = \langle x\,;y'\rangle$ for every $x \in \mathcal{H}$, then $\langle x\,;y\rangle = \langle x\,;y'\rangle$ so that $\langle x\,;y - y'\rangle = 0$ for every $x \in \mathcal{H}$. Therefore, $y - y' \in \mathcal{H}^\perp = \{0\}$.

Finally, as we have already seen in the introduction of this section, if $y \in \mathcal{H}$ is such that $f(x) = \langle x\,;y\rangle$ for every $x \in \mathcal{H}$, then $\|f\| = \|y\|$. □

Corollary 5.63. *For every Hilbert space $\mathcal{H}$ there exists a surjective isometry $\Psi\colon \mathcal{H}^* \to \mathcal{H}$ of the dual $\mathcal{H}^*$ of $\mathcal{H}$ onto $\mathcal{H}$, which is additive and conjugate homogeneous* (i.e., $\Psi(\alpha f) = \overline{\alpha}\Psi(f)$ *for every $f \in \mathcal{H}^*$ and every $\alpha \in \mathbb{F}$*).

Proof. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}^* = \mathcal{B}[\mathcal{H}, \mathbb{F}]$ be the dual of $\mathcal{H}$. By the Riesz Representation Theorem, for each $f \in \mathcal{H}^*$ there exists a unique $y \in \mathcal{H}$ such that $f(x) = \langle x\,;y\rangle$ for every $x \in \mathcal{H}$ and $\|f\| = \|y\|$. Conversely, for each $y \in \mathcal{H}$ the functional $f\colon \mathcal{H} \to \mathbb{F}$ given by $f(x) = \langle x\,;y\rangle$ for every $x \in \mathcal{H}$ is linear and bounded, which means that $f \in \mathcal{H}^*$. This establishes a surjective isometry (i.e., an invertible isometry) $\Psi\colon \mathcal{H}^* \to \mathcal{H}$ of the dual $\mathcal{H}^*$ of $\mathcal{H}$ onto $\mathcal{H}$:

$$\Psi(f) \,=\, y \quad \text{ for every } \quad f \in \mathcal{H}^*,$$

where $y \in \mathcal{H}$ is the (unique) Riesz representation of $f \in \mathcal{H}^*$. Therefore, every f in $\mathcal{H}^*$ is such that

$$f(x) \,=\, \langle x\,;\Psi(f)\rangle \quad \text{ for every } \quad x \in \mathcal{H}.$$

Observe that Ψ is additive. Indeed, if $f, g \in \mathcal{H}^*$, then

$$\begin{aligned}\langle x\,;\Psi(f+g)\rangle \,&=\, (f+g)(x) \,=\, f(x) + g(x)\\ &= \langle x\,;\Psi(f)\rangle + \langle x\,;\Psi(g)\rangle \,=\, \langle x\,;\Psi(f) + \Psi(g)\rangle\end{aligned}$$

for every $x \in \mathcal{H}$, so that $\Psi(f+g) = \Psi(f) + \Psi(g)$. Moreover, if $f \in \mathcal{H}^*$ and $\alpha \in \mathbb{F}$, then

$$\langle x\,;\Psi(\alpha f)\rangle \,=\, \alpha f(x) \,=\, f(\alpha x) \,=\langle \alpha x\,;\Psi(f)\rangle \,=\, \langle x\,;\overline{\alpha}\Psi(f)\rangle$$

for every $x \in \mathcal{H}$, and hence $\Psi(\alpha f) = \overline{\alpha}\Psi(f)$. □

From the above corollary we may conclude: *Every Hilbert space is isometrically equivalent to its dual*. In particular, *every real Hilbert space is isometrically isomorphic to its dual*.

Corollary 5.64. *Every Hilbert space is reflexive.*

Proof. Let $\Psi\colon \mathcal{H}^* \to \mathcal{H}$ be the surjective isometry of Corollary 5.63, which is additive and conjugate homogeneous. Consider the mapping $\langle\ ;\ \rangle_*\colon \mathcal{H}^* \times \mathcal{H}^* \to \mathbb{F}$ given by

$$\langle f\,;g\rangle_* \,=\, \langle \Psi(g)\,;\Psi(f)\rangle$$

for every $f, g \in \mathcal{H}^*$, where $\langle\ ;\ \rangle$ stands for the inner product on $\mathcal{H}$. This defines an inner product on $\mathcal{H}^*$. Indeed, $\langle\ ;\ \rangle_*$ is additive because Ψ is additive. Since Ψ is conjugate homogeneous,

$$\langle \alpha f\,;g\rangle_* = \langle \Psi(g)\,;\Psi(\alpha f)\rangle \,=\, \langle \Psi(g)\,;\overline{\alpha}\Psi(f)\rangle \,=\, \alpha\langle \Psi(g)\,;\Psi(f)\rangle \,=\, \alpha\langle f\,;g\rangle_*$$

for every $f, g \in \mathcal{H}^*$ and every $\alpha \in \mathbb{F}$, and so $\langle\ ;\ \rangle_*$ is homogeneous in the first argument. It is clear that $\langle\ ;\ \rangle_*$ is Hermitian symmetric and positive. Actually,

$$\|f\|_* \;=\; \|\Psi(f)\| \;=\; \|f\|$$

for every $f \in \mathcal{H}^*$, so that the norm $\|\ \|_*$ induced on $\mathcal{H}^*$ by the inner product $\langle\ ;\ \rangle_*$ coincides with the usual (induced uniform) norm on $\mathcal{H}^* = \mathcal{B}[\mathcal{H}, \mathbb{F}]$. Since the dual space of every normed space is a Banach space, $(\mathcal{H}^*, \|\ \|)$ is a Banach space, and hence $(\mathcal{H}^*, \|\ \|_*)$ is a Hilbert space. We shall now apply the Riesz Representation Theorem to the Hilbert space $\mathcal{H}^*$. Take an arbitrary $\varphi \in \mathcal{H}^{**}$. Theorem 5.62 ensures that there exists a unique $g \in \mathcal{H}^*$ such that

$$\varphi(f) \;=\; \langle f\,; g\rangle_* \;=\; \langle \Psi(g)\,; \Psi(f)\rangle$$

for every $f \in \mathcal{H}^*$. According to Theorem 5.62, every $f \in \mathcal{H}^*$ is given by $f(x) = \langle x\,; y\rangle$ for every $x \in \mathcal{H}$, where $y = \Psi(f) \in \mathcal{H}$. Set $z = \Psi(g) \in \mathcal{H}$ so that

$$f(z) \;=\; \langle z\,; y\rangle \;=\; \langle \Psi(g)\,; \Psi(f)\rangle.$$

Therefore, there exists $z \in \mathcal{H}$ such that

$$\varphi(f) \;=\; f(z) \quad \text{for every} \quad f \in \mathcal{H}^*.$$

Hence $\mathcal{H}$ is reflexive by Proposition 4.67. □

Let $T \in \mathcal{B}[\mathcal{H}, \mathcal{Y}]$ be a bounded linear transformation of a Hilbert space $\mathcal{H}$ to an inner product space $\mathcal{Y}$. Take an arbitrary $y \in \mathcal{Y}$ and consider the functional $f_y\colon \mathcal{H} \to \mathbb{F}$ defined by

$$f_y(x) \;=\; \langle Tx\,; y\rangle$$

for every $x \in \mathcal{H}$, where the above inner product is that on $\mathcal{Y}$. It is easy to show that $f_y\colon \mathcal{H} \to \mathbb{F}$ is a bounded linear functional (i.e., $f_y \in \mathcal{H}^*$). Indeed, f_y is linear because T is linear and the inner product is linear in the first argument, and bounded by the Schwarz inequality: $|f_y(x)| \le \|Tx\|\|y\| \le \|T\|\|y\|\|x\|$ for every $x \in \mathcal{H}$. The Riesz Representation Theorem says that there exists a unique z_y in $\mathcal{H}$ such that

$$\langle Tx\,; y\rangle \;=\; f_y(x) \;=\; \langle x\,; z_y\rangle$$

for every $x \in \mathcal{H}$, where the right-hand side inner product is that on $\mathcal{H}$. This establishes a mapping $T^*\colon \mathcal{Y} \to \mathcal{H}$ that assigns to each y in $\mathcal{Y}$ this unique z_y in $\mathcal{H}$ (i.e., $T^*y = z_y$ for every $y \in \mathcal{Y}$), and therefore satisfies the following identity for every x in $\mathcal{H}$ and every y in $\mathcal{Y}$:

$$\langle Tx\,; y\rangle \;=\; \langle x\,; T^*y\rangle.$$

The mapping $T^*\colon \mathcal{Y} \to \mathcal{H}$ is referred to as the *adjoint* of $T \in \mathcal{B}[\mathcal{H}, \mathcal{Y}]$. In fact, as we shall see below, the adjoint T^* of T is unique.

Proposition 5.65. *Take any $T \in \mathcal{B}[\mathcal{H}, \mathcal{Y}]$, where $\mathcal{H}$ is a Hilbert space and $\mathcal{Y}$ is an inner product space.*

(a) *The adjoint* T^* *of* T *is the unique mapping of* $\mathcal{Y}$ *into* $\mathcal{H}$ *such that* $\langle Tx\,;y\rangle = \langle x\,;T^*y\rangle$ *for every* $x \in \mathcal{H}$ *and every* $y \in \mathcal{Y}$.

(b) T^* *is a bounded linear transformation* (i.e., $T^* \in \mathcal{B}[\mathcal{Y},\mathcal{H}]$).

Moreover, if $\mathcal{Y}$ *also is a Hilbert space, then*

(c) $T^{**} = T$, *and*

(d) $\|T^*\|^2 = \|T^*T\| = \|TT^*\| = \|T\|^2$.

Proof. Take $T \in \mathcal{B}[\mathcal{H},\mathcal{Y}]$ and let $T^*\colon \mathcal{Y} \to \mathcal{H}$ be a mapping such that $\langle Tx\,;y\rangle = \langle x\,;T^*y\rangle$ for every $x \in \mathcal{H}$ and every $y \in \mathcal{Y}$.

Proof of (a). If $T^\#\colon \mathcal{Y} \to \mathcal{H}$ satisfies the identity $\langle Tx\,;y\rangle = \langle x\,;T^\#y\rangle$ for every $x \in \mathcal{H}$ and every $y \in \mathcal{Y}$, then for each y in $\mathcal{Y}$ $\langle x\,;T^\#y\rangle = \langle x\,;T^*y\rangle$ — i.e., $\langle x\,;T^\#y - T^*y\rangle = 0$ — for every $x \in \mathcal{H}$. Hence $T^\#y = T^*y$ for every $y \in \mathcal{Y}$.

Proof of (b). Take $y_1, y_2 \in \mathcal{Y}$ and $\alpha_1, \alpha_2 \in \mathbb{F}$ arbitrary. Note that

$$\begin{aligned}\langle x\,;T^*(\alpha_1 y_1 + \alpha_2 y_2)\rangle &= \langle Tx\,;\alpha_1 y_1 + \alpha_2 y_2\rangle\\ &= \overline{\alpha}_1\langle Tx\,;y_1\rangle + \overline{\alpha}_2\langle Tx\,;y_2\rangle\\ &= \overline{\alpha}_1\langle x\,;T^*y_1\rangle + \overline{\alpha}_2\langle x\,;T^*y_2\rangle\\ &= \langle x\,;\alpha_1 T^*y_1 + \alpha_2 T^*y_2\rangle\end{aligned}$$

for every $x \in \mathcal{H}$, and hence $T^*(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 T^*y_1 + \alpha_2 T^*y_2$ so that T^* is linear. Moreover, by the Schwarz inequality,

$$\|T^*y\|^2 = \langle T^*y\,;T^*y\rangle = \langle TT^*y\,;y\rangle \le \|TT^*y\|\,\|y\| \le \|T\|\,\|T^*y\|\,\|y\|$$

for every $y \in \mathcal{Y}$. This implies that $\|T^*y\| \le \|T\|\,\|y\|$ for every $y \in \mathcal{Y}$. Thus T^* is bounded and

$$\|T^*\| \le \|T\|.$$

Now suppose $\mathcal{Y}$ is a Hilbert space so that $T^* \in \mathcal{B}[\mathcal{Y},\mathcal{H}]$ has a unique adjoint $(T^*)^* \in \mathcal{B}[\mathcal{H},\mathcal{Y}]$ (notation: $T^{**} = (T^*)^*$) such that

$$\langle T^*y\,;x\rangle = \langle y\,;T^{**}x\rangle$$

for every $x \in \mathcal{H}$ and every $y \in \mathcal{Y}$.

Proof of (c). Since $\langle y\,;Tx\rangle = \overline{\langle Tx\,;y\rangle} = \overline{\langle x\,;T^*y\rangle} = \langle T^*y\,;x\rangle$, for every $x \in \mathcal{H}$ and every $y \in \mathcal{Y}$, it follows by the above identity that

$$\langle y\,;Tx\rangle = \langle y\,;T^{**}x\rangle$$

for every $x \in \mathcal{H}$ and every $y \in \mathcal{Y}$, and hence $Tx = T^{**}x$ for every $x \in \mathcal{H}$.

Proof of (d). We have already seen that $\|T^*\| \le \|T\|$. Then, as $T = T^{**}$, we get $\|T\| = \|T^{**}\| \le \|T^*\|$. Thus $\|T^*\| = \|T\|$, and so $\|T^*T\| \le \|T^*\|\,\|T\| = \|T\|^2$. However, the Schwarz inequality ensures that

$$\|Tx\|^2 = \langle Tx\,;Tx\rangle = \langle T^*Tx\,;x\rangle \le \|T^*Tx\|\,\|x\| \le \|T^*T\|\,\|x\|^2$$

for every $x \in \mathcal{H}$. Hence $\|T\|^2 \le \|T^*T\|$, and therefore $\|T\|^2 = \|T^*T\|$. Again, as $T^{**} = T$, we get $\|T^*\|^2 = \|T^{**}T^*\| = \|TT^*\|$. □

Let $\mathcal{X}$ be an inner product space and consider the definition of weak convergence (cf. Problem 4.67): An $\mathcal{X}$-valued sequence $\{x_n\}$ converges weakly to $x \in \mathcal{X}$ (notation: $x_n \xrightarrow{w} x$) if the scalar-valued sequence $\{f(x_n)\}$ converges in $\mathbb{F}$ to $f(x)$ for every $f \in \mathcal{X}^*$ (i.e., if $f(x_n - x) \to 0$ for every $f \in \mathcal{X}^*$). Recall from Problem 4.67 that convergence in the norm topology implies weak convergence (to the same and unique limit):

$$x_n \to x \quad \text{implies} \quad x_n \xrightarrow{w} x.$$

Since $\langle \cdot\,; y\rangle\colon \mathcal{X} \to \mathbb{F}$ lies in $\mathcal{X}^*$ for every $y \in \mathcal{X}$, it follows that

$$x_n \xrightarrow{w} x \quad \text{implies} \quad \langle x_n\,; y\rangle \to \langle x\,; y\rangle \ \text{ for every } \ y \in \mathcal{X}.$$

The Riesz Representation Theorem ensures the converse in a Hilbert space: If $\mathcal{X}$ is a Hilbert space, then

$$x_n \xrightarrow{w} x \quad \text{if and only if} \quad \langle x_n\,; y\rangle \to \langle x\,; y\rangle \ \text{ for every } \ y \in \mathcal{X}.$$

In particular, $x_n \xrightarrow{w} x$ implies $\langle x_n\,; x\rangle \to \langle x\,; x\rangle = \|x\|^2$. By recalling that $\|x_n - x\|^2 = \|x_n\|^2 - 2\,\mathrm{Re}\,\langle x_n\,; x\rangle + \|x\|^2$, we may conclude the nontrivial part of the next equivalence, which holds in any inner product space:

$$x_n \to x \quad \text{if and only if} \quad x_n \xrightarrow{w} x \ \text{ and } \ \|x_n\| \to \|x\|.$$

Now suppose $\mathcal{X}$ and $\mathcal{Y}$ are inner product spaces and let $\{T_n\}$ be a $\mathcal{B}[\mathcal{X},\mathcal{Y}]$-valued sequence. If the $\mathcal{Y}$-valued sequence $\{T_n x\}$ converges weakly for every $x \in \mathcal{X}$, then we say that $\{T_n\}$ *converges weakly* (or *converges in the weak operator topology*). If $\mathcal{X}$ is a Hilbert space, then $\{T_n\}$ converges weakly if there exists a unique $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ such that

$$T_n x \xrightarrow{w} Tx \ \text{ in } \ \mathcal{Y} \ \text{ for every } \ x \in \mathcal{X},$$

which is called the *weak limit* of $\{T_n\}$ (see Problem 4.68). Notation: $T_n \xrightarrow{w} T$. Since $\langle \cdot\,; y\rangle\colon \mathcal{Y} \to \mathbb{F}$ lies in $\mathcal{Y}^*$ for every $y \in \mathcal{Y}$, it follows that

$$T_n \xrightarrow{w} T \ \text{ implies } \ \langle (T_n - T)x\,; y\rangle \to 0 \ \text{ for every } \ x \in \mathcal{X} \ \text{ and every } \ y \in \mathcal{Y}.$$

We shall see in Proposition 5.67 that the converse holds if $\mathcal{Y}$ is a Hilbert space.

Remark: Recalling Definition 4.45 and Problem 4.68, uniform convergence implies strong convergence, which implies weak convergence (to the same limit):

$$T_n \xrightarrow{u} T \quad \Longrightarrow \quad T_n \xrightarrow{s} T \quad \Longrightarrow \quad T_n \xrightarrow{w} T.$$

Since $y_n \to y$ if and only if $y_n \xrightarrow{w} y$ and $\|y_n\| \to \|y\|$, it follows that

$$T_n \xrightarrow{s} T \quad \text{if and only if} \quad T_n \xrightarrow{w} T \ \text{ and } \ \|T_n x\| \to \|Tx\| \ \text{ for every } x \in \mathcal{X}.$$

We have already seen that strong and uniform convergence coincide if $\mathcal{X}$ is finite dimensional. In fact, *uniform, strong, and weak convergence coincide if $\mathcal{X}$ and $\mathcal{Y}$ are finite-dimensional inner product spaces.*

Proposition 5.66. (a) *In a finite-dimensional inner product space, weak convergence and convergence in the norm topology coincide.*

(b) *Let $\mathcal{X}$ and $\mathcal{Y}$ be inner product spaces. Consider a $\mathcal{B}[\mathcal{X},\mathcal{Y}]$-valued sequence $\{T_k\}$ and let T be a transformation in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$. If $\mathcal{Y}$ is finite dimensional, then $T_k \xrightarrow{w} T$ if and only if $T_k \xrightarrow{s} T$.*

Proof. (a) Suppose $\mathcal{X}$ is a finite-dimensional inner product space, and so $\mathcal{X}$ is a Hilbert space (Corollary 4.28). Let $B = \{e_i\}_{i=1}^n$ be an orthonormal basis for $\mathcal{X}$. Take an arbitrary weakly convergent sequence $\{x_k\}$ in $\mathcal{X}$ so that $x_k \xrightarrow{w} x$ for some $x \in \mathcal{X}$, and hence $\langle x_k - x\,;e_i\rangle \to 0$ as $k \to \infty$ for each $e_i \in B$. Therefore, for each $i = 1,\dots,n$ and every $\varepsilon > 0$, there exists a positive integer $k_{i,\varepsilon}$ such that $|\langle x_k - x\,;e_i\rangle| < \varepsilon$ whenever $k \ge k_{i,\varepsilon}$. Then the Fourier Series Theorem (Theorem 5.48) ensures that

$$\|x_k - x\|^2 \;=\; \sum_{i=1}^{n} |\langle x_k - x\,;e_i\rangle|^2 \;<\; n\varepsilon^2$$

whenever $k \ge k_\varepsilon = \max\{k_{i,\varepsilon}\}_{i=1}^n$. Hence $\|x_k - x\| \to 0$ as $k \to \infty$. That is, $x_k \to x$. Summing up: $x_k \xrightarrow{w} x$ implies $x_k \to x$ if $\mathcal{X}$ is a finite-dimensional inner product space. This concludes the proof of (a), since norm convergence always implies weak convergence.

(b) $T_k \xrightarrow{w} T$ means $T_k x \xrightarrow{w} Tx$ in $\mathcal{Y}$ for every $x \in \mathcal{X}$. But item (a) says that this is equivalent to $T_k x \to Tx$ in $\mathcal{Y}$ for every $x \in \mathcal{X}$ if $\mathcal{Y}$ is finite dimensional, which means $T_k \xrightarrow{s} T$. The preceding remark ensures the converse. □

Here are equivalent conditions for weak convergence of bounded linear transformations between *Hilbert spaces*.

Proposition 5.67. *Let $\{T_n\}$ be a sequence of bounded linear transformations of a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$* (i.e., *$\{T_n\}$ is a $\mathcal{B}[\mathcal{H},\mathcal{K}]$-valued sequence*). *The following three assertions are pairwise equivalent.*

(a) *There exists $T \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ such that $\{T_n\}$ converges weakly to T* (i.e., *$T_n \xrightarrow{w} T$ or, equivalently, $T_n - T \xrightarrow{w} O$*).

(b) *There exists $T \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ such that $\langle T_n x\,;y\rangle \to \langle Tx\,;y\rangle$ as $n \to \infty$ for every x in $\mathcal{H}$ and every y in $\mathcal{K}$.*

(c) *The scalar-valued sequence $\{\langle T_n x\,;y\rangle\}$ converges in $\mathbb{F}$ for every x in $\mathcal{H}$ and every y in $\mathcal{K}$.*

Now set $\mathcal{K} = \mathcal{H}$ and consider the following further assertions.

(d) *There exists $T \in \mathcal{B}[\mathcal{H}]$ such that $\langle T_n x\,;x\rangle \to \langle Tx\,;x\rangle$ as $n \to \infty$ for every $x \in \mathcal{H}$.*

(e) *The scalar-valued sequence* $\{\langle T_n x\,;x\rangle\}$ *converges in* $\mathbb{F}$ *for every* $x \in \mathcal{H}$.

Clearly, (b) *implies* (d)*, which implies* (e)*. If* $\mathcal{K} = \mathcal{H}$ *is a* complex *Hilbert space, then these five assertions are all pairwise equivalent.*

Proof. If there is $T \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ such that $f(T_n x) \to f(Tx)$ in $\mathbb{F}$ as $n \to \infty$ for every $f \in \mathcal{K}^*$ and every $x \in \mathcal{H}$, then $\langle T_n x\,;y\rangle \to \langle Tx\,;y\rangle$ as $n \to \infty$ for every $y \in \mathcal{K}$ and every $x \in \mathcal{H}$ (since $\langle \cdot\,;y\rangle\colon \mathcal{K} \to \mathbb{F}$ lies in $\mathcal{K}^*$ for every $y \in \mathcal{K}$). Hence (a)⇒(b). Conversely, suppose (b) holds true and take any $f \in \mathcal{K}^*$. Since $\mathcal{K}$ is a Hilbert space, the Riesz Representation Theorem (Theorem 5.62) ensures that $f(T_n x) \to f(Tx)$ in $\mathbb{F}$ as $n \to \infty$ for every $x \in \mathcal{H}$. Thus (b)⇒(a). It is clear that (b)⇒(c). Now suppose assertion (c) holds true. Take an arbitrary y in $\mathcal{K}$ and consider the functional $f_y\colon \mathcal{H} \to \mathbb{F}$ defined by

$$f_y(x) \;=\; \lim_n \langle T_n x\,;y\rangle$$

for every $x \in \mathcal{H}$. Observe that f_y is linear (because T_n is linear for each n, the inner product is linear in the first argument, and the linear operations in $\mathbb{F}$ are continuous). Since $\{\langle T_n x\,;y\rangle\}$ converges in $\mathbb{F}$ for every $x \in \mathcal{H}$, it follows that $\{\langle T_n x\,;y\rangle\}$ is a bounded sequence for every $x \in \mathcal{H}$ (Proposition 3.39). Then $\sup_n |\langle T_n x\,;y\rangle| < \infty$ for every $x \in \mathcal{H}$ and every $y \in \mathcal{K}$, and so $\sup_n \|T_n\| < \infty$ (as a consequence of the Banach–Steinhaus Theorem — see Problem 5.5). Hence, for an arbitrary y in $\mathcal{K}$, $|f_y(x)| = |\lim_n \langle T_n x\,;y\rangle| = \lim_n |\langle T_n x\,;y\rangle| \le \sup_n |\langle T_n x\,;y\rangle| \le \sup_n \|T_n x\|\,\|y\| \le \sup_n \|T_n\|\,\|x\|\,\|y\|$ for every $x \in \mathcal{H}$, so that

$$\|f_y\| \;\le\; \sup_n \|T_n\|\,\|y\|.$$

That is, f_y is bounded, and therefore $f_y \in \mathcal{H}^*$ (f_y is a bounded linear functional on $\mathcal{H}$). Since $\mathcal{H}$ is a Hilbert space, the Riesz Representation Theorem says that there exists a unique $z_y \in \mathcal{H}$ such that

$$f_y(x) \;=\; \langle x\,;z_y\rangle$$

for every $x \in \mathcal{H}$. Consider the mapping $S\colon \mathcal{K} \to \mathcal{H}$ that assigns to each y in $\mathcal{H}$ this unique z_y in $\mathcal{H}$, $Sy = z_y$ for every $y \in \mathcal{K}$, so that

$$\lim_n \langle T_n x\,;y\rangle \;=\; f_y(x) \;=\; \langle x\,;z_y\rangle \;=\; \langle x\,;Sy\rangle$$

for every $x \in \mathcal{H}$ and every $y \in \mathcal{K}$. Note that S is linear and bounded (i.e., $S \in \mathcal{B}[\mathcal{K},\mathcal{H}]$). Indeed, if $y_1, y_2 \in \mathcal{Y}$ and $\alpha_1, \alpha_2 \in \mathbb{F}$, then

$$\begin{aligned}
\langle x\,;S(\alpha_1 y_1 + \alpha_2 y_2)\rangle \;&=\; \lim_n \langle T_n x\,;\alpha_1 y_1 + \alpha_2 y_2\rangle \\
&=\; \overline{\alpha}_1 \lim_n \langle T_n x\,;y_1\rangle + \overline{\alpha}_2 \lim_n \langle T_n x\,;y_2\rangle \\
&=\; \overline{\alpha}_1 \langle x\,;Sy_1\rangle + \overline{\alpha}_2 \langle x\,;Sy_2\rangle \\
&=\; \langle x\,;\alpha_1 Sy_1 + \alpha_2 Sy_2\rangle
\end{aligned}$$

for every $x \in \mathcal{H}$. Hence $S(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 Sy_1 + \alpha_2 Sy_2$, which means that S is linear. Moreover,

$$\|Sy\| \;=\; \|z_y\| \;=\; \|f_y\| \;\le\; \sup_n \|T_n\|\,\|y\|$$

for every $y \in \mathcal{K}$ so that S is bounded with $\|S\| \le \sup_n \|T_n\|$. Setting $T = S^* \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$, the adjoint of S, we get (cf. Proposition 5.65)

$$\lim_n \langle T_n x\,;y\rangle \;=\; \langle x\,;Sy\rangle \;=\; \langle x\,;S^{**}y\rangle \;=\; \langle S^*x\,;y\rangle \;=\; \langle Tx\,;y\rangle$$

for every $x \in \mathcal{H}$ and every $y \in \mathcal{K}$. Therefore, (c)⇒(b). Finally, set $\mathcal{K} = \mathcal{H}$ so that (b)⇒(d) and (c)⇒(e) trivially. By Problem 5.3(b) (with $L = I$), it follows that (d)⇒(b) and (e)⇒(c) whenever $\mathcal{H}$ is a *complex* Hilbert space. □

Carefully note that the latter two conditions are equivalent to the former three equivalent conditions *only if* the Hilbert space $\mathcal{H} = \mathcal{K}$ is *complex*.

Remark: Observe from Proposition 5.67 and Problem 5.5 that

$$T_n \xrightarrow{\;w\;} T \quad\Longrightarrow\quad \sup_n \|T_n\| < \infty \text{ whenever } \mathcal{H} \text{ and } \mathcal{K} \text{ are Hilbert.}$$

Take $T \in \mathcal{B}[\mathcal{X}]$, where $\mathcal{X}$ is an inner product space, and consider the power sequence $\{T^n\}$ so that each T^n lies in $\mathcal{B}[\mathcal{H}]$. The operator T is *weakly stable* if the power sequence $\{T^n\}$ converges weakly to the null operator; that is, $T^n \xrightarrow{\;w\;} O$. If $\mathcal{X}$ is a Hilbert space, then Proposition 5.67 says that this is equivalent to any of those five assertions with T_n replaced with T^n and T replaced with O. In particular, if $\mathcal{X}$ is a complex Hilbert space, then $T^n \xrightarrow{\;w\;} O$ if and only if $\langle T^n x\,;x\rangle \to 0$ as $n \to \infty$ for every $x \in \mathcal{X}$. It is plain that, uniform stability implies strong stability, which implies weak stability,

$$T^n \xrightarrow{\;u\;} O \quad\Longrightarrow\quad T^n \xrightarrow{\;s\;} O \quad\Longrightarrow\quad T^n \xrightarrow{\;w\;} O,$$

which in turn implies power boundedness whenever $\mathcal{X}$ is a Hilbert space:

$$T^n \xrightarrow{\;w\;} O \quad\Longrightarrow\quad \sup_n \|T^n\| < \infty \text{ if } \mathcal{X} \text{ is Hilbert.}$$

The converses, however, fail (see Example 4.K and Problem 5.29(c)).

Let $\mathcal{X}$ and $\mathcal{Y}$ be inner product spaces. We say that a subset Θ of $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is *weakly closed* in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ if every Θ-valued weakly convergent sequence $\{T_n\}$ has its (weak) limit T in Θ. Recall from Proposition 4.48: If Θ is strongly closed in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$, then it is (uniformly) closed in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$.

Proposition 5.68. *If $\Theta \subseteq \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is weakly closed in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$, then it is strongly closed in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$.*

Proof. Take an arbitrary Θ-valued strongly convergent sequence, say $\{T_n\}$, and let $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ be its (strong) limit. Since strong convergence implies weak convergence to the same limit, it follows that $\{T_n\}$ converges weakly to T. If every Θ-valued weakly convergent sequence has its (weak) limit in Θ,

then $T \in \Theta$. Conclusion: Every Θ-valued strongly convergent sequence has its (strong) limit in Θ. □

Remark: If $\mathcal{Y}$ is finite dimensional, then weak convergence coincides with strong convergence (Proposition 5.66(b)), and so the concepts of weakly closed and strongly closed in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ coincide whenever $\mathcal{Y}$ is a finite-dimensional inner product space. Then (see the remark after Proposition 4.48), *if $\mathcal{X}$ and $\mathcal{Y}$ are finite-dimensional inner product spaces, then all three concepts of weakly, strongly, and uniformly closed in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ coincide*.

Recall that a subset A of a metric space is compact if and only if it is sequentially compact (Theorem 3.80), which means by Definition 3.76 that every A-valued sequence has a convergent subsequence (that converges to a point in A). The Heine–Borel Theorem (Theorem 3.83) ensures that every bounded subset B of a finite-dimensional inner product space has a compact closure. Thus every B-valued sequence has a convergent subsequence (that converges to a point in B^-). Hence, *every bounded nonempty subset of a finite-dimensional inner product space has a convergent sequence* (whose limit lies in its closure). This no longer holds in an infinite-dimensional inner product space (e.g., if $\{e_n\}$ is an infinite orthonormal sequence, then $\|e_n - e_m\|^2 = 2$ for every $m \neq n$). Now recall that a finite-dimensional inner product space is a Hilbert space where convergence in the norm topology coincides with weak convergence (Proposition 5.66(a)), so that the next lemma actually is an extension to infinite-dimensional Hilbert spaces of the above italicized result.

Lemma 5.69. *Every bounded nonempty subset of a Hilbert space has a weakly convergent sequence. In particular, every bounded sequence in a Hilbert space has a weakly convergent subsequence*.

Proof. Let B be a bounded nonempty subset of an inner product space $\mathcal{H}$. The result is trivial if $B = \{0\}$. Suppose $B \neq \{0\}$ and let $A \neq \{0\}$ be a nonempty countable subset of B, so that $0 < \sup_{x \in A} \|x\| < \infty$. Set $\mathcal{X} = \bigvee A$, which is a subspace of $\mathcal{H}$, so that $\mathcal{X}$ is a separable inner product space by Proposition 4.9(b). Consider the following subset of the dual of $\mathcal{X}$:

$$\Phi = \big\{ \langle \cdot \, ; x \rangle \colon \mathcal{X} \to \mathbb{F} \colon \ x \in A \big\} \subset \mathcal{X}^*.$$

Recall the definition of "pointwise total boundedness" and "equicontinuity" of Example 3.Z.

Claim 1. Φ is pointwise totally bounded.

Proof. Since total boundedness coincides with plain boundedness in $\mathbb{F}$ (cf. proof of Theorem 3.83), it follows that Φ is pointwise totally bounded if and only if it is pointwise bounded. But $|\langle w \, ; x \rangle| \leq \sup_{x \in A} \|x\| \, \|w\|$ for every $w \in \mathcal{X}$, and so Φ is pointwise bounded. □

Claim 2. Φ is (uniformly) equicontinuous.

Proof. Indeed, $|\langle u\,;x\rangle - \langle v\,;x\rangle| = |\langle u-v\,;x\rangle| \le \sup_{x\in A}\|x\|\,\|u-v\|$ for every $u, v \in \mathcal{X}$ and every $x \in A$. This implies that for every $\varepsilon > 0$ there exists a $\delta = (\sup_{x\in A}\|x\|)^{-1}\varepsilon > 0$ such that $|\langle u\,;x\rangle - \langle v\,;x\rangle| < \varepsilon$ whenever $\|u-v\| < \delta$ for all $u, v \in \mathcal{X}$ and every $x \in A$. □

Then, according to the proof of Example 3.Z (set $f_n = \langle\cdot\,;x_n\rangle \in \Phi$), we may conclude: Φ is totally bounded because $\mathcal{X}$ is separable, which means that every Φ-valued sequence has a Cauchy subsequence (Lemma 3.73). Take an arbitrary A-valued sequence $\{x_n\}$ so that the Φ-valued sequence $\{\langle\cdot\,;x_n\rangle\}$ has a subsequence, say $\{\langle\cdot\,;x_{n_k}\rangle\}$, that is Cauchy in $\mathcal{X}^*$. But $\mathcal{X}^*$ is always complete so that $\{\langle\cdot\,;x_{n_k}\rangle\}$ converges in $\mathcal{X}^*$. That is, there exists an $f \in \mathcal{X}^*$ such that $\langle\cdot\,;x_{n_k}\rangle \to f$ in $\mathcal{X}^*$. If $\mathcal{H}$ is a Hilbert space, then the subspace $\mathcal{X}$ is itself a Hilbert space, and the Riesz Representation Theorem says that there exists an $x \in \mathcal{X}$ for which $f = \langle\cdot\,;x\rangle$. Therefore, $\langle x_{n_k}\,;w\rangle \to \langle x\,;w\rangle$ for every $w \in \mathcal{X}$. Using the Riesz Representation Theorem again, we get $x_{n_k} \xrightarrow{w} x$ in $\mathcal{X}$. Hence $x_{n_k} \xrightarrow{w} x$ in $\mathcal{H}$ (cf. Problem 5.20). □

Theorem 5.70. *Let $\{T_n\}$ be a $\mathcal{B}[\mathcal{H},\mathcal{K}]$-valued sequence, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. If $\mathcal{H}$ is a separable Hilbert space and $\sup_n\|T_n\| < \infty$, then $\{T_n\}$ has a weakly convergent subsequence.*

Proof. Suppose $\mathcal{H} \neq \{0\}$ to avoid trivialities. If $\mathcal{H}$ is separable, then there exists a countably infinite dense subset A of the nonzero linear space $\mathcal{H}$. Let $\{a_i\}_{i\ge1}$ be an A-valued sequence consisting of an enumeration of all points of A. Since $A^- = \mathcal{H}$, it follows by Proposition 3.32 that

$$\inf_i \|x - a_i\| = 0 \quad \text{for every} \quad x \in \mathcal{H}.$$

If $\sup_n\|T_n\| < \infty$, then $\{T_n a_1\}_{n\ge1}$ is a bounded sequence in $\mathcal{K}$. Thus Lemma 5.69 ensures the existence of a subsequence of $\{T_n\}_{n\ge1}$, say $\{T_n^{(1)}\}_{n\ge1}$, such that $\{T_n^{(1)}a_1\}_{n\ge1}$ converges weakly in $\mathcal{K}$. Again, since $\sup_n\|T_n\| < \infty$, it follows that $\{T_n a_2\}_{n\ge1}$ is bounded in $\mathcal{K}$ and, in particular, the subsequence $\{T_n^{(1)}a_2\}_{n\ge1}$ is bounded in $\mathcal{K}$. Hence another application of Lemma 5.69 ensures that there exists a subsequence of $\{T_n^{(1)}\}_{n\ge1}$, say $\{T_n^{(2)}\}_{n\ge1}$, such that $\{T_n^{(2)}a_2\}_{n\ge1}$ converges weakly in $\mathcal{K}$. Note that $\{T_n^{(2)}a_1\}_{n\ge1}$ also converges weakly in $\mathcal{K}$ (reason: $\{T_n^{(2)}\}_{n\ge1}$ is a subsequence of $\{T_n^{(1)}\}_{n\ge1}$ and $\{T_n^{(1)}a_1\}_{n\ge1}$ converges weakly in $\mathcal{K}$ — see Problem 4.67(b)). This leads to an inductive construction of a sequence of $\mathcal{B}[\mathcal{H},\mathcal{K}]$-valued sequences, $\big\{\{T_n^{(k)}\}_{n\ge1}\big\}_{k\ge1}$, with the following properties.

Property (1). $\{T_n^{(k+1)}\}_{n\ge1}$ is a subsequence of $\{T_n^{(k)}\}_{n\ge1}$, which is a subsequence of $\{T_n\}_{n\ge1}$, for every $k \ge 1$.

Property (2). $\{T_n^{(k)}a_i\}_{n\ge1}$ converges weakly in $\mathcal{K}$ whenever $k \ge i$.

Take the "diagonal" sequence $\{T_n^{(n)}\}_{n\ge1}$, which is a subsequence of $\{T_n\}_{n\ge1}$. If $\{T_n^{(n)}\}_{n\ge1}$ is weakly convergent, then the theorem is proved.

Claim. $\{T_n^{(n)}\}_{n\ge1}$ is weakly convergent.

Proof. Take $x \in \mathcal{H}$, $y \in \mathcal{K}$, and $\varepsilon > 0$ arbitrary. Since $\inf_i \|x - a_i\| = 0$, there exists an integer $i_\varepsilon \geq 1$ such that

$$\|x - a_{i_\varepsilon}\| \, < \, \varepsilon.$$

According to Property (1) $\{T_n^{(n)}\}_{n \geq i_\varepsilon}$ is a subsequence of $\{T_n^{(i_\varepsilon)}\}_{n \geq 1}$, and so $\{T_n^{(n)} a_{i_\varepsilon}\}_{n \geq i_\varepsilon}$ is a subsequence of $\{T_n^{(i_\varepsilon)} a_{i_\varepsilon}\}_{n \geq 1}$. Since $\{T_n^{(i_\varepsilon)} a_{i_\varepsilon}\}_{n \geq 1}$ converges weakly in $\mathcal{K}$ by Property (2), it follows that its subsequence $\{T_n^{(n)} a_{i_\varepsilon}\}_{n \geq i_\varepsilon}$ also converges weakly in $\mathcal{K}$. This implies that $\{T_n^{(n)} a_{i_\varepsilon}\}_{n \geq 1}$ converges weakly in $\mathcal{K}$ so that $\{\langle T_n^{(n)} a_{i_\varepsilon} \, ; y \rangle\}_{n \geq 1}$ converges in $\mathbb{F}$, and therefore is a Cauchy sequence in $\mathbb{F}$. That is, there exists an integer $n_\varepsilon \geq 1$ such that

$$m, n \geq n_\varepsilon \quad \text{ implies } \quad |\langle (T_n^{(n)} - T_m^{(m)}) a_{i_\varepsilon} \, ; y \rangle| < \varepsilon.$$

Note that $\|T_n^{(n)} - T_m^{(m)}\| \leq 2 \sup_k \|T_k\|$ for all $m, n \geq 1$ because $\{T_n^{(n)}\}_{n \geq 1}$ is a subsequence of $\{T_n\}_{n \geq 1}$. Hence

$$\begin{aligned} |\langle (T_n^{(n)} - T_m^{(m)}) x \, ; y \rangle| \, &= \, |\langle (T_n^{(n)} - T_m^{(m)})(a_{i_\varepsilon} + x - a_{i_\varepsilon}) ; y \rangle| \\ &\leq \, \langle (T_n^{(n)} - T_m^{(m)}) a_{i_\varepsilon} \, ; y \rangle| + 2 \sup_k \|T_k\| \|x - a_{i_\varepsilon}\| \|y\| \\ &\leq \, \big(1 + 2 \sup_k \|T_k\| \|y\|\big) \varepsilon \end{aligned}$$

whenever $m, n \geq n_\varepsilon$. Conclusion: $\{\langle T_n^{(n)} x \, ; y \rangle\}_{n \geq 1}$ is a Cauchy sequence in $\mathbb{F}$ so that it converges in $\mathbb{F}$ (since $\mathbb{F}$ is complete). As x and y are arbitrary vectors in $\mathcal{H}$ and $\mathcal{K}$, respectively, this implies that the scalar-valued sequence $\{\langle T_n^{(n)} x \, ; y \rangle\}_{n \geq 1}$ converges in $\mathbb{F}$ for every $x \in \mathcal{H}$ and every $y \in \mathcal{K}$, which means that $T_n^{(n)} \xrightarrow{w} T$ for some $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ by Proposition 5.67. □

5.12 The Adjoint Operator

Let $T \in \mathcal{B}[\mathcal{H}, \mathcal{Y}]$ be a bounded linear transformation of a Hilbert space $\mathcal{H}$ into an inner product space $\mathcal{Y}$. The *adjoint* of T was defined in the previous section as the unique mapping $T^* \colon \mathcal{Y} \to \mathcal{H}$ such that

$$\langle Tx \, ; y \rangle \, = \, \langle x \, ; T^* y \rangle$$

for every $x \in \mathcal{H}$ and every $y \in \mathcal{Y}$, whose existence was established in Section 5.11 as a consequence of the Riesz Representation Theorem. The basic facts about the adjoint T^* were stated in Proposition 5.65. In particular, it is linear and bounded (i.e., $T^* \in \mathcal{B}[\mathcal{Y}, \mathcal{H}]$). Here is a useful corollary of Proposition 5.65.

Corollary 5.71. *If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$, then*

$$\|T\| \, = \, \sup_{\|x\| = \|y\| = 1} |\langle Tx \, ; y \rangle|.$$

Proof. Since $\|z\| = \sup_{\|y\|=1} |\langle z \, ; y \rangle|$ for every $z \in \mathcal{K}$ (see Problem 5.1), it follows that $\|Tx\| = \sup_{\|y\|=1} |\langle Tx \, ; y \rangle|$ for every $x \in \mathcal{H}$. Hence

$$\|T\| \;=\; \sup_{\|x\|=1}\,\sup_{\|y\|=1} |\langle Tx\,;y\rangle|.$$

Recalling that $T^{**} = T$ (see Proposition 5.65), it also follows that $\|T^*y\| = \sup_{\|x\|=1}|\langle T^*y\,;x\rangle| = \sup_{\|x\|=1}|\langle y\,;Tx\rangle| = \sup_{\|x\|=1}|\langle Tx\,;y\rangle|$ for every $y \in \mathcal{K}$. Therefore, as $\|T^*\| = \|T\|$ (cf. Proposition 5.65 again),

$$\|T\| \;=\; \|T^*\| \;=\; \sup_{\|y\|=1}\,\sup_{\|x\|=1} |\langle Tx\,;y\rangle|. \qquad \square$$

Let us see some further elementary properties of the adjoint. Consider the linear space $\mathcal{B}[\mathcal{H},\mathcal{Y}]$, where $\mathcal{H}$ is a Hilbert space and $\mathcal{Y}$ is an inner product space. First observe that

$$O^* \;=\; O.$$

That is, the adjoint $O^* \in \mathcal{B}[\mathcal{Y},\mathcal{H}]$ of the null transformation $O \in \mathcal{B}[\mathcal{H},\mathcal{Y}]$ coincides with the null transformation $O \in \mathcal{B}[\mathcal{Y},\mathcal{H}]$. In fact, $\langle x\,;O^*y\rangle = \langle Ox\,;y\rangle = 0$ for every $x \in \mathcal{H}$ and every $y \in \mathcal{Y}$, and hence $O^*y = 0$ for every $y \in \mathcal{Y}$. Now take S and T in $\mathcal{B}[\mathcal{H},\mathcal{Y}]$ so that $S+T$ and αT lie in $\mathcal{B}[\mathcal{H},\mathcal{Y}]$, where α is any scalar. Consider their adjoints: the unique transformations in $\mathcal{B}[\mathcal{Y},\mathcal{H}]$ such that $\langle (S+T)x\,;y\rangle = \langle x\,;(S+T)^*y\rangle$ and $\langle \alpha Tx\,;y\rangle = \langle x\,;(\alpha T)^*y\rangle$ for every $x \in \mathcal{H}$ and $y \in \mathcal{Y}$, respectively. These identities imply that $\langle x\,;(S+T)^*y\rangle = \langle Sx\,;y\rangle + \langle Tx\,;y\rangle = \langle x\,;(S^*+T^*)y\rangle$ and $\langle x\,;(\alpha T)^*y\rangle = \langle x\,;\overline{\alpha}T^*y\rangle$ for every $x \in \mathcal{H}$ and every $y \in \mathcal{Y}$. Therefore,

$$(S+T)^* \;=\; S^*+T^* \quad \text{and} \quad (\alpha T)^* \;=\; \overline{\alpha}T^*.$$

Next take T in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ and S in $\mathcal{B}[\mathcal{K},\mathcal{Y}]$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $\mathcal{Y}$ is an inner product space, so that ST lies in $\mathcal{B}[\mathcal{H},\mathcal{Y}]$ by Proposition 4.16. Consider its adjoint, which is the unique transformation in $\mathcal{B}[\mathcal{Y},\mathcal{H}]$ such that $\langle STx\,;y\rangle = \langle x\,;(ST)^*y\rangle$ for every $x \in \mathcal{H}$ and $y \in \mathcal{Y}$. This implies that $\langle x\,;(ST)^*y\rangle = \langle x\,;T^*S^*y\rangle$ for every $x \in \mathcal{H}$ and every $y \in \mathcal{Y}$. Then

$$(ST)^* \;=\; T^*S^*.$$

Finally, consider the algebra $\mathcal{B}[\mathcal{H}]$, where $\mathcal{H}$ is a Hilbert space. It is clear by the very definition of adjoint that

$$I^* \;=\; I,$$

where I is the identity operator in $\mathcal{B}[\mathcal{H}]$. If $T \in \mathcal{G}[\mathcal{H},\mathcal{K}]$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces (i.e., if T is invertible in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ so that $T^{-1} \in \mathcal{B}[\mathcal{K},\mathcal{H}]$ by the Inverse Mapping Theorem), then the above identities ensure that $I = I^* = (T^{-1}T)^* = T^*(T^{-1})^*$, the identity operator in $\mathcal{B}[\mathcal{H}]$, and $I = I^* = (TT^{-1})^* = (T^{-1})^*T^*$, the identity operator in $\mathcal{B}[\mathcal{K}]$. Hence $T^* \in \mathcal{G}[\mathcal{K},\mathcal{H}]$. Dually (since $T^{**} = T$), this implies that $T \in \mathcal{G}[\mathcal{H},\mathcal{K}]$. Therefore, $T \in \mathcal{G}[\mathcal{H},\mathcal{K}]$ if and only if $T^* \in \mathcal{G}[\mathcal{K},\mathcal{H}]$ and

$$(T^*)^{-1} \;=\; (T^{-1})^*.$$

Example 5.P. Consider the Hilbert spaces $\mathbb{F}^n$ and $\mathbb{F}^m$ (for arbitrary positive integers m and n) equipped with their usual inner products as in Example 5.A. Take $A \in \mathcal{B}[\mathbb{F}^n, \mathbb{F}^m]$ and recall that $\mathcal{B}[\mathbb{F}^n, \mathbb{F}^m] = \mathcal{L}[\mathbb{F}^n, \mathbb{F}^m]$ by Corollary 4.30. As usual (cf. Examples 2.L and 2.N), we shall identify the linear transformation $A \in \mathcal{L}[\mathbb{F}^n, \mathbb{F}^m]$ with its matrix

$$[A] \,=\, [\alpha_{ij}] \,=\, \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix} \,\in\, \mathbb{F}_{m\times n}$$

relative to the canonical bases for $\mathbb{F}^n$ and $\mathbb{F}^m$. Thus for every $x = (\xi_1, \dots, \xi_n)$ in $\mathbb{F}^n$ the vector $y = Ax = (\upsilon_1, \dots, \upsilon_m)$ in $\mathbb{F}^m$ is such that

$$\upsilon_i \,=\, \sum_{j=1}^{n} \alpha_{ij}\xi_j$$

for every $i = 1, \dots, m$. In terms of common matrix notation, and according to ordinary matrix operations, the matrix equation

$$[y] \,=\, [A][x]$$

represents the identity $y = Ax$. Here

$$[y] \,=\, \begin{pmatrix} \upsilon_1 \\ \vdots \\ \upsilon_m \end{pmatrix} \in \mathbb{F}_{m\times 1} \qquad \text{and} \qquad [x] \,=\, \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in \mathbb{F}_{n\times 1}$$

are the matrices of y and x with respect to the canonical bases for $\mathbb{F}^m$ and $\mathbb{F}^n$, respectively. Recall that $\overline{[A]} = [\overline{\alpha}_{ij}] \in \mathbb{F}_{m\times n}$, $[A]^T = [\alpha_{ji}] \in \mathbb{F}_{n\times m}$, and $\overline{[A]}^{\,T} = \overline{[A]^T} = [\overline{\alpha}_{ji}] \in \mathbb{F}_{n\times m}$ denote conjugate, transpose, and transpose conjugate of $[A] = [\alpha_{ij}] \in \mathbb{F}_{m\times n}$, respectively. Now observe that the inner product on $\mathbb{F}^m$ can be written as

$$\langle y\,;z\rangle \,=\, \sum_{i=1}^{m} \upsilon_i\,\overline{\zeta}_i \,=\, [y]^T\,\overline{[z]}$$

for every $y = (\upsilon_1, \dots, \upsilon_m) \in \mathbb{F}^m$ and every $z = (\zeta_1, \dots, \zeta_m) \in \mathbb{F}^m$. Using standard matrix algebra, we get

$$\langle Ax\,;y\rangle \,=\, \big([A][x]\big)^T\,\overline{[y]} \,=\, [x]^T\,[A]^T\,\overline{[y]} \,=\, [x]^T\,\overline{\big(\overline{[A]}^{\,T}[y]\big)}$$

for every $x \in \mathbb{F}^n$ and every $y \in \mathbb{F}^m$. Next consider the adjoint $A^* \in \mathcal{B}[\mathbb{F}^m, \mathbb{F}^n]$ of $A \in \mathcal{B}[\mathbb{F}^n, \mathbb{F}^m]$, and let $[A^*] \in \mathbb{F}_{n\times m}$ denote the matrix of A^* relative to the canonical bases for $\mathbb{F}^m$ and $\mathbb{F}^n$. Then

$$\langle Ax\,;y\rangle \,=\, \langle x\,;A^*y\rangle \,=\, [x]^T\,\overline{\big([A^*][y]\big)}$$

for every $x \in \mathbb{F}^n$ and every $y \in \mathbb{F}^m$. Therefore, by uniqueness of the matrix representation (with respect to fixed bases — cf. Example 2.L), we get

$$[A^*] \,=\, \overline{[A]}^T.$$

That is, *the matrix of the adjoint A^* of A is the transpose conjugate of the matrix of A.*

Example 5.Q. Let S be a nondegenerate interval of the real line $\mathbb{R}$. Consider the Hilbert space $L^2(S)$ equipped with its usual inner product (see Example 5.D). As always, we write $x \in L^2(S)$ instead of $[x] \in L^2(S)$, where x is any representative of the equivalence class $[x]$. Take an arbitrary $a \in L^2(S{\times}S)$ so that $a(s,\cdot) \in L^2(S)$ for each $s \in S$, $a(\cdot,t) \in L^2(S)$ for each $t \in S$, and

$$\begin{aligned}\|a\|^2 \,&=\, \int_S \|a(s,\cdot)\|^2\,ds \,=\, \int_S \|a(\cdot,t)\|^2\,dt \\ &=\, \int_S\int_S |a(s,t)|^2\,ds\,dt \,=\, \int_S\int_S |a(s,t)|^2\,dt\,ds \,<\, \infty.\end{aligned}$$

The preceding identities are due to a well-known result in integration theory referred to as the *Fubini Theorem* (which ensures that the order of the integrals can be interchanged). Now define the integral mapping $A\colon L^2(S) \to L^2(S)$ as follows. For each $x \in L^2(S)$ let $z = Ax \in L^2(S)$ be given by

$$z(s) \,=\, \int_S a(s,t)\,x(t)\,dt \,=\, \big\langle a(s,\cdot)\,;\overline{x}\,\big\rangle \,=\, \big\langle x\,;\overline{a(s,\cdot)}\,\big\rangle$$

for every $s \in S$. Note that $z = Ax$ actually lies in $L^2(S)$ because a lies in $L^2(S{\times}S)$. Indeed, by the Schwarz inequality,

$$\|z\|^2 \,=\, \int_S \big|\big\langle x\,;\overline{a(s,\cdot)}\,\big\rangle\big|^2 ds \,\le\, \int_S \|x\|^2 \big\|\overline{a(s,\cdot)}\,\big\|^2 ds \,=\, \|x\|^2\|a\|^2,$$

and hence, $\|Ax\| \le \|a\|\,\|x\|$ for every $x \in L^2(S)$ so that A is bounded. Since A is certainly linear, it follows that $A \in \mathcal{B}[L^2(S)]$. Also note that

$$\begin{aligned}\langle Ax\,;y\rangle \,&=\, \int_S \Big(\int_S a(s,t)\,x(t)\,dt\Big)\,\overline{y(s)}\,ds \\ &=\, \int_S\int_S a(s,t)\,x(t)\,\overline{y(s)}\,dt\,ds \,=\, \int_S x(t)\,\overline{\Big(\int_S \overline{a(s,t)}\,y(s)\,ds\Big)}\,dt\end{aligned}$$

for every $x, y \in L^2(S)$. Consider the adjoint $A^* \in \mathcal{B}[L^2(S)]$ of $A \in \mathcal{B}[L^2(S)]$. For each $y \in L^2(S)$ set $w = A^*y \in L^2(S)$ so that

$$\langle Ax\,;y\rangle \,=\, \langle x\,;A^*y\rangle \,=\, \langle x\,;w\rangle \,=\, \int_S x(t)\,\overline{w(t)}\,dt,$$

for every $x, y \in L^2(S)$. Then $w = A^*y \in L^2(S)$ is given by

$$w(s) \;=\; \int_S \overline{a(t,s)}\, y(t)\, dt \;=\; \int_S a^*(s,t)\, y(t)\, dt$$

for every $s \in S$. Therefore, the adjoint A^* of the integral operator A is again an integral operator whose kernel $a^* \in L^2(S{\times}S)$ is related to the kernel $a \in L^2(S{\times}S)$ of A as follows. For every $(s,t) \in S{\times}S$,

$$a^*(s,t) \;=\; \overline{a(t,s)}.$$

An isometry between metric spaces is a map that preserves distance, and so every isometry is an injective contraction. A linear isometry between normed spaces is a linear transformation that preserves norm and, between inner product spaces, a linear isometry is a linear transformation that preserves inner product. Propositions 4.37 and 5.21 gave necessary and sufficient conditions for a linear transformation to be an isometry. Here is another one for linear transformations between Hilbert spaces that is stated in terms of the adjoint.

Proposition 5.72. *A transformation $V \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ of a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$ is an isometry if and only if $V^*V = I$.*

Proof. According to Proposition 5.21, V is an isometry if and only if it preserves inner product; that is, $\langle Vx\,;Vy\rangle = \langle x\,;y\rangle$ for every $x, y \in \mathcal{H}$. Equivalently, $\langle (V^*V - I)x\,;y\rangle = 0$ for every $x, y \in \mathcal{H}$, and hence $(V^*V - I)x = 0$ for every $x \in \mathcal{H}$, where I is the identity on $\mathcal{H}$. □

A *coisometry* is a transformation T in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ such that its adjoint T^* in $\mathcal{B}[\mathcal{K},\mathcal{H}]$ is an isometry. By the previous proposition, *T is a coisometry if and only if $TT^* = I$* (identity on $\mathcal{K}$). Recall that a unitary transformation U in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ is an isometric isomorphism between $\mathcal{H}$ and $\mathcal{K}$. Equivalently, U in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ is *unitary* if it is an *invertible isometry* (i.e., an isometry in $\mathcal{G}[\mathcal{H},\mathcal{K}]$), which means a *surjective isometry* (since isometries are always injective).

Proposition 5.73. *Take $U \in \mathcal{B}[\mathcal{H},\mathcal{K}]$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. The following assertions are pairwise equivalent.*

(a) *U is unitary* (i.e., *U is a surjective isometry, or an invertible isometry*).

(b) *U is invertible and $U^{-1} = U^*$* (i.e., *U lies in $\mathcal{G}[\mathcal{H},\mathcal{K}]$ and $U^{-1} = U^*$*).

(c) *U is invertible and $\|U\| = \|U^{-1}\| = 1$.*

(d) *U is invertible, $\|U\| \le 1$, and $\|U^{-1}\| \le 1$.*

(e) *$U^*U = I$ (identity on $\mathcal{H}$) and $UU^* = I$ (identity on $\mathcal{K}$).*

(f) *U is an isometry and a coisometry.*

(g) *U is invertible and both U and U^{-1} are isometries.*

(h) *$\|Ux\| = \|x\|$ and $\|U^*y\| = \|y\|$ for every $x \in \mathcal{H}$ and every $y \in \mathcal{K}$.*

If $\mathcal{H} = \mathcal{K}$, then each of the above is equivalent to each of the following.

(i) $U^*U = UU^* = I$.

(j) *U is an isometry such that* $U^*U = UU^*$.

(k) $\|U^*x\| = \|Ux\| = \|x\|$ *for every* $x \in \mathcal{H}$.

(l) $\|U^{*n}x\| = \|U^nx\| = \|x\|$ *for every* $x \in \mathcal{H}$ *and every integer* $n \geq 1$.

(m) $U^{*n}U^n = U^nU^{*n} = I$ *for every integer* $n \geq 1$.

(n) *U is invertible and* $\|U^n\| = \|U^{-n}\| = 1$ *for every integer* $n \geq 1$.

Proof. Take $U \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$. First we show that (a) and (b) are equivalent. If U is unitary, then it is an isometry in $\mathcal{G}[\mathcal{H}, \mathcal{K}]$. Thus there exists $U^{-1} \in \mathcal{G}[\mathcal{K}, \mathcal{H}]$ such that $UU^{-1} = I$ (identity on $\mathcal{K}$). Proposition 5.21 ensures that

$$\langle Ux_1 ; Ux_2 \rangle \;=\; \langle x_1 ; x_2 \rangle \quad \text{for every} \quad x_1, x_2 \in \mathcal{H}.$$

Hence

$$\langle x ; U^{-1}y \rangle \;=\; \langle Ux ; UU^{-1}y \rangle \;=\; \langle Ux ; y \rangle \;=\; \langle x ; U^*y \rangle$$

for every $x \in \mathcal{H}$ and $y \in \mathcal{K}$. Then, as the adjoint is unique (Proposition 5.65), $U^{-1} = U^*$. Conversely, if $U \in \mathcal{G}[\mathcal{H}, \mathcal{K}]$ and $U^{-1} = U^*$, then $U^*U = I$ (identity on $\mathcal{H}$), and so U is a surjective isometry (Proposition 5.72). Thus (a) $\Leftrightarrow$ (b). It is plain that (b) $\Leftrightarrow$ (e) by Theorem 4.22, (e) $\Leftrightarrow$ (f) by Proposition 5.72, and (f) $\Leftrightarrow$ (h) by Proposition 4.37. Tautologically (g) $\Rightarrow$ (a), and (e) $\Rightarrow$ (g) by Proposition 5.72 since $(U^{-1})^*U^{-1} = (UU^*)^{-1}$. If (b) holds, then $1 = \|U^{-1}U\| = \|U^*U\| = \|U\|^2 = \|U^*\|^2 = \|U^{-1}\|^2$ by Proposition 5.65, and so (b) $\Rightarrow$ (c). Trivially (c) $\Rightarrow$ (d). If (d) holds, then $\|x\| = \|U^{-1}Ux\| \leq \|Ux\| \leq \|x\|$ so that $\|Ux\| = \|x\|$ for each x in $\mathcal{H}$, and therefore (d) $\Rightarrow$ (a). Suppose $\mathcal{H} = \mathcal{K}$. Thus (i) $\Leftrightarrow$ (e) $\Leftrightarrow$ (j) by Proposition 5.72, (l) $\Leftrightarrow$ (f) by Proposition 4.37(c), (k) $\Leftrightarrow$ (h) trivially, (m) $\Rightarrow$ (i) and the converse follows by induction, and (b, l) $\Rightarrow$ (n) $\Rightarrow$ (c). $\square$

Let T be an operator in $\mathcal{B}[\mathcal{X}]$, where $\mathcal{X}$ is an inner product space, and let $\mathcal{M}$ be a subspace of $\mathcal{X}$. Recall that $\mathcal{M}$ is an *invariant subspace for* T (or *invariant under* T, or T*-invariant*) if $T(\mathcal{M}) \subseteq \mathcal{M}$ (i.e., $Tx \in \mathcal{M}$ whenever $x \in \mathcal{M}$). A *nontrivial invariant subspace* for T is an invariant subspace $\mathcal{M}$ for T such that $\{0\} \neq \mathcal{M} \neq \mathcal{X}$ (see Problems 4.18 to 4.20). If $\mathcal{M}$ and its orthogonal complement $\mathcal{M}^\perp$ are both invariant for T (i.e., if $T(\mathcal{M}) \subseteq \mathcal{M}$ and $T(\mathcal{M}^\perp) \subseteq \mathcal{M}^\perp$), then we say that $\mathcal{M}$ *reduces* T (or $\mathcal{M}$ is a *reducing subspace* for T). Accordingly, a *nontrivial reducing subspace* for T is a reducing subspace for T such that $\{0\} \neq \mathcal{M} \neq \mathcal{X}$. An operator is *reducible* if it has a nontrivial reducing subspace. Now let $\mathcal{X} = \mathcal{H}$ be a Hilbert space and consider the orthogonal direct sum $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ of Theorem 5.25. Observe from Example 2.O (also see Problem 4.16) that the following assertions are pairwise equivalent.

(a) $\mathcal{M}$ reduces T,

(b) $T = T|_{\mathcal{M}} \oplus T|_{\mathcal{M}^\perp} = \left(\begin{smallmatrix} T|_{\mathcal{M}} & O \\ O & T|_{\mathcal{M}^\perp} \end{smallmatrix}\right) \colon \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp \to \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$.

(c) $PT = TP$, where $P = \left(\begin{smallmatrix} I & O \\ O & O \end{smallmatrix}\right)$: $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp \to \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ is the orthogonal projection onto $\mathcal{M}$.

If $\mathcal{M}$ is nontrivial, then T is reducible. This suggests that, if $\mathcal{M}$ reduces T, then the investigation of T is *reduced* to the investigation of smaller operators (viz., $T|_{\mathcal{M}}$ and $T|_{\mathcal{M}^\perp}$), which justifies the terminology "reducing subspace".

Proposition 5.74. *Let T be any operator on a Hilbert space $\mathcal{H}$. A subspace $\mathcal{M}$ of $\mathcal{H}$ is invariant for T if and only if $\mathcal{M}^\perp$ is invariant for T^*. Thus T has a nontrivial invariant subspace if and only if T^* has.*

Proof. Let $\mathcal{M}$ be a subspace of a Hilbert space $\mathcal{H}$, let T be any operator in $\mathcal{B}[\mathcal{H}]$, and take any $y \in \mathcal{M}^\perp$. If $Tx \in \mathcal{M}$ whenever $x \in \mathcal{M}$, then $\langle x\,; T^*y\rangle = \langle Tx\,; y\rangle = 0$ for every $x \in \mathcal{M}$ so that $T^*y \perp \mathcal{M}$. Then $T^*y \in \mathcal{M}^\perp$ for every $y \in \mathcal{M}^\perp$. Conclusion: $T(\mathcal{M}) \subseteq \mathcal{M}$ implies $T^*(\mathcal{M}) \subseteq \mathcal{M}^\perp$. Conversely, since the above implication holds for every $T \in \mathcal{B}[\mathcal{H}]$, it follows that $T^*(\mathcal{M}^\perp) \subseteq \mathcal{M}^\perp$ implies $T^{**}(\mathcal{M}^{\perp\perp}) \subseteq \mathcal{M}^{\perp\perp}$. But $T^{**} = T$ and $\mathcal{M}^{\perp\perp} = \mathcal{M}^- = \mathcal{M}$ (Propositions 5.15 and 5.65(c)).Thus $T^*(\mathcal{M}^\perp) \subseteq \mathcal{M}^\perp$ implies $T(\mathcal{M}) \subseteq \mathcal{M}$. Finally, note that $\{0\} \neq \mathcal{M} \neq \mathcal{H}$ if and only if $\{0\} \neq \mathcal{M}^\perp \neq \mathcal{H}$ (Proposition 5.15). □

Corollary 5.75. *A subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ reduces $T \in \mathcal{B}[\mathcal{H}]$ if and only if it is invariant for both T and T^*. In this case $(T|_{\mathcal{M}})^* = T^*|_{\mathcal{M}}$.*

Proof. Since $\mathcal{M}^{\perp\perp} = \mathcal{M}$, the previous proposition says that $T(\mathcal{M}^\perp) \subseteq \mathcal{M}^\perp$ if and only if $T^*(\mathcal{M}) \subseteq \mathcal{M}$. Therefore, $T(\mathcal{M}) \subseteq \mathcal{M}$ and $T(\mathcal{M}^\perp) \subseteq \mathcal{M}^\perp$ if and only if $T(\mathcal{M}) \subseteq \mathcal{M}$ and $T^*(\mathcal{M}) \subseteq \mathcal{M}$. Moreover, in this case, $\langle (T|_{\mathcal{M}})x\,; y\rangle = \langle Tx\,; y\rangle = \langle x\,; T^*y\rangle = \langle x\,; (T^*|_{\mathcal{M}})y\rangle$ for every x, y in the Hilbert space $\mathcal{M}$. □

Recall that $\mathcal{N}(T)$ and $\mathcal{R}(T)^-$ are invariant subspaces for $T \in \mathcal{B}[\mathcal{H}]$ (see Problems 4.20 to 4.22). Null spaces of Hilbert space operators constitute an important source of invariant subspaces. The next result shows that $\mathcal{N}(T^*)^\perp$, $\mathcal{R}(T^*)^\perp$, $\mathcal{N}(T^*T)$, and $\mathcal{R}(TT^*)^-$ also are invariant subspaces for $T \in \mathcal{B}[\mathcal{H}]$.

Proposition 5.76. *If T is a bounded linear transformation of a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$, then*

(a) $$\mathcal{N}(T) = \mathcal{R}(T^*)^\perp = \mathcal{N}(T^*T),$$

(b) $$\mathcal{R}(T)^- = \mathcal{N}(T^*)^\perp = \mathcal{R}(TT^*)^-,$$

(a*) $$\mathcal{N}(T^*) = \mathcal{R}(T)^\perp = \mathcal{N}(TT^*),$$

(b*) $$\mathcal{R}(T^*)^- = \mathcal{N}(T)^\perp = \mathcal{R}(T^*T)^-.$$

Proof. Note that $x \in \mathcal{R}(T^*)^\perp$ if and only if $\langle x\,; T^*y\rangle = 0$ for every $y \in \mathcal{K}$. By the definition of adjoint, this is equivalent to $\langle Tx\,; y\rangle = 0$ for every $y \in \mathcal{K}$, which means that $Tx = 0$; that is, $x \in \mathcal{N}(T)$. Hence

$$\mathcal{R}(T^*)^\perp = \mathcal{N}(T).$$

Moreover, since $\|Tx\|^2 = \langle Tx\,;Tx\rangle = \langle T^*Tx\,;x\rangle$ for every $x \in \mathcal{H}$, it follows that $\mathcal{N}(T^*T) \subseteq \mathcal{N}(T)$. But $\mathcal{N}(T) \subseteq \mathcal{N}(T^*T)$ trivially, and so

$$\mathcal{N}(T) = \mathcal{N}(T^*T),$$

which completes the proof of (a). Since (a) holds true for every $T \in \mathcal{B}[\mathcal{H},\mathcal{K}]$, it also holds for $T^* \in \mathcal{B}[\mathcal{K},\mathcal{H}]$ and $TT^* \in \mathcal{B}[\mathcal{K}]$. Therefore (cf. Propositions 5.15 and 5.65(c)),

$$\begin{aligned}\mathcal{R}(T)^- &= \mathcal{R}(T^{**})^{\perp\perp} = \mathcal{N}(T^*)^\perp = \mathcal{N}(T^{**}T^*)^\perp \\ &= \mathcal{N}(TT^*)^\perp = \mathcal{R}((TT^*)^*)^{\perp\perp} = \mathcal{R}(T^{**}T^*)^{\perp\perp} = \mathcal{R}(TT^*)^-,\end{aligned}$$

which proves (b). Since $T^{**} = T$ we get the dual expressions (a*) and (b*). □

Here is a useful result concerning closed ranges and adjoints.

Proposition 5.77. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and take any $T \in \mathcal{B}[\mathcal{H},\mathcal{K}]$. The following assertions are pairwise equivalent.*

(a) $\mathcal{R}(T) = \mathcal{R}(T)^-$.

(b) $\mathcal{R}(T^*) = \mathcal{R}(T^*)^-$.

(c) $\mathcal{R}(T^*T) = \mathcal{R}(T^*T)^-$.

(d) $\mathcal{R}(TT^*) = \mathcal{R}(TT^*)^-$.

Proof. Let T be an arbitrary bounded linear transformation of a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$.

Proof of (a)⇔(b). Set $T_0 = T|_{\mathcal{N}(T)^\perp} \in \mathcal{B}[\mathcal{N}(T)^\perp,\mathcal{K}]$, the restriction of T to $\mathcal{N}(T)^\perp$. Since

$$\mathcal{H} = \mathcal{N}(T) + \mathcal{N}(T)^\perp$$

(cf. Proposition 4.13 and Theorem 5.20), it follows that every $x \in \mathcal{H}$ can be written as $x = u + v$ with $u \in \mathcal{N}(T)$ and $v \in \mathcal{N}(T)^\perp$. If $y \in \mathcal{R}(T)$, then $y = Tx = Tu + Tv = Tv = T|_{\mathcal{N}(T)^\perp}v$ for some $x \in \mathcal{H}$, and hence $y \in \mathcal{R}(T|_{\mathcal{N}(T)^\perp})$. Thus $\mathcal{R}(T) \subseteq \mathcal{R}(T|_{\mathcal{N}(T)^\perp})$. Since $\mathcal{R}(T|_{\mathcal{N}(T)^\perp}) \subseteq \mathcal{R}(T)$ trivially, we get

$$\mathcal{R}(T_0) = \mathcal{R}(T) \quad \text{and} \quad \mathcal{N}(T_0) = \{0\}$$

because $\mathcal{N}(T|_{\mathcal{N}(T)^\perp}) = \{0\}$. If $\mathcal{R}(T) = \mathcal{R}(T)^-$, then Corollary 4.24 ensures the existence of $T_0^{-1} \in \mathcal{B}[\mathcal{R}(T),\mathcal{N}(T)^\perp]$. Now take an arbitrary $w \in \mathcal{N}(T)^\perp$ and consider the functional $f_w\colon \mathcal{R}(T) \to \mathbb{F}$ defined by

$$f_w(y) = \langle T_0^{-1}y\,;w\rangle$$

for every $y \in \mathcal{R}(T)$, which is linear (reason: T_0^{-1} is linear and the inner product is linear in the first argument) and bounded (in fact, $|f_w(y)| \le \|T_0^{-1}\|\,\|w\|\,\|y\|$ for every $y \in \mathcal{R}(T)$). The Riesz Representation Theorem (Theorem 5.62) says

that there is a z_w in the Hilbert space $\mathcal{R}(T)$ (recall: $\mathcal{R}(T)$ is a subspace of the Hilbert space $\mathcal{K}$, and so a Hilbert space itself, if $\mathcal{R}(T) = \mathcal{R}(T)^-$) such that

$$f_w(y) \,=\, \langle y\,;z_w\rangle$$

for every $y \in \mathcal{R}(T)$. Consider the decomposition $\mathcal{H} = \mathcal{N}(T) + \mathcal{N}(T)^\perp$ (again) and take any $x \in \mathcal{H}$ so that $x = u + v$ with $u \in \mathcal{N}(T)$ and $v \in \mathcal{N}(T)^\perp$. Then

$$\begin{aligned}\langle x\,;T^*z_w\rangle \,&=\, \langle Tx\,;z_w\rangle \,=\, \langle Tu\,;z_w\rangle + \langle Tv\,;z_w\rangle \,=\, \langle Tv\,;z_w\rangle\\ &=\, f_w(Tv) \,=\, \langle T_0^{-1}Tv\,;w\rangle \,=\, \langle T_0^{-1}T_0\,v\,;w\rangle\\ &=\, \langle v\,;w\rangle \,=\, \langle u\,;w\rangle + \langle v\,;w\rangle \,=\, \langle x\,;w\rangle.\end{aligned}$$

Thus $\langle x\,;T^*z_w - w\rangle = 0$ for every $x \in \mathcal{H}$, which means that $T^*z_w = w$. Therefore, $w \in \mathcal{R}(T^*)$. This ensures the inclusion $\mathcal{N}(T)^\perp \subseteq \mathcal{R}(T^*)$. On the other hand, $\mathcal{R}(T^*) \subseteq \mathcal{R}(T^*)^- = \mathcal{N}(T)^\perp$ by Proposition 5.76(b*) so that

$$\mathcal{R}(T^*) \,=\, \mathcal{N}(T)^\perp.$$

Hence (a) implies (b) by Proposition 5.12 (or by Proposition 5.76(b*)). Since (a) implies (b), it follows that (b) implies (a) because $T^{**} = T$.

Proof of (a)⇒(c) *and* (b)⇒(d). Let $T_1 \in \mathcal{B}[\mathcal{H}, \mathcal{R}(T)]$ be defined by $T_1x = Tx$ for every $x \in \mathcal{H}$ (i.e., T_1 is surjective and coincides with T on $\mathcal{H}$). Clearly,

$$\mathcal{R}(T) \,=\, \mathcal{R}(T_1).$$

Let $T_1^* \in \mathcal{B}[\mathcal{R}(T), \mathcal{H}]$ be the adjoint of $T_1 \in \mathcal{B}[\mathcal{H}, \mathcal{R}(T)]$, consider the restriction of $T^* \in \mathcal{B}[\mathcal{K}, \mathcal{H}]$ of T to $\mathcal{R}(T)$, viz., $T^*|_{\mathcal{R}(T)} \in \mathcal{B}[\mathcal{R}(T), \mathcal{H}]$, and note that

$$\langle x\,;T_1^*y\rangle \,=\, \langle T_1x\,;y\rangle \,=\, \langle Tx\,;y\rangle \,=\, \langle x\,;T^*y\rangle \,=\, \langle x\,;T^*|_{\mathcal{R}(T)}y\rangle$$

for every $x \in \mathcal{H}$ and every $y \in \mathcal{R}(T)$. Then $T_1^*y = T^*|_{\mathcal{R}(T)}y$ for every $y \in \mathcal{R}(T)$ (i.e., $T_1^* = T^*|_{\mathcal{R}(T)}$), and hence

$$\mathcal{R}(T_1^*) \,=\, \mathcal{R}(T^*|_{\mathcal{R}(T)}).$$

Observe that x lies in $\mathcal{R}(T^*|_{\mathcal{R}(T)})$ if and only if $x = T^*|_{\mathcal{R}(T)}y = T^*y$ for some y in $\mathcal{R}(T)$. But this is equivalent to saying that $x = T^*Tu$ for some u in $\mathcal{H}$, which means that x lies in $\mathcal{R}(T^*T)$. That is,

$$\mathcal{R}(T^*|_{\mathcal{R}(T)}) \,=\, \mathcal{R}(T^*T),$$

so that

$$\mathcal{R}(T_1^*) \,=\, \mathcal{R}(T^*T).$$

If $\mathcal{R}(T) = \mathcal{R}(T)^-$, then $\mathcal{R}(T_1) = \mathcal{R}(T_1)^-$. Since (a) implies (b), it follows that $\mathcal{R}(T_1^*) = \mathcal{R}(T_1^*)^-$. Therefore,

$$\mathcal{R}(T^*T) \,=\, \mathcal{R}(T^*T)^-.$$

Conclusion: (a) implies (c), and hence (b) implies (d) (because $T^{**} = T$).

Proof of (d)⇒(a) *and* (c)⇒(b). According to Proposition 5.76(b),

$$\mathcal{R}(TT^*) \subseteq \mathcal{R}(T) \subseteq \mathcal{R}(T)^- = \mathcal{R}(TT^*)^-.$$

Then (d) implies (a), and so (c) implies (b) (because $T^{**} = T$). □

We close this section by introducing an important notion. Let $\mathcal{A}$ be a unital (complex) Banach algebra (cf. Definition 4.17) and suppose there exists a mapping $\mathcal{A} \to \mathcal{A}$ defined by $A \mapsto A^*$ that satisfies the following conditions for all $A, B \in \mathcal{A}$ and all $\alpha \in \mathbb{C}$.

$$\begin{aligned} &\text{(i)} && (A^*)^* = A, \\ &\text{(ii)} && (AB)^* = B^*A^*, \\ &\text{(iii)} && (A+B)^* = A^* + B^*, \\ &\text{(iv)} && (\alpha A)^* = \overline{\alpha} A^*. \end{aligned}$$

Such a mapping is called an *involution* on $\mathcal{A}$. A C*-*algebra* is a unital Banach algebra with an involution $*\colon \mathcal{A} \to \mathcal{A}$ such that

$$\text{(v)} \qquad \|A^*A\| = \|A\|^2$$

for every $A \in \mathcal{A}$. It is clear that *if $\mathcal{H}$ is a complex Hilbert space, then $\mathcal{B}[\mathcal{H}]$ is a* C*-*algebra, where $*$ is the adjoint operation*. Every $T \in \mathcal{B}[\mathcal{H}]$ determines a C*-subalgebra of $\mathcal{B}[\mathcal{H}]$, denoted by $C^*(T)$, which is the smallest C*-algebra of operators from $\mathcal{B}[\mathcal{H}]$ containing both T and the identity I. It can be shown that $C^*(T) = P(T, T^*)^-$, the closure in $\mathcal{B}[\mathcal{H}]$ of all polynomials in T and T^* with complex coefficients. We mention the *Gelfand–Naimark Theorem* that asserts the converse: *Every* C*-*algebra is isometrically $*$-isomorphic to a* C*-*subalgebra of $\mathcal{B}[\mathcal{H}]$*. That is, for every C*-algebra $\mathcal{A}$ there exists an isometric algebra isomorphism of $\mathcal{A}$ onto a C*-subalgebra of $\mathcal{B}[\mathcal{H}]$ that preserves the involution $*$. A great deal of the rest of this book can be posed in an abstract C*-algebra. However, we shall stick to $\mathcal{B}[\mathcal{H}]$.

5.13 Self-Adjoint Operators

Throughout this section $\mathcal{H}$ and $\mathcal{K}$ will stand for Hilbert spaces. An operator $T \in \mathcal{B}[\mathcal{H}]$ is *self-adjoint* (or *Hermitian*) if $T^* = T$. By the definition of the adjoint operator, $T \in \mathcal{B}[\mathcal{H}]$ is self-adjoint if and only if

$$\langle Tx\,;y\rangle = \langle x\,;Ty\rangle \quad \text{for every} \quad x, y \in \mathcal{H}.$$

Proposition 5.78. *If $T \in \mathcal{B}[\mathcal{H}]$ is self-adjoint, then*

$$\|T\| = \sup_{\|x\|=1} |\langle Tx\,;x\rangle|.$$

Proof. Let T be any operator in $\mathcal{B}[\mathcal{H}]$. The Schwarz inequality says that $|\langle Tx\,;x\rangle| \le \|Tx\|\,\|x\| \le \|T\|\,\|x\|^2$ for every $x \in \mathcal{H}$. Then

$$\sup_{\|u\|=1} |\langle Tu\,;u\rangle| \;\le\; \|T\|.$$

On the other hand, $\langle T(x \pm y)\,;x \pm y\rangle = \langle Tx\,;x\rangle \pm \langle Tx\,;y\rangle \pm \langle Ty\,;x\rangle + \langle Ty\,;y\rangle$ for every $x, y \in \mathcal{H}$. Therefore, if $T = T^*$, then $\langle Ty\,;x\rangle = \langle y\,;Tx\rangle = \overline{\langle Tx\,;y\rangle}$, and hence $\langle Tx\,;y\rangle + \langle Ty\,;x\rangle = 2\,\mathrm{Re}\,\langle Tx\,;y\rangle$, which implies that

$$\langle T(x \pm y)\,;x \pm y\rangle \;=\; \langle Tx\,;x\rangle \pm 2\,\mathrm{Re}\,\langle Tx\,;y\rangle + \langle Ty\,;y\rangle.$$

Thus

$$\langle T(x+y)\,;x+y\rangle - \langle T(x-y)\,;x-y\rangle \;=\; 4\,\mathrm{Re}\,\langle Tx\,;y\rangle$$

for every $x, y \in \mathcal{H}$. But $|\langle Tz\,;z\rangle| = |\langle T(\|z\|^{-1}z)\,;\|z\|^{-1}z\rangle|\,\|z\|^2$ for every non-zero vector z in $\mathcal{H}$, and so

$$|\langle Tz\,;z\rangle| \;\le\; \sup_{\|u\|=1} |\langle Tu\,;u\rangle|\,\|z\|^2$$

for every $z \in \mathcal{H}$. By the above two relations and the parallelogram law,

$$\begin{aligned} 4\,\mathrm{Re}\,\langle Tx\,;y\rangle \;&\le\; |\langle T(x+y)\,;x+y\rangle| + |\langle T(x-y)\,;x-y\rangle| \\ &\le\; \sup_{\|u\|=1} |\langle Tu\,;u\rangle| \left(\|x+y\|^2 + \|x-y\|^2\right) \\ &=\; 2 \sup_{\|u\|=1} |\langle Tu\,;u\rangle| \left(\|x\|^2 + \|y\|^2\right) \;=\; 4 \sup_{\|u\|=1} |\langle Tu\,;u\rangle| \end{aligned}$$

if $\|x\| = \|y\| = 1$. Consider the polar representation $\langle Tx\,;y\rangle = |\langle Tx\,;y\rangle|e^{i\theta}$, set $x' = e^{-i\theta}x$, and get $|\langle Tx\,;y\rangle| = e^{-i\theta}\langle Tx\,;y\rangle = \langle Tx'\,;y\rangle$. Since $\|x'\| = \|x\|$, and since $|\langle Tx\,;y\rangle| = \langle Tx'\,;y\rangle = \mathrm{Re}\,\langle Tx'\,;y\rangle$, it follows that

$$|\langle Tx\,;y\rangle| \;\le\; \sup_{\|u\|=1} |\langle Tu\,;u\rangle| \quad \text{whenever} \quad \|x\| = \|y\| = 1.$$

Therefore, according to Corollary 5.71,

$$\|T\| \;=\; \sup_{\|x\|=\|y\|=1} |\langle Tx\,;y\rangle| \;\le\; \sup_{\|u\|=1} |\langle Tu\,;u\rangle|. \qquad \square$$

It is worth noticing now that there are non-self-adjoint operators T in $\mathcal{B}[\mathcal{H}]$ such that $\|T\| = \sup_{\|x\|=1}|\langle Tx\,;x\rangle|$. The class of all operators for which this norm identity holds will be characterized in Chapter 6 (these are the *normaloid* operators; a class of operators that includes the class of the *normal* operators of Section 6.1). The next proposition gives a necessary and sufficient condition for an operator to be self-adjoint on a *complex* Hilbert space.

Proposition 5.79. *If $\mathcal{H}$ is a complex Hilbert space, then $T \in \mathcal{B}[\mathcal{H}]$ is self-adjoint if and only if $\langle Tx\,;x\rangle \in \mathbb{R}$ for every $x \in \mathcal{H}$.*

Proof. If $T = T^*$, then $\langle Tx\,;x\rangle = \langle x\,;Tx\rangle = \overline{\langle Tx\,;x\rangle}$, and so $\langle Tx\,;x\rangle$ is a real number, for every $x \in \mathcal{H}$. Conversely, if $\langle Tx\,;x\rangle \in \mathbb{R}$ for every $x \in \mathcal{H}$, then $\langle Tx\,;x\rangle = \overline{\langle Tx\,;x\rangle} = \langle x\,;Tx\rangle$ for every $x \in \mathcal{H}$, and hence (cf. Problem 5.3(b))

$$\langle Tx\,;y\rangle \;=\; \langle x\,;Ty\rangle$$

for every $x, y \in \mathcal{H}$; that is, T is self-adjoint. □

Remark: If $\mathcal{H}$ is a real Hilbert space, then $\langle Tx\,;x\rangle \in \mathbb{R}$ for every $x \in \mathcal{H}$ for all $T \in \mathcal{B}[\mathcal{H}]$. Since there are non-self-adjoint operators on real Hilbert spaces, the above proposition fails when $\mathcal{H}$ is a real Hilbert space. Example: If $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathcal{B}[\mathbb{R}^2]$, then $O \neq A \neq A^* = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\langle Ax\,;x\rangle = 0$ for all $x \in \mathbb{R}^2$.

Take any z in $\mathcal{K}$, where $\mathcal{K}$ is a (real or complex) Hilbert space. Recall that $\langle z\,;y\rangle = 0$ for every $y \in \mathcal{K}$ if and only if $z = 0$. Now take T in $\mathcal{B}[\mathcal{H},\mathcal{K}]$. Since $T = O$ if and only if $Tx = 0$ in $\mathcal{K}$ for all $x \in \mathcal{H}$, it follows that $T = O$ if and only if $\langle Tx\,;y\rangle = 0$ for every $x \in \mathcal{H}$ and every $y \in \mathcal{K}$. Next set $\mathcal{K} = \mathcal{H}$ and take T in $\mathcal{B}[\mathcal{H}]$. If $\mathcal{H}$ is a *complex* Hilbert space, then $T = O$ if and only if $\langle Tx\,;x\rangle = 0$ for all $x \in \mathcal{H}$ (cf. Problem 5.4). This is in general false for a *real* Hilbert space (see the above example). However, this certainly holds for every operator (on any Hilbert space) that satisfies the norm identity of Proposition 5.78. In particular, it holds for self-adjoint operators on a real Hilbert space.

Corollary 5.80. *Let $\mathcal{H}$ be any Hilbert space. If $T \in \mathcal{B}[\mathcal{H}]$ is self-adjoint, then $T = O$ if and only if $\langle Tx\,;x\rangle = 0$ for all $x \in \mathcal{H}$.*

Proposition 5.79 leads to a partial ordering of the set of all self-adjoint operators. Let $Q \in \mathcal{B}[\mathcal{H}]$ be a *self-adjoint* operator. We say that Q is *nonnegative* (notation: $O \leq Q$ or $Q \geq O$) if $0 \leq \langle Qx\,;x\rangle$ for every $x \in \mathcal{H}$. If $0 < \langle Qx\,;x\rangle$ for every nonzero x in $\mathcal{H}$, then Q is called *positive* (notation: $O < Q$ or $Q > O$). If there exists a real number $\alpha > 0$ such that $\alpha\|x\|^2 \leq \langle Qx\,;x\rangle$ for every $x \in \mathcal{H}$, then Q is called *strictly positive* (notation: $O \prec Q$ or $Q \succ O$). Trivially,

$$O \prec Q \quad\Longrightarrow\quad O < Q \quad\Longrightarrow\quad O \leq Q.$$

Observe that $T^*T \in \mathcal{B}[\mathcal{H}]$ and $TT^* \in \mathcal{B}[\mathcal{K}]$ are self-adjoint operators for every $T \in \mathcal{B}[\mathcal{H},\mathcal{K}]$. In fact, since $0 \leq \|Tx\|^2 = \langle Tx\,;Tx\rangle = \langle T^*Tx\,;x\rangle$ for every $x \in \mathcal{H}$, it follows that $O \leq T^*T$. Dually, since $T^{**} = T$, it also follows that $O \leq TT^*$. The next proposition uses this fact to give necessary and sufficient conditions that a projection be an orthogonal projection.

Proposition 5.81. *If $P \in \mathcal{B}[\mathcal{H}]$ is a nonzero projection, then the following assertions are pairwise equivalent.*

(a) *P is an orthogonal projection.*

(b) *P is self-adjoint.*

(c) *P is nonnegative.*

(d) $\|P\| = 1$.

(e) $\|P\| \le 1$.

Proof. Let P be an orthogonal projection. Proposition 5.51 says that $\mathcal{R}(P) = \mathcal{R}(P)^{--}$ (and so $\mathcal{R}(P^*) = \mathcal{R}(P^*)^{--}$ by Proposition 5.77) and $\mathcal{R}(P) = \mathcal{N}(P)^{\perp}$. But $\mathcal{N}(P)^{\perp} = \mathcal{R}(P^*)^{--}$ by Proposition 5.76. Therefore,

$$\mathcal{R}(P) \;=\; \mathcal{R}(P^*).$$

Now take an arbitrary $x \in \mathcal{H}$. The above identity ensures that $Px = P^*z$ for some $z \in \mathcal{H}$, and hence $P^*Px = (P^*)^2z = (P^2)^*z = P^*z = Px$. That is, $P^*P = P$, which implies that P is self-adjoint. Therefore (a)⇒(b). If $P = P^*$ (i.e., if P is self-adjoint), then $\langle Px\,;x\rangle = \langle P^2x\,;x\rangle = \langle Px\,;Px\rangle = \|Px\|^2 \ge 0$ for every $x \in \mathcal{H}$. This shows that (b)⇒(c). If $P \ge O$ (i.e., if P is nonnegative), then $P = P^*$ (by definition) so that $\|P\|^2 = \|P^*P\| = \|P^2\| = \|P\|$ (cf. Proposition 5.65), and so $\|P\| = 1$ (since $P \ne O$), which implies that $\|P\| \le 1$. Thus (c)⇒(d)⇒(e). Finally, suppose $\|P\| \le 1$ and take an arbitrary $v \in \mathcal{N}(P)^{\perp}$. Since $\mathcal{R}(I-P) = \mathcal{N}(P)$, we get $(I-P)v \in \mathcal{N}(P)$. Then $(I-P)v \perp v$ so that $0 = \langle (I-P)v\,;v\rangle = \|v\|^2 - \langle Pv\,;v\rangle$ and hence $\|v\|^2 = \langle Pv\,;v\rangle \le \|Pv\|\|v\| \le \|P\|\|v\|^2 \le \|v\|^2$. This implies that $\|Pv\| = \|v\| = \langle Pv\,;v\rangle^{\frac{1}{2}}$. Therefore,

$$\|(I-P)v\|^2 \;=\; \|Pv - v\|^2 \;=\; \|Pv\|^2 - 2\mathrm{Re}\langle Pv\,;v\rangle + \|v\|^2 \;=\; 0,$$

and so $v \in \mathcal{N}(I-P) = \mathcal{R}(P)$. Thus $\mathcal{N}(P)^{\perp} \subseteq \mathcal{R}(P)$. On the other hand, if $y \in \mathcal{R}(P)$, then $y = u + v$, where $u \in \mathcal{N}(P)$ and $v \in \mathcal{N}(P)^{\perp}$ (because $\mathcal{R}(P) \subseteq \mathcal{H} = \mathcal{N}(P) + \mathcal{N}(P)^{\perp}$). Since $\mathcal{N}(P)^{\perp} \subseteq \mathcal{R}(P) = \{x \in \mathcal{H}\colon\, Px = x\}$, we get $y = Py = Pu + Pv = Pv = v$, and so $y \in \mathcal{N}(P)^{\perp}$. Hence $\mathcal{R}(P) \subseteq \mathcal{N}(P)^{\perp}$. Then $\mathcal{R}(P) = \mathcal{N}(P)^{\perp}$ so that $\mathcal{R}(P) \perp \mathcal{N}(P)$. Outcome: (e)⇒(a). □

Take $S, T \in \mathcal{B}[\mathcal{H}]$. If $T - S$ is self-adjoint (in particular, if T and S are self-adjoint) and $O \le T - S$, then we write $S \le T$ so that $S \le O$ means $O \le -S$. It is easy to show that $\le$ defines a reflexive, transitive, and antisymmetric relation on the set of all self-adjoint operators on $\mathcal{H}$, and so a partial ordering of it. Similarly, we write $S < T$ if $O < T - S$ and $S \prec T$ if $O \prec T - S$ (and hence $S < O$ means $O < -S$ and $S \prec O$ means $O \prec -S$). Observe that

$$O \prec Q \quad\Longleftrightarrow\quad \alpha I \le Q \text{ for some } \alpha > 0.$$

Moreover, $\|T\| \le 1$ (i.e., T is a contraction) if and only if $T^*T \le I$, and $\|T\| < 1$ (i.e., T is a strict contraction) if and only if $T^*T \prec I$ (i.e., $T^*T \le \beta I$ for some $\beta \in (0,1)$). Why?

Proposition 5.82. *If $Q \in \mathcal{B}[\mathcal{H}]$ is nonnegative, then*

$$|\langle Qx\,;y\rangle|^2 \;\le\; \langle Qx\,;x\rangle\langle Qy\,;y\rangle$$

for every $x, y \in \mathcal{H}$, and hence

$$\|Qx\|^2 \;\le\; \|Q\|\langle Qx\,;x\rangle \quad \textit{for every} \quad x \in \mathcal{H}.$$

Proof. Take $O \le Q \in \mathcal{B}[\mathcal{H}]$. Consider the function $\langle\ ;\ \rangle_Q : \mathcal{H}\times\mathcal{H} \to \mathbb{F}$ given by

$$\langle x\,;y\rangle_Q \,=\, \langle Qx\,;y\rangle \quad \text{for every} \quad x, y \in \mathcal{H}.$$

It is easy to verify that $\langle\ ;\ \rangle_Q$ is a semi-inner product on $\mathcal{H}$. Let $\|\ \|_Q$ be the seminorm induced on $\mathcal{H}$ by $\langle\ ;\ \rangle_Q$ so that $\|x\|_Q^2 = \langle Qx\,;x\rangle$ for every $x \in \mathcal{H}$. Since the Schwarz inequality holds in a semi-inner product space, we get

$$|\langle Qx\,;y\rangle|^2 \,=\, |\langle x\,;y\rangle_Q|^2 \,\le\, \|x\|_Q^2\,\|y\|_Q^2 \,=\, \langle Qx\,;x\rangle\langle Qy\,;y\rangle$$

for every $x, y \in \mathcal{H}$. In particular, by setting $y = Qx$,

$$\begin{aligned}\|Qx\|^4 \,=\, \langle Qx\,;Qx\rangle^2 \,&\le\, \langle Qx\,;x\rangle\langle Q^2x\,;Qx\rangle \\ &\le\, \langle Qx\,;x\rangle\|Q^2x\|\,\|Qx\| \,\le\, \langle Qx\,;x\rangle\|Q\|\,\|Qx\|^2,\end{aligned}$$

which implies that $\|Qx\|^2 \le \|Q\|\,\langle Qx\,;x\rangle$, for every $x \in \mathcal{H}$. □

Positive operators do not necessarily have a bounded inverse. Indeed,

$$O < Q \quad\Longleftrightarrow\quad O \le Q \text{ and } \mathcal{N}(Q) = \{0\} \quad\Longleftrightarrow\quad O \le Q \text{ and } \mathcal{R}(Q)^- = \mathcal{H}.$$

The nontrivial part of the first equivalence follows from Proposition 5.82, and the second one is a straightforward consequence of Propositions 5.15 and 5.76(a*): $\mathcal{R}(Q)^\perp = \mathcal{N}(Q^*) = \mathcal{N}(Q)$ so that $\mathcal{R}(Q)^- = \mathcal{H}$ if and only if $\mathcal{N}(Q) = \{0\}$. Therefore, $Q > O$ has an inverse on its range, which is not necessarily bounded. However, strictly positive operators are invertible (i.e., have a bounded inverse). In fact, if $Q \succ O$, then Q is bounded below (Schwarz inequality). Conversely, if $Q \ge O$ is bounded below, then $Q \succ O$ (cf. Proposition 5.82). Thus, by the above displayed equivalence and Proposition 4.24,

$$O \prec Q \quad\Longleftrightarrow\quad O \le Q \in \mathcal{G}[\mathcal{H}] \quad\Longleftrightarrow\quad O \le Q \text{ and } \mathcal{R}(Q) = \mathcal{H}.$$

Note: If $\mathcal{H}$ is finite dimensional, then $Q \succ O$ if and only if $Q > O$. Why?

Example 5.R. Take a sequence $a = \{\alpha_k\}_{k=0}^\infty$ in ℓ_+^∞ and consider the diagonal operator $D_a = \operatorname{diag}(\{\alpha_k\}_{k=0}^\infty)$ in $\mathcal{B}[\ell_+^2]$ of Examples 4.H and 4.J, now acting on the Hilbert space ℓ_+^2. It is readily verified that

$$\begin{aligned} D_a = D_a^* \quad &\text{if and only if} \quad \alpha_k \in \mathbb{R} \text{ for all } k \ge 0, \\ O \le D_a \quad &\text{if and only if} \quad \alpha_k \ge 0 \text{ for all } k \ge 0, \\ O < D_a \quad &\text{if and only if} \quad \alpha_k > 0 \text{ for all } k \ge 0, \\ O \prec D_a \quad &\text{if and only if} \quad D_a = D_a^* \text{ and } \inf_k \alpha_k > 0. \end{aligned}$$

Let $\mathcal{B}^+[\mathcal{H}]$ denote the weakly closed convex cone of all nonnegative operators in $\mathcal{B}[\mathcal{H}]$ (see Problem 5.49) and set $\mathcal{G}^+[\mathcal{H}] = \mathcal{B}^+[\mathcal{H}] \cap \mathcal{G}[\mathcal{H}]$: the class of all strictly positive operators on $\mathcal{H}$. The following is a rather useful corollary of Proposition 5.82.

Corollary 5.83. *If $\{Q_n\}$ is a sequence of nonnegative operators on $\mathcal{H}$* (i.e., *if $Q_n \in \mathcal{B}^+[\mathcal{H}]$ for every integer n), then*

$$Q_n \xrightarrow{w} O \quad \textit{if and only if} \quad Q_n \xrightarrow{s} O.$$

Proof. Recall that strong convergence always implies weak convergence (to the same limit). On the other hand, since $\mathcal{H}$ is a Hilbert space, it follows that $\sup_n \|Q_n\| < \infty$ whenever $\{Q_n\}$ converges weakly. Therefore, according to Proposition 5.82, $Q_n \xrightarrow{w} O$ implies $Q_n \xrightarrow{s} O$, because

$$\|Q_n x\|^2 \le \sup_k \|Q_k\| \, \langle Q_n x \,; x \rangle$$

for every $x \in \mathcal{H}$ and every integer n (see Proposition 5.67). □

An immediate consequence of the above corollary reads as follows. *If T_n in $\mathcal{B}[\mathcal{H}]$ is such that, for each $n \ge 1$, either $O \le T - T_n$ or $O \le T_n - T$, then $T_n \xrightarrow{w} T$ if and only if $T_n \xrightarrow{s} T$.* This is what is behind the next proposition. Let $\{T_n\}$ be a $\mathcal{B}[\mathcal{H}]$-valued sequence. We say that $\{T_n\}$ is *increasing* or *decreasing* if $O \le T_{n+1} - T_n$ or $O \le T_n - T_{n+1}$, respectively, for every integer n. If it is increasing or decreasing, then we say that $\{T_n\}$ is *monotone*.

Proposition 5.84. *A bounded monotone sequence of self-adjoint operators converges strongly.*

Proof. Suppose $\{T_n\}$ is a bounded decreasing sequence of self-adjoint operators in $\mathcal{B}[\mathcal{H}]$ and take an arbitrary $x \in \mathcal{H}$. Thus $\{\langle T_n x \,; x \rangle\}$ is a real-valued bounded decreasing sequence (see Proposition 5.79 and recall that $\sup_n |\langle T_n x \,; x \rangle| \le \sup_n \|T_n\| \, \|x\|^2$). Therefore, $\{\langle T_n x \,; x \rangle\}$ converges in $\mathbb{R}$ (Problem 3.10), and so in $\mathbb{F}$, which implies that $T_n - T \xrightarrow{w} O$ for some $T \in \mathcal{B}[\mathcal{H}]$ (Problem 5.48). Moreover, as $\{T_n\}$ is decreasing, $\langle T_n x \,; x \rangle \ge \langle T_{n+k} x \,; x \rangle \to \langle T x \,; x \rangle$ as $k \to \infty$ so that $T_n - T \in \mathcal{B}^+[\mathcal{H}]$ for every integer n, and hence Corollary 5.83 ensures that $T_n - T \xrightarrow{s} O$. If $\{T_n\}$ is a bounded increasing sequence of self-adjoint operators, then it is clear that $\{-T_n\}$ is a bounded decreasing sequence of self-adjoint operators. The above argument ensures that $\{-T_n\}$ also converges strongly, and so does $\{T_n\}$. □

5.14 Square Root and Polar Decomposition

Throughout this section $\mathcal{H}$ and $\mathcal{K}$ will again stand for Hilbert spaces. If T is an operator in $\mathcal{B}[\mathcal{H}]$, and if there exists an operator S in $\mathcal{B}[\mathcal{H}]$ such that $S^2 = T$, then S is referred to as a *square root* of T. Nonnegative operators have a unique nonnegative square root.

Theorem 5.85. *Every operator Q in $\mathcal{B}^+[\mathcal{H}]$ has a unique square root $Q^{\frac{1}{2}}$ in $\mathcal{B}^+[\mathcal{H}]$, which commutes with every operator in $\mathcal{B}[\mathcal{H}]$ that commutes with Q.*

Proof. (a) First we show that there is no loss of generality in assuming that Q is a nonnegative contraction (i.e., that $O \le Q \le I$ — cf. Problem 5.55). In fact, if $O \le Q$ and $O \ne Q$, then $O \le \|Q\|^{-1}Q \le I$ (because $\langle \|Q\|^{-1}Qx\,;x\rangle = \|Q\|^{-1}\langle Qx\,;x\rangle \le \|Q\|^{-1}\|Q\|\|x\|^2 = \langle x\,;x\rangle$ for each $x \in \mathcal{H}$). If there is a unique $S \ge O$ such that $S^2 = \|Q\|^{-1}Q$, then the operator $Q^{\frac{1}{2}} = \|Q\|^{\frac{1}{2}}S$ is such that $Q^{\frac{1}{2}} \ge O$ and $(Q^{\frac{1}{2}})^2 = \|Q\|S^2 = Q$. Moreover, $Q^{\frac{1}{2}}$ is the unique nonnegative operator such that $(Q^{\frac{1}{2}})^2 = Q$ (since $S = \|Q\|^{-\frac{1}{2}}Q^{\frac{1}{2}}$ is the unique nonnegative operator such that $S^2 = \|Q\|^{-1}Q$). Finally, if $\|Q\|^{-1}QT = T\|Q\|^{-1}Q$ implies $ST = TS$, then (as $S = \|Q\|^{-\frac{1}{2}}Q^{\frac{1}{2}}$) $QT = TQ$ implies $Q^{\frac{1}{2}}T = TQ^{\frac{1}{2}}$.

(b) Thus suppose $O \le Q \le I$ and set $R = I - Q$ so that $O \le R \le I$. Consider the sequence $\{B_n\}_{n=0}^{\infty}$ in $\mathcal{B}[\mathcal{H}]$ recursively defined by

$$B_{n+1} \;=\; \tfrac{1}{2}(R + B_n^2) \quad \text{with} \quad B_0 = O.$$

Claim 1. $O \le B_n$ for every integer $n \ge 0$.

Proof. The proof goes by induction. First observe that B_0 is trivially self-adjoint. If B_n is self-adjoint for some $n \ge 0$, then B_{n+1} is self-adjoint because $(B_n^2)^* = (B_n^*)^2 = B_n^2$ and $R = R^*$. Hence B_n is self-adjoint for every $n \ge 0$. Now note that $O \le B_0$ trivially. If $O \le B_n$ for some $n \ge 0$, then $\langle B_{n+1}x\,;x\rangle = \frac{1}{2}(\langle Rx\,;x\rangle + \|B_nx\|^2) \ge 0$ for every $x \in \mathcal{H}$ because $O \le R$. □

Claim 2. $\|B_n\| \le 1$ so that $B_n \le I$ for all $n \ge 0$.

Proof. Since $O \le R \le I$ it follows that $\|R\| \le 1$ by Proposition 5.78: $\|R\| = \sup_{\|x\|=1}\langle Rx\,;x\rangle \le \sup_{\|x\|=1}\langle x\,;x\rangle = 1$. Trivially, $\|B_0\| \le 1$. If $\|B_n\| \le 1$ for some $n \ge 0$, then $\|B_{n+1}\| \le \frac{1}{2}(\|R\| + \|B_n\|^2) \le 1$, proving by induction that $\|B_n\| \le 1$ for all $n \ge 0$. Then $O \le B_n \le I$ for all $n \ge 0$ (cf. Problem 5.55). □

Claim 3. $\{B_n\}$ is an increasing sequence.

Proof. It is readily verified by induction that each B_n is a polynomial in R with positive coefficients. Hence $B_nB_m = B_mB_n$ for every $m, n \ge 0$. Thus

$$B_{n+2} - B_{n+1} \;=\; \tfrac{1}{2}\big(B_{n+1}^2 - B_n^2\big) \;=\; \tfrac{1}{2}(B_{n+1} - B_n)(B_{n+1} + B_n)$$

for each $n \ge 0$. Note that $B_1 - B_0 = \frac{1}{2}R$ and $B_2 - B_1 = \frac{1}{8}R^2$ are polynomials in R with positive coefficients. Since $B_{n+1} + B_n$ is a polynomial in R with positive coefficients (because each B_n is), it follows by the above displayed identity that $B_{n+2} - B_{n+1}$ is a polynomial in R with positive coefficients whenever $B_{n+1} - B_n$ is. This proves by induction that $B_{n+1} - B_n$ is a polynomial in R with positive coefficients for every $n \ge 0$, and so $B_{n+1} - B_n \ge O$ for every $n \ge 0$ because $R \ge O$ (cf. Problem 5.52(e)). □

Since $\{B_n\}$ is a bounded monotone sequence of self-adjoint operators, it converges strongly by Proposition 5.84. That is,

$$B_n \xrightarrow{\;s\;} B \quad \text{and hence} \quad I - B_n \xrightarrow{\;s\;} I - B$$

for some $B \in \mathcal{B}[\mathcal{H}]$. Since $O \le B_n \le I$ (i.e., B_n and $I - B_n$ lie in $\mathcal{B}^+[\mathcal{H}]$) for each n, and since $\mathcal{B}^+[\mathcal{H}]$ is weakly (thus strongly) closed in $\mathcal{B}[\mathcal{H}]$ (cf. Problem 5.49 and Proposition 5.68), it follows that $O \le B \le I$. Moreover, as $B_n \xrightarrow{s} B$, we get $B_n^2 \xrightarrow{s} B^2$ (cf. Problem 4.46(a)). Therefore,

$$O \le B = \tfrac{1}{2}\big(R + B^2\big) \le I.$$

Thus $R = 2B - B^2$, and so $Q = I - R = B^2 - 2B + I = (I - B)^2$. Recalling that $O \le I - B \le I$, set $Q^{\frac{1}{2}} = I - B$. Then $O \le Q^{\frac{1}{2}} \le I$ and $(Q^{\frac{1}{2}})^2 = Q$ so that there exists a nonnegative square root $Q^{\frac{1}{2}}$ of Q.

(c) Suppose $TQ = QT$ for some $T \in \mathcal{B}[\mathcal{H}]$. Thus $T(I - R) = (I - R)T$ so that $TR = RT$, and hence $Tp(R) = p(R)T$ for every polynomial p in R. Therefore $TB_n = B_nT$ (cf. proof of Claim 3) for each n. Since $TB_n \xrightarrow{s} TB$ and since $B_nT \xrightarrow{s} BT$, it follows that $TB = BT$ (the strong limit is unique). Then $T(I - B) = (I - B)T$ so that $TQ^{\frac{1}{2}} = Q^{\frac{1}{2}}T$. That is, $Q^{\frac{1}{2}}$ commutes with every operator that commutes with Q.

(d) We show that the nonnegative square root $Q^{\frac{1}{2}}$ of Q is unique. If $A \ge O$ is such that $A^2 = Q$, then $AQ = A^3 = QA$, and so $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}}A$ by (c). Then

$$\begin{aligned}(Q^{\frac{1}{2}} - A)\, Q^{\frac{1}{2}}(Q^{\frac{1}{2}} - A) + (Q^{\frac{1}{2}} - A)A(Q^{\frac{1}{2}} - A)\\ = (Q^{\frac{1}{2}} - A)(Q^{\frac{1}{2}} + A)(Q^{\frac{1}{2}} - A) = (Q^{\frac{1}{2}} - A)(Q - A^2) = O.\end{aligned}$$

But $(Q^{\frac{1}{2}} - A)\, Q^{\frac{1}{2}}(Q^{\frac{1}{2}} - A) \ge O$ and $(Q^{\frac{1}{2}} - A)A(Q^{\frac{1}{2}} - A) \ge O$ (since $Q^{\frac{1}{2}}$ and A are nonnegative — cf. Problem 5.51(a)), which implies that these two operators are null, and so is their difference. That is,

$$(Q^{\frac{1}{2}} - A)^3 = (Q^{\frac{1}{2}} - A)\, Q^{\frac{1}{2}}(Q^{\frac{1}{2}} - A) - (Q^{\frac{1}{2}} - A)A(Q^{\frac{1}{2}} - A) = O,$$

and therefore $(Q^{\frac{1}{2}} - A)^4 = O$. Since $(Q^{\frac{1}{2}} - A)$ is self-adjoint, $\|(Q^{\frac{1}{2}} - A)\|^4 = \|(Q^{\frac{1}{2}} - A)^4\| = 0$ (see Problem 5.45(e)) so that $A = Q^{\frac{1}{2}}$. □

Proposition 5.86. *If $Q \in \mathcal{B}^+[\mathcal{H}]$, then*

(a) $$\|Q^{\frac{1}{2}}\|^2 = \|Q\| = \|Q^2\|^{\frac{1}{2}},$$

(b) $$\mathcal{N}(Q^{\frac{1}{2}}) = \mathcal{N}(Q) = \mathcal{N}(Q^2) \quad \textit{and} \quad \mathcal{R}(Q^{\frac{1}{2}})^- = \mathcal{R}(Q)^- = \mathcal{R}(Q^2)^-.$$

Proof. If Q is nonnegative, then Q^2 is nonnegative (Problem 5.52). Hence Q is the unique nonnegative square root of Q^2 (Theorem 5.85). Therefore, the identities involving Q and Q^2 follow at once if the identities involving $Q^{\frac{1}{2}}$ and Q are established.

(a) $Q^{\frac{1}{2}} = (Q^{\frac{1}{2}})^*$ implies $\|Q^{\frac{1}{2}}\|^2 = \|(Q^{\frac{1}{2}})^2\| = \|Q\|$ (Proposition 5.65).

(b) Since $Q = Q^{\frac{1}{2}}Q^{\frac{1}{2}}$, we get $\mathcal{N}(Q^{\frac{1}{2}}) \subseteq \mathcal{N}(Q)$. On the other hand, $\|Q^{\frac{1}{2}}x\|^2 = \langle Q^{\frac{1}{2}}x\,; Q^{\frac{1}{2}}x\rangle = \langle Qx\,; x\rangle$ for each $x \in \mathcal{H}$ and so $\mathcal{N}(Q) \subseteq \mathcal{N}(Q^{\frac{1}{2}})$. Thus $\mathcal{N}(Q) = \mathcal{N}(Q^{\frac{1}{2}})$. Then $\mathcal{R}(Q)^- = \mathcal{N}(Q)^\perp = \mathcal{N}(Q^{\frac{1}{2}})^\perp = \mathcal{R}(Q^{\frac{1}{2}})^-$ (Proposition 5.76). □

A *partial isometry* is a bounded linear transformation that acts isometrically on the orthogonal complement of its null space. That is, $W \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ is a partial isometry if $W|_{\mathcal{N}(W)^\perp}: \mathcal{N}(W)^\perp \to \mathcal{K}$, the restriction of W to $\mathcal{N}(W)^\perp$, is an isometry. Note: In this context, "isometry" means "linear isometry".

Proposition 5.87. *If $W: \mathcal{H} \to \mathcal{K}$ is a partial isometry, then*

$$W = VP,$$

where $V: \mathcal{N}(W)^\perp \to \mathcal{K}$ is an isometry and $P: \mathcal{H} \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{N}(W)^\perp$. Conversely, let $\mathcal{M}$ be any subspace of $\mathcal{H}$. If $V: \mathcal{M} \to \mathcal{H}$ is an isometry and $P: \mathcal{H} \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{M}$, then $W = VP : \mathcal{H} \to \mathcal{K}$ is a partial isometry.

Proof. If $W: \mathcal{H} \to \mathcal{K}$ is a partial isometry, then let $P: \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{N}(W)^\perp$ (so that $\mathcal{R}(P) = \mathcal{N}(W)^\perp$, and therefore $\mathcal{R}(I-P) = \mathcal{N}(P) = \mathcal{R}(P)^\perp = \mathcal{N}(W)$ — cf. Propositions 4.13, 5.15, and 5.51). Set $V = W|_{\mathcal{N}(W)^\perp}: \mathcal{N}(W)^\perp \to \mathcal{K}$, which is an isometry. Thus, for every $x \in \mathcal{H}$,

$$\begin{aligned} Wx = W(Px + (I-P)x) &= WPx + W(I-P)x \\ &= WPx = W|_{\mathcal{N}(W)^\perp} Px = VPx. \end{aligned}$$

Conversely, let $V: \mathcal{M} \to \mathcal{K}$ be an isometry, and let $P: \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{M}$, where $\mathcal{M}$ is an arbitrary subspace of $\mathcal{H}$. If $W = VP$, then $W|_{\mathcal{M}} v = Wv = VPv = Vv$ for every $v \in \mathcal{R}(P) = \mathcal{M}$, and hence W acts isometrically on $\mathcal{M}$. Moreover, if $Wu = 0$, then $VPu = 0$ so that $Pu \in \mathcal{N}(V) = \{0\}$ (recall that V is an isometry), which implies that $u \in \mathcal{N}(P)$. Therefore, $\mathcal{N}(W) \subseteq \mathcal{N}(P)$, and so $\mathcal{N}(W) = \mathcal{N}(P)$ (reason: $\mathcal{N}(P) \subseteq \mathcal{N}(W)$ because $W = VP$). Thus $\mathcal{M} = \mathcal{R}(P) = \mathcal{N}(P)^\perp = \mathcal{N}(W)^\perp$. Conclusion: $W = VP : \mathcal{H} \to \mathcal{K}$ is a bounded linear transformation (composition of bounded linear transformations) that acts isometrically on $\mathcal{N}(W)^\perp$; that is, W is a partial isometry. □

Since $W = VP$ and $V = W|_{\mathcal{N}(W)^\perp}$, $\|V\| \le \|W\| = \|VP\| \le \|V\|\|P\|$ and $\mathcal{R}(W) \subseteq \mathcal{R}(V) = \mathcal{R}(W|_{\mathcal{N}(W^\perp)}) \subseteq \mathcal{R}(W)$. Then

$$\|W\| = \|V\| = 1 \quad \text{and} \quad \mathcal{R}(W) = \mathcal{R}(V)$$

(Proposition 5.81(e) and Problem 4.41(a)). Recall that the range of a linear isometry on a Banach space is a subspace (Problem 4.41(b)). That is,

$$\mathcal{R}(V) = \mathcal{R}(V)^- \quad \text{and hence} \quad \mathcal{R}(W) = \mathcal{R}(W)^-.$$

Thus the range of a partial isometry is also a subspace. The subspaces $\mathcal{N}(W)^\perp$ and $\mathcal{R}(W)$ (of $\mathcal{H}$ and $\mathcal{K}$) are called *initial* and *final spaces* of the partial isometry $W \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$, respectively.

Proposition 5.88. *Take $W \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$. The following assertions are pairwise equivalent.*

(a) *W is a partial isometry with initial space $\mathcal{M}$ and final space $\mathcal{R}$.*

(b) *W^*W is the orthogonal projection onto $\mathcal{M}$ and $\mathcal{R}(W) = \mathcal{R}$.*

(c) $WW^*W = W$.

(d) $W^*WW^* = W^*$.

(e) *WW^* is the orthogonal projection onto $\mathcal{R}$ and $\mathcal{N}(W)^\perp = \mathcal{M}$.*

(f) *W^* is a partial isometry with initial space $\mathcal{R}$ and final space $\mathcal{M}$.*

Proof. Let W be a bounded linear transformation of $\mathcal{H}$ into $\mathcal{K}$.

Proof of (a)⇒(c). If (a) holds, then $W = VP$, where $V\colon \mathcal{M} \to \mathcal{K}$ is an isometry, $P\colon \mathcal{H} \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{M}$, and $\mathcal{M} = \mathcal{N}(W)^\perp$ (Proposition 5.87). Consider the adjoint $V^*\colon \mathcal{K} \to \mathcal{M}$ of V, and recall from Proposition 5.72 that $V^*V = I$, the identity on the Hilbert space $\mathcal{M}$ (since $\mathcal{M} = \mathcal{N}(W)^\perp$ is a subspace of the Hilbert space $\mathcal{H}$ by Proposition 5.12, and hence a Hilbert space itself). Since P is an orthogonal projection, it follows by Proposition 5.81 that $P = P^*$ and therefore $W^* = P^*V^* = PV^*$. Hence $WW^*W = VPPV^*VP = VP = W$.

Proof of (c)⇒(b). If $WW^*W = W$, then $(W^*W)^2 = W^*WW^*W = W^*W$ so that W^*W is a (continuous) projection, and so it has a closed range (reason: if $E = E^2$, then $\mathcal{R}(E) = \mathcal{N}(I - E)$, and $\mathcal{N}(I - E)$ is closed whenever the linear transformation E is bounded — see Proposition 4.13). Therefore, $\mathcal{R}(W^*W) = \mathcal{R}(W^*W)^- = \mathcal{N}((W^*W)^*)^\perp = \mathcal{N}(W^*W)^\perp = \mathcal{N}(W)^\perp$ (Proposition 5.76). Thus $\mathcal{R}(W^*W) = \mathcal{N}(W)^\perp$ and $\mathcal{R}(W^*W) \perp \mathcal{N}(W^*W)$, which implies that W^*W is the orthogonal projection onto $\mathcal{M} = \mathcal{N}(W)^\perp$.

Proof of (b)⇒(a). If (b) holds, then $\mathcal{M} = \mathcal{R}(W^*W) = \mathcal{R}(W^*W)^- = \mathcal{N}(W)^\perp$ (Propositions 5.51 and 5.76). If $v \in \mathcal{M} = \mathcal{R}(W^*W)$, then $W^*Wv = v$, and hence $\|Wv\|^2 = \langle Wv\,;Wv\rangle = \langle W^*Wv\,;v\rangle = \langle v\,;v\rangle = \|v\|^2$ so that $W|_{\mathcal{M}}$ is an isometry. Since $\mathcal{M} = \mathcal{N}(W)^\perp$, it follows that (b) implies (a).

Conclusion: Assertions (a), (b), and (c) are pairwise equivalent. If W^*W is an orthogonal projection, then $\mathcal{R}(W^*W)$ is closed, which implies that $\mathcal{R}(W^*)$ and $\mathcal{R}(W)$ are closed as well. Similarly, if W^* is a partial isometry, then $\mathcal{R}(W^*)$ is closed, and so is $\mathcal{R}(W)$. Therefore, in both cases, $\mathcal{R}(W^*) = \mathcal{N}(W)^\perp$ and $\mathcal{N}(W^*)^\perp = \mathcal{R}(W)$. Since assertions (a), (b), and (c) are pairwise equivalent, it follows that the dual assertions (f), (e), and (d) are pairwise equivalent too. Finally, observe that (c) and (d) are trivially equivalent. □

If a transformation $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ is the product of a partial isometry $W \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ and a nonnegative operator $Q \in \mathcal{B}[\mathcal{H}]$, and if $\mathcal{N}(W) = \mathcal{N}(Q)$, then the representation $T = WQ$ is called the *polar decomposition* of T. The next theorem says that every bounded linear transformation in $\mathcal{B}[\mathcal{H}, \mathcal{K}]$ has a unique polar decomposition.

Theorem 5.89. *If T lies in $\mathcal{B}[\mathcal{H},\mathcal{K}]$, then there exists a partial isometry W in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ with initial space $\mathcal{N}(T)^\perp$ and final space $\mathcal{R}(T)^-$ such that*

$$T = W(T^*T)^{\frac{1}{2}}$$

and $\mathcal{N}(W) = \mathcal{N}((T^*T)^{\frac{1}{2}})$. Moreover, if $T = ZQ$, where Q is a nonnegative operator in $\mathcal{B}[\mathcal{H}]$ and Z is a partial isometry in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ with $\mathcal{N}(Z) = \mathcal{N}(Q)$, then $Q = (T^*T)^{\frac{1}{2}}$ and $Z = W$.

Proof. Take T in $\mathcal{B}[\mathcal{H},\mathcal{K}]$. Recall that (Propositions 5.76 and 5.86) T^*T is in $\mathcal{B}^+[\mathcal{H}]$ and $\mathcal{R}((T^*T)^{\frac{1}{2}})^- = \mathcal{R}(T^*T)^- = \mathcal{N}(T^*T)^\perp = \mathcal{N}((T^*T)^{\frac{1}{2}})^\perp = \mathcal{N}(T)^\perp$.

Existence. Consider a mapping $V_0\colon \mathcal{R}((T^*T)^{\frac{1}{2}}) \subseteq \mathcal{H} \to \mathcal{K}$ defined by

$$V_0(T^*T)^{\frac{1}{2}}x = Tx \quad \text{for every} \quad x \in \mathcal{H}$$

so that $\mathcal{R}(V_0) \subseteq \mathcal{R}(T)$. Note that V_0 is linear. Indeed, if $y, z \in \mathcal{R}((T^*T)^{\frac{1}{2}})$, then $y = (T^*T)^{\frac{1}{2}}u$ and $z = (T^*T)^{\frac{1}{2}}v$ for some u and v in $\mathcal{H}$, and hence

$$\begin{aligned} V_0(\alpha y + \beta z) &= V_0(\alpha(T^*T)^{\frac{1}{2}}u + \beta(T^*T)^{\frac{1}{2}}v) \\ &= V_0(T^*T)^{\frac{1}{2}}(\alpha u + \beta v) = T(\alpha u + \beta v) = \alpha Tu + \beta Tv \\ &= \alpha V_0(T^*T)^{\frac{1}{2}}u + \beta V_0(T^*T)^{\frac{1}{2}}v = \alpha V_0 y + \beta V_0 z \end{aligned}$$

for every $\alpha, \beta \in \mathbb{F}$. Moreover, since

$$\|Tx\|^2 = \langle Tx\,;Tx\rangle = \langle T^*Tx\,;x\rangle = \langle (T^*T)^{\frac{1}{2}}x\,;(T^*T)^{\frac{1}{2}}x\rangle = \|(T^*T)^{\frac{1}{2}}x\|^2,$$

it follows that $\|V_0(T^*T)^{\frac{1}{2}}x\| = \|Tx\| = \|(T^*T)^{\frac{1}{2}}x\|$, for every $x \in \mathcal{H}$. Hence $\|V_0 y\| = \|y\|$ for every $y \in \mathcal{R}((T^*T)^{\frac{1}{2}})$ so that $V_0\colon \mathcal{R}((T^*T)^{\frac{1}{2}}) \to \mathcal{K}$ is a linear isometry. Extend it to $\mathcal{R}(T^*T)^- = \mathcal{R}((T^*T)^{\frac{1}{2}})^-$ and get the mapping

$$V\colon \mathcal{R}(T^*T)^- \to \mathcal{K}.$$

This is a linear isometry with $\mathcal{R}(V) \subseteq \mathcal{R}(V_0)^- \subseteq \mathcal{R}(T)^-$. Indeed, since the mapping $V_0\colon \mathcal{R}((T^*T)^{\frac{1}{2}}) \to \mathcal{R}(V_0)$ is an isometric isomorphism, it follows by Corollary 4.38 that $V\colon \mathcal{R}((T^*T)^{\frac{1}{2}})^- \to \mathcal{R}(V_0)^-$ is again an isometric isomorphism. Since $V(T^*T)^{\frac{1}{2}}x = V_0(T^*T)^{\frac{1}{2}}x = Tx$ for every $x \in \mathcal{H}$,

$$T = V(T^*T)^{\frac{1}{2}},$$

and so $\mathcal{R}(T) \subseteq \mathcal{R}(V)$, where

$$V\colon \mathcal{N}(T)^\perp \to \mathcal{K} \text{ is a linear isometry with } \mathcal{R}(V) = \mathcal{R}(T)^-.$$

(Recall: $\mathcal{N}(T)^\perp$ is a Hilbert space so that $\mathcal{R}(V) = \mathcal{R}(V)^-$.) Let $P\colon \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{N}(T)^\perp$. Then $\mathcal{R}(P) = \mathcal{N}(T)^\perp$, and so $VPy = Vy$ for every $y \in \mathcal{N}(T)^\perp = \mathcal{R}((T^*T)^{\frac{1}{2}})$. Thus $VP(T^*T)^{\frac{1}{2}}x = V(T^*T)^{\frac{1}{2}}x$ for each $x \in \mathcal{H}$. That is, $VP(T^*T)^{\frac{1}{2}} = V(T^*T)^{\frac{1}{2}}$. With $W = VP : \mathcal{H} \to \mathcal{K}$ we get

$$T = W(T^*T)^{\frac{1}{2}}.$$

Since $V = W|_{\mathcal{R}(P)} = W|_{\mathcal{N}(T)^\perp}$ is an isometry and P is an orthogonal projection, we get $\mathcal{N}(W) = \mathcal{N}(VP) = \mathcal{N}(P) = \mathcal{R}(P)^\perp = \mathcal{N}(T) = \mathcal{N}((T^*T)^{\frac{1}{2}})$ and $\mathcal{R}(W) = \mathcal{R}(V) = \mathcal{R}(T)^-$. Therefore,

$$W\colon \mathcal{H} \to \mathcal{K} \text{ is a partial isometry}$$

with initial space $\mathcal{N}(T)^\perp$, final space $\mathcal{R}(T)^-$, and $\mathcal{N}(W) = \mathcal{N}((T^*T)^{\frac{1}{2}})$.

Uniqueness. Let $Z \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ be a partial isometry with $\mathcal{N}(Z) = \mathcal{N}(Q)$, where Q is a nonnegative operator in $\mathcal{B}[\mathcal{H}]$. Suppose $T = ZQ$. Since Z^*Z is the orthogonal projection onto $\mathcal{N}(Z)^\perp = \mathcal{N}(Q)^\perp = \mathcal{R}(Q)^-$ (Propositions 5.76(b*) and 5.88(b)), we get $T^*T = QZ^*ZQ = Q^2$. Thus $Q = (T^*T)^{\frac{1}{2}}$ by uniqueness of the nonnegative square root. Thus $Z(T^*T)^{\frac{1}{2}} = T = W(T^*T)^{\frac{1}{2}}$ so that $Z|_{\mathcal{R}((T^*T)^{1/2})} = W|_{\mathcal{R}((T^*T)^{1/2})}$, and therefore

$$\mathcal{N}(Z) = \mathcal{N}(W) \quad \text{and} \quad Z|_{\mathcal{N}(Z)^\perp} = W|_{\mathcal{N}(W)^\perp},$$

for $\mathcal{R}((T^*T)^{\frac{1}{2}})^- = \mathcal{R}(T^*T)^- = \mathcal{N}(T)^\perp$ and $\mathcal{N}(Z) = \mathcal{N}(Q) = \mathcal{N}((T^*T)^{\frac{1}{2}}) = \mathcal{N}(T) = \mathcal{N}(W)$. Conclusion: The partial isometries $Z\colon \mathcal{H} \to \mathcal{K}$ and $W\colon \mathcal{H} \to \mathcal{K}$ have the same initial space and they coincide there. That is, $Z = W$. □

If $T = WQ$ is the polar decomposition of T, then W^*W is the orthogonal projection onto $\mathcal{N}(W)^\perp = \mathcal{N}(Q)^\perp = \mathcal{R}(Q)^-$ so that $W^*WQ = Q$. Thus

$$T = WQ \quad \text{implies} \quad W^*T = Q.$$

Here is another corollary of Theorem 5.89.

Corollary 5.90. *Let $T = WQ$ be the polar decomposition of $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$.*

(a) *$W \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ is an isometry if and only if $\mathcal{N}(T) = \{0\}$.*

(b) *$W \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ is a coisometry if and only if $\mathcal{R}(T)^- = \mathcal{K}$.*

Proof. Recall that $\mathcal{N}(T) = \mathcal{N}(W)$ and $\mathcal{R}(W) = \mathcal{R}(T)^-$ whenever $T = WQ$ is the polar decomposition of T.

(a) A partial isometry W is an isometry if and only if $\mathcal{N}(W)^\perp = \mathcal{H}$, which means that $\mathcal{N}(W) = \{0\}$.

(b) W is a coisometry if and only if W^* is an isometry. But W^* is a partial isometry with $\mathcal{N}(W^*)^\perp = \mathcal{R}(W)$ (Proposition 5.88). Hence W^* is an isometry if and only if $\mathcal{R}(W) = \mathcal{K}$. □

Recall that a unitary transformation is precisely a surjective isometry or, equivalently, an isometry and a coisometry. Also recall that a transformation is called quasiinvertible if it is injective and has dense range. An immediate consequence of the preceding results reads as follows.

Corollary 5.91. *If $T = WQ$ is the polar decomposition of $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$, then W is unitary if and only if T is quasiinvertible.*

In particular, if T is invertible (i.e., if $T \in \mathcal{G}[\mathcal{H}, \mathcal{K}]$), then $T = U(T^*T)^{\frac{1}{2}}$, where $U \in \mathcal{G}[\mathcal{H}, \mathcal{K}]$ is unitary. If T is an operator on $\mathcal{H}$ (i.e., if $T \in \mathcal{B}[\mathcal{H}]$), then both W and Q are again operators on $\mathcal{H}$. However, the polar decomposition of an operator T does not necessarily commute (i.e., in general $WQ \neq QW$). The class of all operators for which the polar decomposition commutes will be characterized in Chapter 6 (cf. Proposition 6.4). These are the *quasinormal* operators; a class of operators that includes the class of the *normal* operators of Section 6.1.

Suggested Reading

Akhiezer and Glazman [1]
Arveson [1]
Bachman and Narici [1]
Balakrishnan [1]
Beals [1]
Berberian [3]
Berezansky, Sheftel, and Us [1]
Conway [1], [2]
Davidson [1]
Dunford and Schwartz [2]
Fillmore [1], [2]
Furuta [1]
Gohberg and Kreĭn [1]
Goldberg [1]
Halmos [1], [4]
Kato [1]
Kubrusly [1], [2]
Lax [1]
Murphy [1]
Naylor and Sell [1]
Pearcy [1], [2]
Putnam [1]
Radjavi and Rosenthal [1]
Reed and Simon [1]
Riesz and Sz.-Nagy [1]
Schatten [1]
Stone [1]
Sz.-Nagy and Foiaş [1]
Weidmann [1]
Yoshino [1]

Problems

Problem 5.1. Let $\| \ \|$ be the norm induced by an inner product $\langle \ ; \ \rangle$ on a linear space $\mathcal{X} \neq \{0\}$. Show that, for every $x \in \mathcal{X}$,

$$\|x\| = \sup_{\|y\|=1} |\langle x\,; y\rangle| = \sup_{\|y\|\leq 1} |\langle x\,; y\rangle| = \sup_{y\neq 0} \tfrac{|\langle x\,; y\rangle|}{\|y\|}.$$

Hint: If $\|x\| \neq 0$, then $\|x\| = \langle \frac{x}{\|x\|}\,; x\rangle$. Use the Schwarz inequality.

Problem 5.2. Let $(\mathcal{X}, \langle \ ; \ \rangle)$ be an inner product space and let $\| \ \|$ be the norm on $\mathcal{X}$ induced by $\langle \ ; \ \rangle$. Take arbitrary vectors x and y in $\mathcal{X}$ and consider the following assertions.

(a) $\|x + y\| = \|x\| + \|y\|$.

(a′) $\|x - y\| = \big|\|x\| - \|y\|\big|$.

(b) $|\langle x\,;y\rangle| = \|x\|\,\|y\|$.

(b′) $\langle x\,;y\rangle = \|x\|\,\|y\|$.

(c) x and y are *collinear* (i.e., either one of them is null or $y = \alpha x$ for some nonzero α in $\mathbb{F}$).

(c′) x and y are *proportional* (i.e., either one of them is null or $y = \alpha x$ for some real $\alpha > 0$).

Show that the diagram below exhibits *all* possible implications among the above assertions.

$$\begin{array}{ccccccc} \text{(a)} & \Longleftrightarrow & \text{(a}') & \Longleftrightarrow & \text{(b}') & \Longleftrightarrow & \text{(c}') \\ & & & & \Downarrow & & \Downarrow \\ & & & & \text{(b)} & \Longleftrightarrow & \text{(c)} \end{array}$$

Hint: Consider the auxiliary assertion.

(b″) $\operatorname{Re}\langle x\,;y\rangle = \|x\|\,\|y\|$.

Recall that $\|x \pm y\|^2 = \|x\|^2 \pm 2\operatorname{Re}\langle x\,;y\rangle + \|y\|^2$ for every $x, y \in \mathcal{X}$. Use this identity to show that (a)⇔(b″), (a′)⇔(b″) and, by setting $z = \langle x\,;y\rangle\,\|y\|^{-2}y$ and $z = |\langle x\,;y\rangle|\,\|y\|^{-2}y$, that (b)⇒(c) and (b′)⇒(c′). Now use the Schwarz inequality to show that (b″)⇔(b′). The remaining implications are trivially verified. Show that (c)$\not\Rightarrow$(c′) and conclude that there may be no other implication in the above diagram.

Problem 5.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be inner product spaces and take arbitrary T and L in $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$. Prove the following identities.

(a) $$\langle Tx\,;Ly\rangle + \langle Ty\,;Lx\rangle = \tfrac{1}{2}\big(\langle T(x+y)\,;L(x+y)\rangle - \langle T(x-y)\,;L(x-y)\rangle\big)$$

for every $x, y \in \mathcal{X}$. If $\mathcal{X}$ is a *complex* inner product space, then

(b) $$\begin{aligned}\langle Tx\,;Ly\rangle = \tfrac{1}{4}\big(&\langle T(x+y)\,;L(x+y)\rangle - \langle T(x-y)\,;L(x-y)\rangle \\ &+ i\langle T(x+iy)\,;L(x+iy)\rangle - i\,\langle T(x-iy)\,;L(x-iy)\rangle\big)\end{aligned}$$

for every $x, y \in \mathcal{X}$. These yield the polarization identities of Proposition 5.4.

Hint: Verify: $\langle T(x+\alpha y)\,;L(x+\alpha y)\rangle = \langle Tx\,;Lx\rangle + \overline{\alpha}\langle Tx\,;Ly\rangle + \alpha\langle Ty\,;Lx\rangle + |\alpha|^2\langle Ty\,;Ly\rangle$. Then set $\alpha = \pm 1$ to get (a), and also $\alpha = \pm i$ to get (b).

Problem 5.4. Take $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$, where $\mathcal{X}$ and $\mathcal{Y}$ are inner product spaces. Show that the following assertions are pairwise equivalent.

(a) $T = O$.

(b) $Tx = 0$ for all $x \in \mathcal{X}$.

(c) $\langle Tx\,;y\rangle = 0$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Now set $\mathcal{Y} = \mathcal{X}$ so that $T \in \mathcal{B}[\mathcal{X}]$, and consider the following further assertion.

(d) $\langle Tx\,;x\rangle = 0$ for all $x \in \mathcal{X}$.

Clearly, (c) implies (d). If $\mathcal{X}$ is a *complex* inner product space, then prove the converse: if $\mathcal{X} = \mathcal{Y}$ is complex, these four assertions are all pairwise equivalent.

Hint: Use Problem 5.3(b) to show that (d) implies (c) if $\mathcal{X} = \mathcal{Y}$ is complex.

Problem 5.5. Let $\{T_n\}$ be a sequence of bounded linear transformations of a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$ (i.e., $T_n \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ for each n). Show that uniform, strong, and weak boundedness coincide. In other words, show that the following assertions are pairwise equivalent.

(a) $\sup_n \|T_n\| < \infty$.

(b) $\sup_n \|T_n x\| < \infty$ for every $x \in \mathcal{H}$.

(c) $\sup_n |\langle T_n x\,;y\rangle| < \infty$ for every $x \in \mathcal{H}$ and every $y \in \mathcal{K}$.

Now set $\mathcal{K} = \mathcal{H}$ so that $T_n \in \mathcal{B}[\mathcal{H}]$, and consider the following further assertion.

(d) $\sup_n |\langle T_n x\,;x\rangle| < \infty$ for every $x \in \mathcal{H}$.

Clearly, (c) implies (d). If $\mathcal{H} = \mathcal{K}$ is a *complex* Hilbert space, then prove the converse: if $\mathcal{H} = \mathcal{K}$ is complex, these four assertions are all pairwise equivalent.

Hint: Check that (a)⇒(b)⇒(c). Now take an arbitrary $x \in \mathcal{H}$. For each n consider the functional $x_n\colon \mathcal{K} \to \mathbb{F}$ given by

$$x_n(y) \;=\; \langle y\,;T_n x\rangle \quad \text{for every} \quad y \in \mathcal{K}.$$

Show that each x_n is linear and bounded. If (c) holds (so that $\sup_n |x_n(y)| < \infty$ for every $y \in \mathcal{K}$), then the Banach–Steinhaus Theorem (Theorem 4.43) ensures that $\sup_n \|x_n\| < \infty$ because $\mathcal{K}$ is a Banach space. But

$$\|T_n x\| \;=\; \sup_{\|y\|=1} |x_n(y)| \;=\; \|x_n\|$$

for each n (cf. Problem 5.1). Thus conclude that (c)⇒(b). A straightforward application of the Banach–Steinhaus Theorem ensures that (b)⇒(a) because $\mathcal{H}$ is a Banach space. Finally, if $\mathcal{K} = \mathcal{H}$ is complex, then use Problem 5.3(b) to show that (d)⇒(c).

Problem 5.6. Let $(\mathcal{X}, \langle\ ;\ \rangle)$ be an inner product space and equip the field $\mathbb{F}$ with its usual metric. Use the Schwarz inequality and Corollary 3.8 to prove the following assertions.

(a) $\langle\,\cdot\,;y\rangle\colon \mathcal{X} \to \mathbb{F}$ is a continuous function for every $y \in \mathcal{X}$.

(b) $\langle x\,;\cdot\,\rangle\colon \mathcal{X} \to \mathbb{F}$ is a continuous function for every $x \in \mathcal{X}$.

Equip the direct sum $\mathcal{X} \oplus \mathcal{X}$ with the inner product $\langle\ ;\ \rangle_{\oplus}$ of Examples 5.E and 5.I. That is, set $\langle \boldsymbol{x}\,;\boldsymbol{y}\rangle_{\oplus} = \langle x_1\,;y_1\rangle + \langle x_2\,;y_2\rangle$ for every $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ in $\mathcal{X} \oplus \mathcal{X}$. Show that

(c) $\langle\ ;\ \rangle_{\oplus} \colon \mathcal{X} \oplus \mathcal{X} \to \mathbb{F}$ is a continuous function.

Hint: Take an arbitrary convergent sequence $\{\boldsymbol{x}_{(n)} = (x_{1(n)}, x_{2(n)})\}$ in $\mathcal{X} \oplus \mathcal{X}$ and let $\boldsymbol{x} = (x_1, x_2) \in \mathcal{X} \oplus \mathcal{X}$ be its limit. Verify that

$$\begin{aligned}\big\langle x_{1(n)} - x_1 + x_1\,;x_{2(n)} - x_2 + x_2\big\rangle \;&=\; \big\langle x_{1(n)} - x_1\,;x_{2(n)} - x_2\big\rangle \\ &\quad+\; \big\langle x_{1(n)} - x_1\,;x_2\big\rangle + \big\langle x_1\,;x_{2(n)} - x_2\big\rangle + \big\langle x_1\,;x_2\big\rangle\end{aligned}$$

and use the Schwarz inequality to show that

$$\begin{aligned}\big|\langle x_{1(n)}\,;x_{2(n)}\rangle - \langle x_1\,;x_2\rangle\big| \;&\le\; \|x_{1(n)} - x_1\|\,\|x_{2(n)} - x_2\| \\ &\quad+\; \|x_2\|\,\|x_{1(n)} - x_1\| \;+\; \|x_1\|\,\|x_{2(n)} - x_2\|.\end{aligned}$$

Since $\|\boldsymbol{x}_{(n)} - \boldsymbol{x}\|_{\oplus}^2 = \|x_{1(n)} - x_1\|^2 + \|x_{2(n)} - x_2\|^2$, we get $\boldsymbol{x}_{(n)} \to \boldsymbol{x}$ in $\mathcal{X} \oplus \mathcal{X}$ if and only if $x_{1(n)} \to x_1$ and $x_{2(n)} \to x_2$ in $\mathcal{X}$, which implies that $\langle x_{1(n)}\,;x_{2(n)}\rangle \to \langle x_1\,;x_2\rangle$ in $\mathbb{F}$.

Problem 5.7. Let A and B be subsets of a Hilbert space $\mathcal{H}$. Show that

(a) $A \subseteq B$ and $A^{\perp} \subseteq B$ implies $\bigvee B = \mathcal{H}$.

Hint: $B^{\perp\perp} = \mathcal{H}$ because $B^{\perp} \subseteq A^{\perp} \cap A^{\perp\perp} = \{0\}$. Use Proposition 5.15.

In particular, if $\mathcal{M}$ is a subspace of $\mathcal{H}$, then

(b) $A \subseteq \mathcal{M}$ and $A^{\perp} \subseteq \mathcal{M}$ implies $\mathcal{M} = \mathcal{H}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of an inner product space $\mathcal{X}$. Show that

(c) $\mathcal{M} \perp \mathcal{N}$ implies $\mathcal{M}^- \perp \mathcal{N}^-$.

Hint: Verify that if $u = \lim_m u_m \in \mathcal{M}^-$ and $v = \lim_m u_m \in \mathcal{N}^-$ for u_m in $\mathcal{M}$ and v_m in $\mathcal{M}$, then $\langle u\,;v\rangle = \lim_m \langle u_m\,;v\rangle = \lim_m \lim_n \langle u_m\,;v_n\rangle = 0$.

In particular, if $\mathcal{X}$ is a Hilbert space, then

(d) $\mathcal{M} \perp \mathcal{N}$ implies $(\mathcal{M} + \mathcal{N})^- = \mathcal{M}^- + \mathcal{N}^-$.

Hint: Apply the remark after Proposition 4.9, Theorem 5.10(b), and item (c) to conclude that $(\mathcal{M} + \mathcal{N})^- = (\mathcal{M}^- + \mathcal{N}^-)^- = \mathcal{M}^- + \mathcal{N}^-$ if $\mathcal{M} \perp \mathcal{N}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, and consider the (orthogonal) direct sum $\mathcal{H} \oplus \mathcal{K}$. Prove the following assertions.

(e) $(\mathcal{M} \oplus \mathcal{N})^- = \mathcal{M}^- \oplus \mathcal{N}^-$.

Hint: By Proposition 5.24 and item (d).

(f) $(\mathcal{M} \oplus \mathcal{N})^- = \mathcal{M} \oplus \mathcal{N} \neq \mathcal{H} \oplus \mathcal{K}$ if and only if $\mathcal{M}^- = \mathcal{M}$, $\mathcal{N}^- = \mathcal{N}$, and $\mathcal{M} \neq \mathcal{H}$ or $\mathcal{N} \neq \mathcal{K}$.

Hint: By (e) the result is equivalent to $\mathcal{M}^- \oplus \mathcal{N}^- = \mathcal{M} \oplus \mathcal{N} \neq \mathcal{H} \oplus \mathcal{K}$.

Problem 5.8. Let $\mathcal{X}$ be an inner product space and let $\mathcal{H}$ be a Hilbert space. Prove the following propositions.

If $\{A_\gamma\}_{\gamma\in\Gamma}$ is a nonempty family of subsets of $\mathcal{X}$, then

(a) $\bigcap_{\gamma\in\Gamma} A_\gamma^\perp = \big(\bigcup_{\gamma\in\Gamma} A_\gamma\big)^\perp$.

If $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ is a nonempty family of linear manifolds of $\mathcal{X}$, then

(b) $\bigcap_{\gamma\in\Gamma} \mathcal{M}_\gamma^\perp = \big(\sum_{\gamma\in\Gamma} \mathcal{M}_\gamma\big)^\perp$.

Hint: Proposition 5.12. Recall: $\bigvee\big(\bigcup_{\gamma\in\Gamma} \mathcal{M}_\gamma\big) = \big(\sum_{\gamma\in\Gamma} \mathcal{M}_\gamma\big)^-$.

If $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ is a nonempty family of subspaces of $\mathcal{H}$, then

(c) $\bigcap_{\gamma\in\Gamma} \mathcal{M}_\gamma = \{0\}$ if and only if $\big(\sum_{\gamma\in\Gamma} \mathcal{M}_\gamma^\perp\big)^- = \mathcal{H}$,

(d) $\big(\sum_{\gamma\in\Gamma} \mathcal{M}_\gamma\big)^- = \mathcal{H}$ if and only if $\bigcap_{\gamma\in\Gamma} \mathcal{M}_\gamma^\perp = \{0\}$.

Hint: Propositions 5.12 and 5.15 and item (b).

If $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ is a nonempty family of orthogonal subspaces of $\mathcal{H}$, then

(e) $\mathcal{H} = \big(\sum_{\gamma\in\Gamma} \mathcal{M}_\gamma\big)^-$ implies $\mathcal{M}_\alpha = \bigcap_{\alpha\neq\gamma\in\Gamma} \mathcal{M}_\gamma^\perp$ for every $\alpha \in \Gamma$.

Hint: Show that $\mathcal{M}_\alpha \perp \sum_{\alpha\neq\gamma\in\Gamma} \mathcal{M}_\gamma$, and so $\mathcal{M}_\alpha \perp \big(\sum_{\alpha\neq\gamma\in\Gamma} \mathcal{M}_\gamma\big)^-$ (cf. Proposition 5.12). Verify each of the following steps (see Theorem 5.10(b)): $\big(\sum_{\gamma\in\Gamma} \mathcal{M}_\gamma\big)^- = \bigvee\big(\mathcal{M}_\alpha \cup \bigcup_{\alpha\neq\gamma\in\Gamma} \mathcal{M}_\gamma\big) \subseteq \bigvee\big(\mathcal{M}_\alpha \cup \bigvee\big(\bigcup_{\alpha\neq\gamma\in\Gamma} \mathcal{M}_\gamma\big)\big) = \mathcal{M}_\alpha + \big(\sum_{\alpha\neq\gamma\in\Gamma} \mathcal{M}_\gamma\big)^- \subseteq \mathcal{H}$. Hence $\mathcal{H} = \mathcal{M}_\alpha + \big(\sum_{\alpha\neq\gamma\in\Gamma} \mathcal{M}_\gamma\big)^-$. Conclude that $\mathcal{M}_\alpha^\perp = \big(\sum_{\alpha\neq\gamma\in\Gamma} \mathcal{M}_\gamma\big)^-$ by Proposition 5.19, and therefore $\mathcal{M}_\alpha = \big(\sum_{\alpha\neq\gamma\in\Gamma} \mathcal{M}_\gamma\big)^\perp$ (Propositions 5.12 and 5.15). Apply (b).

Remark: The collection $\mathrm{Lat}(\mathcal{H})$ of all subspaces of $\mathcal{H}$ is a complete lattice in the inclusion ordering, where $\mathcal{M} \wedge \mathcal{N} = \mathcal{M} \cap \mathcal{N}$ and $\mathcal{M} \vee \mathcal{N} = (\mathcal{M} + \mathcal{N})^-$ (Section 4.3). Thus, by item (b) and Proposition 5.15, $\mathcal{M} \vee \mathcal{N} = (\mathcal{M}^\perp \cap \mathcal{N}^\perp)^\perp$.

Problem 5.9. Let $\mathcal{X}$ and $\mathcal{Y}$ be inner product spaces and consider the setup of Problem 4.42. If there exists a unitary transformation $U \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$ intertwining $T \in \mathcal{B}[\mathcal{X}]$ and $S \in \mathcal{B}[\mathcal{Y}]$; that is, if

$$UT = SU,$$

then we say that T and S are *unitarily equivalent* (notation: $T \cong S$). If this happens, then it is clear that $\mathcal{X}$ and $\mathcal{Y}$ are unitarily equivalent inner product spaces. Unitary equivalence is a rather important special case of isometric equivalence. In particular (see Problems 4.41 and 4.42),

$$T \cong S \quad \text{implies} \quad \|T\| = \|S\|.$$

First prove the following assertion that mirrors similarity on linear spaces, and isometric equivalence on normed spaces.

(a) Unitary equivalence has the defining properties of an equivalence relation.

Now take any polynomial p in one variable (cf. Problem 2.20) and show that

(b) $Up(T) = p(S)U$ whenever $UT = SU$ for every unitary $U \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$.

In particular, $T \cong S$ implies $T^n \cong S^n$ for every $n \geq 0$ (see Problem 4.24). Next show that product does not preserve unitary equivalence. That is, show that

(c) $T \cong \widetilde{T}$ and $S \cong \widetilde{S}$ do not imply $TS \cong \widetilde{T}\widetilde{S}$.

Hint: Let T be a nilpotent and let U be a unitary, both operators acting on the same Hilbert space, such that UT is a nonzero idempotent (e.g., see Problem 5.43(e)). Take $S = T$ and set $\widetilde{S} = ISI = S$ and $\widetilde{T} = UTU$ so that $T \cong \widetilde{T}$ and $S \cong \widetilde{S}$. Verify that $TS \ncong UT = (UT)^2 = UTUT = \widetilde{T}\widetilde{S}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and let $\mathcal{M}$ be a subspace of $\mathcal{X}$. Suppose $T \cong S$ and let $U \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$ be any unitary transformation intertwining T to S. Theorem 3.24 ensures that $U(\mathcal{M})$ is a subspace of $\mathcal{Y}$. Show that

(d) $\mathcal{M}$ is invariant for T (or reduces T) if and only if $U(\mathcal{M})$ is invariant for S (or reduces S). Moreover, $\mathcal{M}$ is nontrivial if and only if $U(\mathcal{M})$ is.

Hint: Use Proposition 5.73 to show that

$$UT = SU \quad \text{implies} \quad UT^* = S^*U.$$

Corollary 5.75 says that $\mathcal{M}$ reduces T if and only if $T(\mathcal{M}) \subseteq \mathcal{M}$ and $T^*(\mathcal{M}) \subseteq \mathcal{M}$. Thus verify that (cf. Problem 1.2(c)) $T(\mathcal{M}) \subseteq \mathcal{M}$ implies $S[U(\mathcal{M})] \subseteq U(\mathcal{M})$ and $T^*(\mathcal{M}) \subseteq \mathcal{M}$ implies $S^*[U(\mathcal{M})] \subseteq U(\mathcal{M})$.

Therefore we may infer that T has a nontrivial invariant subspace (or a reducing subspace) if and only if S has (also see Problem 4.25 through 4.29).

Problem 5.10. This is the uncountable version of the Orthogonal Structure Theorem. Prove it.

(a) Let $\mathcal{H}$ be a Hilbert space. Suppose $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ is an uncountable family of pairwise orthogonal subspaces of $\mathcal{H}$. If $x \in \big(\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma\big)^-$, then there exists a unique summable family $\{u_\gamma\}_{\gamma\in\Gamma}$ of vectors in $\mathcal{H}$ with $u_\gamma \in \mathcal{M}_\gamma$ for each γ such that $x = \sum_{\gamma\in\Gamma} u_\gamma$. Moreover, $\|x\|^2 = \sum_{\gamma\in\Gamma}\|u_\gamma\|^2$. Conversely, if $\{u_\gamma\}_{\gamma\in\Gamma}$ is a square-summable family of vectors in $\mathcal{H}$ with $u_\gamma \in \mathcal{M}_\gamma$ for each $\gamma \in \Gamma$, then $\{u_\gamma\}_{\gamma\in\Gamma}$ is summable and $\sum_{\gamma\in\Gamma} u_\gamma$ lies in $\big(\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma\big)^-$.

Hint: Set $x_{(n)} = \sum_{k\in N_n} x_{(n)k}$ in $\sum_{k\in N_n}\mathcal{M}_k$, where each N_n is a finite subset of Γ such that $N_n \subseteq N_{n+1}$ and $\#N_n \geq n$. Consider the set $N_\infty = \bigcup_{n\geq 1} N_n$ and construct a countable family of orthogonal vectors in $\mathcal{H}$, say $\{u_k\}_{k\in N_\infty}$, as in the proof of Theorem 5.16.

(b) Show that $\mathcal{H}$ must be nonseparable if $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ has uncountably many nonzero subspaces. (*Hint*: Proposition 5.43 and the Axiom of Choice.) Even in this case, the sum $x = \sum_{\gamma\in\Gamma} u_\gamma$ has only a countable number of nonzero vectors. (Why?)

(c) Restate Corollaries 5.17 and 5.18 in light of item (a). Rewrite Examples 5.I and 5.J for an uncountable family $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ of pairwise orthogonal subspaces of a Hilbert space $(\mathcal{H}, \langle\ ;\ \rangle)$ and conclude that

$$\bigoplus_{\gamma\in\Gamma}\mathcal{M}_\gamma \cong \Big(\sum_{\gamma\in\Gamma}\mathcal{M}_\gamma\Big)^-.$$

Problem 5.11. Let $\{\mathcal{M}_\gamma\}$ be a collection of orthogonal subspaces of a Hilbert space $\mathcal{H}$ that spans $\mathcal{H}$ (i.e., $\bigvee_\gamma\mathcal{M}_\gamma = \big(\sum_\gamma\mathcal{M}_\gamma\big)^- = \mathcal{H}$) and let B_γ be an orthonormal basis for each $\mathcal{M}_\gamma$. Show that $\bigcup_\gamma B_\gamma$ is an orthonormal basis for $\mathcal{H}$.

Hint: $\bigcup_\gamma B_\gamma$ is an orthonormal set in $\mathcal{H}$ and $\bigcup_\gamma\big(\bigvee B_\gamma\big) \subseteq \bigvee\big(\bigcup_\gamma B_\gamma\big)$ — Sections 2.3, 3.5, and 4.3. Verify each of the following steps: $\big(\sum_\gamma\mathcal{M}_\gamma\big)^- = \bigvee\big(\bigcup_\gamma\mathcal{M}_\gamma\big) = \bigvee\big(\bigcup_\gamma\bigvee B_\gamma\big) \subseteq \bigvee\big(\bigvee\big(\bigcup_\gamma B_\gamma\big)\big) = \bigvee\big(\bigcup_\gamma B_\gamma\big) \subseteq \mathcal{H}$.

Problem 5.12. Completeness is necessary in Theorem 5.10(b): *If $\mathcal{M}$ and $\mathcal{N}$ are orthogonal subspaces of an incomplete inner product space $\mathcal{X}$, then it may happen that $\mathcal{M}+\mathcal{N}$ is not closed in $\mathcal{X}$.* Indeed, set

$$\mathcal{X} \;=\; \operatorname{span}\{e_k\}_{k=0}^\infty,$$

where $e_0 = \{\frac{1}{i}\}_{i=1}^\infty \in \ell_+^2$ and $\{e_k\}_{k=1}^\infty$ is the canonical orthonormal basis for the Hilbert space ℓ_+^2 (cf. Example 5.L(b)). Recall that $\mathcal{X}$ is the linear manifold of ℓ_+^2 consisting of all (finite) linear combinations of vectors from $\{e_k\}_{k=0}^\infty$. Now consider the following linear manifolds of $\mathcal{X}$.

$$\mathcal{M} \;=\; \big\{u \in \{\upsilon_k\}_{k=1}^\infty \in \mathcal{X}\colon\ \upsilon_{2k-1} = 0 \ \text{ for all } \ k \geq 1\big\},$$

$$\mathcal{N} \;=\; \big\{v \in \{\nu_k\}_{k=1}^\infty \in \mathcal{X}\colon\ \nu_{2k} = 0 \ \text{ for all } \ k \geq 1\big\}.$$

It is clear that $\mathcal{M} \perp \mathcal{N}$, isn't it? Moreover, they are subspaces of $\mathcal{X}$.

(a) Show that $\mathcal{M}$ and $\mathcal{N}$ are both closed in the inner product space $\mathcal{X}$.

Hint: Take a sequence $\{u_n\}_{n=1}^\infty$, with each $u_n = \{\upsilon_{k(n)}\}_{k=1}^\infty$ in $\mathcal{M}$ such that $u_n \to x = \{\xi_k\}_{k=1}^\infty$ in $\mathcal{X}$. Split the series $\|u_n - x\|^2 = \sum_{k=1}^\infty|\upsilon_{k(n)} - \xi_k|^2$ into the sum of a series running over the odd integers and a series running over the even integers and conclude that $\|u_n - x\|^2 = \sum_{k=1}^\infty|\xi_{2k-1}|^2 + \sum_{k=1}^\infty|\upsilon_{2k(n)} - \xi_{2k}|^2$. This implies that $x \in \mathcal{M}$.

(b) Exhibit an $\mathcal{M}+\mathcal{N}$-valued sequence that converges in $\mathcal{X}$ to $e_0 \in \mathcal{X}$.

Hint: For each $n \geq 1$ define $u_n = \{\upsilon_{k(n)}\}_{k=1}^\infty$ and $v_n = \{\nu_{k(n)}\}_{k=1}^\infty$ as follows. $\upsilon_{k(n)} = \frac{1}{k}$ if $1 \leq k \leq n$ is an even integer, and $\upsilon_{k(n)} = 0$ otherwise. $\nu_{k(n)} = \frac{1}{k}$ if $1 \leq k \leq n$ is an odd integer, and $\nu_{k(n)} = 0$ otherwise. Show that $u_n, v_n \in \operatorname{span}\{e_k\}_{k=1}^n \subset \mathcal{X}$, $u_n \in \mathcal{M}$, $v_n \in \mathcal{N}$, and that $u_n + v_n = (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots)$ for every $n \geq 1$.

(c) Show (by contradiction) that $e_0 \notin \mathcal{M}+\mathcal{N}$.

Conclude from (b) and (c) that $\mathcal{M}+\mathcal{N}$ is not closed in $\mathcal{X}$.

Problem 5.13. Orthogonality is necessary in Theorem 5.10(b): *If $\mathcal{M}$ and $\mathcal{N}$ are nonorthogonal subspaces of a Hilbert space $\mathcal{H}$, then it may happen that $\mathcal{M}+\mathcal{N}$ is not closed in $\mathcal{H}$ even if $\mathcal{M}\cap\mathcal{N}=\{0\}$.* In fact, set

$$v_k \;=\; \big(1-\tfrac{1}{k^2}\big)^{\frac{1}{2}} e_{2k-1} + \tfrac{1}{k}\, e_{2k} \quad \text{in} \quad \ell^2_+$$

for every $k\geq 1$, where $\{e_k\}_{k=1}^{\infty}$ is the canonical orthonormal basis for the Hilbert space ℓ^2_+, and consider the following subspaces of ℓ^2_+.

$$\mathcal{M} = \bigvee\{e_{2k-1}\}_{k=1}^{\infty} \quad \text{and} \quad \mathcal{N} = \bigvee\{v_k\}_{k=1}^{\infty}.$$

(a) Show that $\{v_k\}_{k=1}^{\infty}$ is an orthonormal sequence, and hence an orthonormal basis for the Hilbert space $\mathcal{N}$.

(b) Apply the Fourier Series Theorem to show that $\mathcal{M}\cap\mathcal{N}=\{0\}$.

Hint: Take any x in $\mathcal{M}\cap\mathcal{N}$ and consider its Fourier series expansions: $x=\sum_{k=1}^{\infty}\langle x\,;e_{2k-1}\rangle e_{2k-1}=\sum_{k=1}^{\infty}\langle x\,;v_k\rangle v_k$. Verify that $0=\langle x\,;e_{2j}\rangle=\sum_{k=1}^{\infty}\langle x\,;v_k\rangle\langle v_k\,;e_{2j}\rangle=\frac{1}{j}\langle x\,;v_j\rangle$ for all $j\geq 1$. Conclude that $x=0$.

Show that the series $\sum_{k=1}^{\infty}\frac{1}{k}e_{2k}$ converges in ℓ^2_+ and set $x=\sum_{k=1}^{\infty}\frac{1}{k}e_{2k}$ in ℓ^2_+. Note that $x_n=\sum_{k=1}^{n}\frac{1}{k}e_{2k}=\sum_{k=1}^{n}v_k-\sum_{k=1}^{n}(1-\frac{1}{k^2})^{\frac{1}{2}}e_{2k-1}$ lies in $\mathcal{M}+\mathcal{N}$ for each $n\geq 1$. Moreover, check that $x_n\to x$ in ℓ^2_+ as $n\to\infty$.

(c) Apply the Fourier Series Theorem to show that $x\notin\mathcal{M}+\mathcal{N}$

Hint: Suppose $x=u+v$ with $u\in\mathcal{M}$ and $v\in\mathcal{N}$. Verify that $\langle x\,;e_{2n}\rangle=\frac{1}{n}$, $\langle u\,;e_{2n}\rangle=0$, and $\langle v\,;e_{2n}\rangle=\langle v\,;v_n\rangle\frac{1}{n}$ for each $n\geq 1$. Now conclude that $\frac{1}{n}\langle v\,;v_n\rangle=\frac{1}{n}$, and hence $\langle v\,;v_n\rangle=1$ for every $n\geq 1$, which is a contradiction (since $\|v\|^2=\sum_{n=1}^{\infty}|\langle v\,;v_n\rangle|^2$).

Therefore, $\mathcal{M}+\mathcal{N}$ is not closed in ℓ^2_+ by the Closed Set Theorem.

Problem 5.14. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be any orthonormal basis for a complex Hilbert space $\mathcal{H}$. For each $x\in\mathcal{H}$ consider its Fourier series expansion with respect to $\{e_\gamma\}_{\gamma\in\Gamma}$; that is, $x=\sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma$. Verify that $\{\mathrm{Re}\,\langle x\,;e_\gamma\rangle e_\gamma\}_{\gamma\in\Gamma}$ and $\{\mathrm{Im}\,\langle x\,;e_\gamma\rangle e_\gamma\}_{\gamma\in\Gamma}$ are both summable families of vectors in $\mathcal{H}$. Set

$$\mathrm{Re}\,x=\textstyle\sum_{\gamma\in\Gamma}\mathrm{Re}\,\langle x\,;e_\gamma\rangle e_\gamma \quad \text{and} \quad \mathrm{Im}\,x=\sum_{\gamma\in\Gamma}\mathrm{Im}\,\langle x\,;e_\gamma\rangle e_\gamma$$

in $\mathcal{H}$. Show that $x=\mathrm{Re}\,x+i\,\mathrm{Im}\,x$ and prove the following identities.

(a) $\langle \mathrm{Re}\,x\,;e_\gamma\rangle=\mathrm{Re}\,\langle x\,;e_\gamma\rangle$ and $\langle \mathrm{Im}\,x\,;e_\gamma\rangle=\mathrm{Im}\,\langle x\,;e_\gamma\rangle$ for every $\gamma\in\Gamma$.

(b) $\langle \mathrm{Re}\,x\,;\mathrm{Im}\,x\rangle=\langle \mathrm{Im}\,x\,;\mathrm{Re}\,x\rangle$ and $\|x\|^2=\|\mathrm{Re}\,x\|^2+\|\mathrm{Im}\,x\|^2$.

(c) $\mathrm{Re}\,(\alpha x)=\mathrm{Re}\,\alpha\,\mathrm{Re}\,x-\mathrm{Im}\,\alpha\,\mathrm{Im}\,x$ and $\mathrm{Im}\,(\alpha x)=\mathrm{Re}\,\alpha\,\mathrm{Im}\,x-\mathrm{Im}\,\alpha\,\mathrm{Re}\,x$.

(d) $\big\langle \mathrm{Re}\,(\alpha x)\,;\mathrm{Im}\,(\alpha x)\big\rangle =$
$\mathrm{Re}\,\alpha\,\mathrm{Im}\,\alpha\big(\|\mathrm{Re}\,x\|^2-\|\mathrm{Im}\,x\|^2\big)+\langle \mathrm{Re}\,x\,;\mathrm{Im}\,x\rangle\big((\mathrm{Re}\,\alpha)^2-(\mathrm{Im}\,\alpha)^2\big)$.

Now prove the *Orthogonal Normalization Lemma*:

> For every nonzero vector x in a complex Hilbert space there exists an $\alpha \in \mathbb{C}$ such that $\|\alpha x\| = 1$ and $\mathrm{Re}(\alpha x) \perp \mathrm{Im}(\alpha x)$.

Hint: Use the above results and the polar representation for $\alpha \in \mathbb{C}$.

Problem 5.15. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal family of vectors e_γ in an infinite-dimensional Hilbert space $\mathcal{H}$, and let $\{e_k\}_{k=1}^{\infty}$ be a countably infinite subset of $\{e_\gamma\}_{\gamma\in\Gamma}$. Set $\mathcal{M} = \bigvee\{e_k\}_{k=1}^{\infty}$ and $\mathcal{M}_n = \mathrm{span}\,\{e_k\}_{k=1}^{n}$ for each integer $n \geq 1$. Let $P\colon \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{M}$, and let $P_n\colon \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto each finite-dimensional subspace $\mathcal{M}_n$. Show that $P_n \xrightarrow{s} P$. (*Hint*: Corollary 5.55.)

Problem 5.16. Let $\{P_k\}_{k=0}^{\infty}$ be a resolution of the identity on a Hilbert space $\mathcal{H}$ so that $\sum_{k=0}^{\infty} P_k x = x$ for every $x \in \mathcal{H}$; that is, $\sum_{k=0}^{n} P_k \xrightarrow{s} I$ as $n \to \infty$. Suppose $P_k \neq O$ for every $k \geq 0$ and let $\{\lambda_k\}_{k=0}^{\infty}$ be a bounded sequence of scalars. By Proposition 5.61 the weighted sum of projections $\sum_{k=0}^{\infty}\lambda_k P_k$ is the operator on $\mathcal{H}$ for which $\sum_{k=0}^{n}\lambda_k P_k \xrightarrow{s} \sum_{k=0}^{\infty}\lambda_k P_k$ as $n \to \infty$. Theorem 5.59 says that $\mathcal{H} = \big(\sum_{k=0}^{\infty}\mathcal{R}(P_k)\big)^{-}$, and $\big(\sum_{k=0}^{\infty}\mathcal{R}(P_k)\big)^{-}$ is unitarily equivalent to the orthogonal direct sum $\bigoplus_{k=0}^{\infty}\mathcal{R}(P_k)$ equipped with its usual inner product (see Examples 5.I and 5.J). Moreover, the natural unitary transformation

$$\Phi\colon \bigoplus_{k=0}^{\infty}\mathcal{R}(P_k) \;\to\; \Big(\sum_{k=0}^{\infty}\mathcal{R}(P_k)\Big)^{-}$$

between the Hilbert spaces $\bigoplus_{k=0}^{\infty}\mathcal{R}(P_k)$ and $\big(\sum_{k=0}^{\infty}\mathcal{R}(P_k)\big)^{-}$ is given by

$$\Phi\big(\{u_k\}_{k=0}^{\infty}\big) \;=\; \sum_{k=0}^{\infty} u_k \quad \text{for every} \quad \{u_k\}_{k=0}^{\infty} \in \bigoplus_{k=0}^{\infty}\mathcal{R}(P_k).$$

Set $I_k = P_k|_{\mathcal{R}(P_k)}$ in $\mathcal{B}[\mathcal{R}(P_k)]$ for each k and consider the operator

$$\bigoplus_{k=0}^{\infty}\lambda_k I_k \quad \text{in} \quad \mathcal{B}\big[\bigoplus_{k=0}^{\infty}\mathcal{R}(P_k)\big]$$

given by $\big(\bigoplus_{k=0}^{\infty}\lambda_k I_k\big)\{u_k\}_{k=0}^{\infty} = \{\lambda_k u_k\}_{k=0}^{\infty}$ for every $\{u_k\}_{k=0}^{\infty}$ in $\bigoplus_{k=0}^{\infty}\mathcal{R}(P_k)$ (cf. Problem 4.16). Show that

$$\Phi\Big(\bigoplus_{k=0}^{\infty}\lambda_k I_k\Big)\Phi^{-1}x \;=\; \Big(\sum_{k=0}^{\infty}\lambda_k P_k\Big)x$$

for every x in $\mathcal{H}$. (*Hint*: $\Phi^{-1}\big(\sum_{k=0}^{\infty}P_k x\big) = \{P_k x\}_{k=0}^{\infty}$.) In other words, as the orthogonal projections $\{P_k\}_{k=0}^{\infty}$ are orthogonal to each other, *a weighted sum of projections is identified with an orthogonal direct sum of scalar operators*. In fact, these are unitarily equivalent operators:

$$\sum_{k=0}^{\infty} \lambda_k P_k \cong \bigoplus_{k=0}^{\infty} \lambda_k I_k \quad \text{with} \quad I_k = P_k|_{\mathcal{R}(P_k)}.$$

Problem 5.17. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for a (separable infinite-dimensional) Hilbert space $\mathcal{H}$ and let $\{\lambda_k\}_{k=1}^{\infty}$ be a sequence of scalars. For each integer $m \geq 1$ consider the mapping $T_m \colon \mathcal{H} \to \mathcal{H}$ defined by

$$T_m x = \sum_{k=1}^{m} \lambda_k \langle x\,; e_k \rangle e_k \quad \text{for every} \quad x \in \mathcal{H}.$$

(a) Verify that T_m lies in $\mathcal{B}[\mathcal{H}]$ for each m and show that

$$\sup_k |\lambda_k| < \infty \quad \text{if and only if} \quad T_m \xrightarrow{s} T,$$

where $T \in \mathcal{B}[\mathcal{H}]$ is the weighted sum of projections (cf. Definition 5.60)

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x\,; e_k \rangle e_k \quad \text{for every} \quad x = \sum_{k=1}^{\infty} \langle x\,; e_k \rangle e_k \in \mathcal{H}$$

with $\|T\| = \sup_k |\lambda_k|$ (see Theorem 5.48, and Propositions 5.57 and 5.61). In this case, T is called a *diagonal operator with respect to the basis* $\{e_k\}_{k=1}^{\infty}$.

(b) Conversely, take $T \in \mathcal{B}[\mathcal{H}]$. If there exists an orthonormal basis for $\mathcal{H}$ and a bounded sequence of scalars such that T is the strong limit of $\{T_m\}$, then T is a *diagonalizable operator*. Let $\sup_k |\lambda_k| < \infty$. Show that (see Problem 4.53)

$$\lim_k \lambda_k = 0 \quad \text{if and only if} \quad T_m \xrightarrow{u} T.$$

(c) Still under the assumption that $\sup_k |\lambda_k| < \infty$, which ensures the existence of T in $\mathcal{B}[\mathcal{H}]$, prove the following assertions (see Example 4.J).

(c_1) There exists $T^{-1} \in \mathcal{L}[\mathcal{R}(T), \mathcal{H}]$ if and only if $\lambda_k \neq 0$ for every $k \geq 1$. In this case,

$$\mathcal{R}(T)^- = \mathcal{H}.$$

Hint: $\operatorname{span}\{e_k\}_{k=1}^{\infty} \subseteq \mathcal{R}(T)$ if $\lambda_k \neq 0$ for all k (see Problem 3.47).

(c_2) There exists $T^{-1} \in \mathcal{B}[\mathcal{H}]$ if and only if $\inf_k |\lambda_k| > 0$. In this case,

$$T^{-1} x = \sum_{k=1}^{\infty} \lambda_k^{-1} \langle x\,; e_k \rangle e_k \quad \text{for every} \quad x \in \mathcal{H}.$$

Problem 5.18. Consider the setup of the previous problem under the assumption that $\sup_k |\lambda_k| < \infty$. Use the Fourier expansion of $x \in \mathcal{H}$ (Theorem 5.48) to show by induction that

$$T^n x = \sum_{k=1}^{\infty} \lambda_k^n \langle x\,; e_k \rangle e_k \quad \text{for every} \quad x \in \mathcal{H}$$

and every positive integer n. Now prove the following propositions.

(a) $T^n \xrightarrow{u} O$ if and only if $\sup_k|\lambda_k| < 1$.

(b) $T^n \xrightarrow{s} O$ if and only if $|\lambda_k| < 1$ for every $k \geq 1$.

(c) $T^n \xrightarrow{w} O$ if and only if $T^n \xrightarrow{s} O$.

(d) $\lim_n \|T^n x\| = \infty$ for every $x \neq 0$ if and only if $1 < |\lambda_k|$ for every $k \geq 1$.

Hint: For (a) and (b), see Example 4.H. For (c), note that $T^n e_j = \lambda_j^n e_j$, and so $|\langle T^n e_j \,;\, e_j\rangle| = |\lambda_j|^n$. If $|\lambda_j| \geq 1$ for some j, then $T^n \not\xrightarrow{w} O$. For (d), note that the expansion $\|T^n x\|^2 = \sum_k |\langle\, x;\, e_k\rangle|^2 |\lambda_k|^{2n}$ has nonzero terms for every $x \neq 0$ and if and only if $0 < |\lambda_k|$ for every k (see Example 4.J).

Suppose T has an inverse $T^{-1} \in \mathcal{L}[\mathcal{R}(T), \mathcal{H}]$ on its range. Prove the assertion.

(e) $\lim_n \|T^n x\| = \infty$ or $\lim_n \|T^{-n} x\| = \infty$ for every $x \neq 0$ if and only if $0 \neq |\lambda_k| \neq 1$ for every $k \geq 1$.

Problem 5.19. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Show that $\mathcal{M}$ (defined below) is a dense linear manifold of $\mathcal{H}$.

$$\mathcal{M} = \big\{x \in \mathcal{H}\colon\ \textstyle\sum_{k=1}^{\infty}|\langle x \,;\, e_k\rangle| < \infty\big\}.$$

Hint: Let T be a diagonal operator (Problem 5.17) with $\lambda_k \neq 0$ for all k (so that $\mathcal{R}(T)^- = \mathcal{H}$) and $\sum_{k=1}^{\infty}|\lambda_k|^2 < \infty$. Show that (Schwarz inequality in ℓ_+^2) $\sum_{j=1}^{\infty}|\langle Tx \,;\, e_j\rangle| = \sum_{j=1}^{\infty}|\lambda_j||\langle x \,;\, e_j\rangle| \leq \big(\sum_{j=1}^{\infty}|\lambda_j|^2\big)^{\frac{1}{2}}\big(\sum_{j=1}^{\infty}|\langle x \,;\, e_j\rangle|^2\big)^{\frac{1}{2}} < \infty$ for all $x \in \mathcal{H}$. Hence $\mathcal{R}(T) \subseteq \mathcal{M}$.

Problem 5.20. Let $\{x_n\}$ be an $\mathcal{M}$-valued sequence and let x be a vector in $\mathcal{M}$, where $\mathcal{M}$ is a subspace of a Hilbert space $\mathcal{H}$. Show that

$$x_n \xrightarrow{w} x \ \text{ in } \ \mathcal{M} \quad \text{ if and only if } \quad x_n \xrightarrow{w} x \ \text{ in } \ \mathcal{H}.$$

Hint: $x_n \xrightarrow{w} x$ in $\mathcal{M}$ if and only if $\langle x_n \,;\, u\rangle \to \langle x \,;\, u\rangle$ for every $u \in \mathcal{M}$ because $\mathcal{M}$ is a Hilbert space (Theorem 5.62). Recall: $\mathcal{H} = \mathcal{M} + \mathcal{M}^{\perp}$ (Theorem 5.20), and show that $x_n \xrightarrow{w} x$ in $\mathcal{M}$ implies $x_n \xrightarrow{w} x$ in $\mathcal{H}$.

Problem 5.21. Let $\{T_n\}$ be a sequence of transformations in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ where $\mathcal{X}$ and $\mathcal{Y}$ are inner product spaces. Prove the following propositions.

(a) If $T_n \xrightarrow{w} T$ for some $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$, then $\|T\| \leq \liminf_n \|T_n\|$.

Hint: If $\{T_{n_k}\}$ is a subsequence of $\{T_n\}$ such that $\lim_k \|T_{n_k}\| = \liminf_n \|T_n\|$ and if $T_n \xrightarrow{w} T$, then $\langle Tx \,;\, y\rangle = \lim_k \langle T_{n_k} x \,;\, y\rangle$ (why?) and so $|\langle Tx \,;\, y\rangle| \leq \liminf_n \|T_n\|\,\|x\|\,\|y\|$, for every $x, y \in \mathcal{H}$. Now apply Corollary 5.71.

(b) If $\sup_n \|T_n\| < \infty$ and $\{\langle T_n a \,;\, b\rangle\}$ converges in $\mathbb{F}$ for every a in a dense subset A of $\mathcal{X}$ and every b in a dense subset B of $\mathcal{Y}$, then $\{\langle T_n x \,;\, y\rangle\}$ converges in $\mathbb{F}$ for every $x \in \mathcal{X}$ and every $y \in \mathcal{Y}$.

Hint: See the hint to Problem 4.45(b) and recall that $\mathbb{F}$ is complete.

(c) Take $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$. If $\sup_n\|T_n\| < \infty$ and $\langle T_n a\,;b\rangle \to \langle Ta\,;b\rangle$ for every a in a dense subset A of $\mathcal{X}$ and every b in a dense subset B of $\mathcal{Y}$, then $\langle T_n x\,;y\rangle \to \langle Tx\,;y\rangle$ for every $x \in \mathcal{X}$ and every $y \in \mathcal{Y}$.

Hint: $\langle (T_n - T)x\,;y\rangle = \langle (T_n - T)(x - a_\varepsilon) + (T_n - T)a_\varepsilon\,;y - b_\varepsilon + b_\varepsilon\rangle$.

(d) If $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, and if the hypothesis of (b) or (c) holds true, then $T_n \xrightarrow{w} T$.

Problem 5.22. Let $\{T_n\}$ and $\{S_n\}$ be sequences of transformations in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ and in $\mathcal{B}[\mathcal{Y},\mathcal{Z}]$, respectively, where $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are inner product spaces. Take $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$, $S \in \mathcal{B}[\mathcal{Y},\mathcal{Z}]$, and prove the following propositions.

(a) If $\sup_n\|S_n\| < \infty$, $S_n \xrightarrow{w} S$, and $T_n \xrightarrow{s} T$, then $S_nT_n \xrightarrow{w} ST$.

Hint: $S_nT_n - ST = S_n(T_n - T) + (S_n - S)T$.

(b) If $\mathcal{Y}$ and $\mathcal{Z}$ are Hilbert spaces, $S_n \xrightarrow{w} S$, and $T_n \xrightarrow{s} T$, then $S_nT_n \xrightarrow{w} S$.

(c) If $S_n \xrightarrow{u} S$, $\sup_n\|T_n\| < \infty$, and $T_n \xrightarrow{w} T$, then $S_nT_n \xrightarrow{w} ST$.

Hint: $S_nT_n - ST = (S_n - S)T_n + S(T_n - T)$.

(d) If $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, $S_n \xrightarrow{u} S$, and $T_n \xrightarrow{w} T$, then $S_nT_n \xrightarrow{w} ST$.

Show that addition of weakly convergent sequences of bounded linear transformations is again a weakly convergent sequence of bounded linear transformations whose limit is the sum of the limits of each summand.

Remarks: It is easy to show, even if $\mathcal{X} = \mathcal{Y} = \mathcal{Z}$ is a Hilbert space, that

$$S_n \xrightarrow{w} S \text{ and } T_n \xrightarrow{u} T \quad \text{do not imply} \quad S_nT_n \xrightarrow{s} ST.$$

For instance, if $\{T_n\}$ is a constant sequence, say $T_n = I$ for all n, and $\{S_n\}$ is any operator sequence that converges weakly to zero but does not converge strongly, then $S_nT_n = S_n \xrightarrow{s} O = OI$. It is also easy to show that

$$S_n \xrightarrow{s} S \text{ and } T_n \xrightarrow{w} T \quad \text{do not imply} \quad S_nT_n \xrightarrow{w} ST.$$

Indeed, there exists an isometry S_+ (thus $S_+^{*n}S_+^n = I$ and $\|S_+^n x\| = 1$ for all n and all x, and so $S_+^{*n}S^n \xrightarrow{w} O$ and $S_+^n \xrightarrow{s} O$) for which $S_+^{*n} \xrightarrow{s} O$ (and hence $S_+^{*n} \xrightarrow{w} O$ and $S_+^n \xrightarrow{w} O$) — the isometry S_+ is introduced in Problem 5.29.

Problem 5.23. This problem deals with a characterization of weak convergence for sequences of Hilbert space operators, and also with its relation to strong convergence. Prove the following proposition.

(a) Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$. Let $\{T_n\}$ be a sequence of operators in $\mathcal{B}[\mathcal{H}]$, and let T be an operator in $\mathcal{B}[\mathcal{H}]$.

$$T_n \xrightarrow{w} T \quad \text{if and only if}$$

$$\sup_n \|T_n\| < \infty \quad \text{and} \quad \langle T_n e_j\,;e_k\rangle \to \langle Te_j\,;e_k\rangle \text{ as } n \to \infty \text{ for every } j,k \geq 1.$$

Hint: The "only if" part is easy (cf. Problem 5.5 and Proposition 5.67). Conversely, suppose $\sup_n \|T_n\| < \infty$ and $\lim_n \langle (T_n - T)e_j \,;\, e_k \rangle = 0$ for every $j, k \geq 1$. Use Theorem 5.48 and Problem 4.14(b) to show that

$$\limsup_n |\langle (T_n - T)e_j \,;\, y \rangle| \leq \limsup_n \sum_{k=1}^{\infty} |\langle (T_n - T)e_j \,;\, e_k \rangle| \, |\langle e_k \,;\, y \rangle| = 0$$

for each $j \geq 1$ and every $y \in \mathcal{M}$, where $\mathcal{M}$ is the linear manifold of Problem 5.19. Repeat the argument to show that

$$\begin{aligned} \limsup_n |\langle (T_n - T)x \,;\, y \rangle| &= \limsup_n |\langle x \,;\, (T_n - T)^* y \rangle| \\ &\leq \limsup_n \sum_{k=1}^{\infty} |\langle x \,;\, e_k \rangle| \, |\langle (T_n - T)e_k \,;\, y \rangle| = 0, \end{aligned}$$

and so $\lim_n |\langle (T_n - T)x \,;\, y \rangle| = 0$, for every $x, y \in \mathcal{M}$. Now use Problems 5.19 and 5.21(d) to conclude that $T_n \xrightarrow{w} T$.

(b) *Weak convergence from below implies strong convergence, which coincides with weak convergence and norm convergence together.* That is, if $\{T_n\}$ is a sequence of operators in $\mathcal{B}[\mathcal{H}]$ and T is an operator in $\mathcal{B}[\mathcal{H}]$, then

$$\left.\begin{array}{c} T_n \xrightarrow{w} T \quad \text{and} \\ \|T_n x\| \leq \|Tx\| \text{ for every } x \end{array}\right\} \Longrightarrow T_n \xrightarrow{s} T \Longleftrightarrow \left\{\begin{array}{c} T_n \xrightarrow{w} T \quad \text{and} \\ \|T_n x\| \to \|Tx\| \text{ for every } x. \end{array}\right.$$

- (b$_1$) *If a sequence of Hilbert space contractions converges weakly to an isometry, then it converges strongly.*
- (b$_2$) It may happen that $T_n \xrightarrow{s} T$ and $\|Tx\| < \|T_n x\|$ for every $x \neq 0$ and all n.

Hint: Show that if $\|T_n x\|^2 \leq \|Tx\|^2 = \langle Tx \,;\, Tx \rangle$ for every $x \in \mathcal{H}$ and every n, then $\|(T_n - T)x\|^2 \leq 2 \operatorname{Re} \langle (T - T_n)x \,;\, Tx \rangle$ for every $x \in \mathcal{H}$ and every n. Thus conclude the claimed implication in (b). For the equivalence in (b), see the remark between Propositions 5.65 and 5.66. Apply Proposition 4.37 to verify that (b$_1$) is a straightforward corollary of (b). Set $T_n = \frac{n+1}{n} I$ and $T = I$ to get an example for (b$_2$).

Problem 5.24. Take $L \in \mathcal{L}[\mathcal{X}]$ on a linear space $\mathcal{X}$ and consider its nth power $L^n \in \mathcal{L}[\mathcal{X}]$ for each integer $n \geq 0$ with $L^0 = I$ (see Problem 2.20). Recall that if $\mathcal{X}$ is a normed space, then each L^n lies in $\mathcal{B}[\mathcal{X}]$ whenever L lies in $\mathcal{B}[\mathcal{X}]$ (see Problem 4.22). Verify that the the power sequence $\{L^n\}_{n \geq 0}$ can be recursively defined by $L^{n+1} = LL^n$ for every $n \geq 0$. Show by induction:

(a) $L^{n+k} = L^n L^k = L^k L^n$ and $(L^n)^k = (L^k)^n = L^{nk}$ for every $k, n \geq 0$.

Now let $\mathcal{X}$ be an inner product space and take any power bounded operator $T \in \mathcal{B}[\mathcal{X}]$ (i.e., any T such that $\sup_n \|T^n\| < \infty$). If $T^n \xrightarrow{w} P$ for some operator $P \in \mathcal{B}[\mathcal{X}]$, then use Problem 5.22 to show that

(b) $PT^k = T^kP = P = P^k$ for every $k \geq 1$, so that P is a projection (not necessarily an orthogonal projection). Show by induction that

(c) $(T-P)^n = T^n - P$ for every $n \geq 1$, so that $(T-P)^n \xrightarrow{w} O$.

If $\mathcal{X}$ is a Hilbert space and $T \in \mathcal{B}[\mathcal{X}]$, then show that

(d) $T^{n*} = T^{*n}$ and $\|T^{*n}\| = \|T^n\|$ for every $n \geq 0$.

Problem 5.25. Let $\mathcal{X}$ and $\mathcal{Y}$ be two normed spaces. Recall that $T \in \mathcal{B}[\mathcal{X}]$ and $S \in \mathcal{B}[\mathcal{Y}]$ are similar if there exists $W \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$ such that $WT = SW$ (i.e., $T = W^{-1}SW$ — see Problem 4.42). Consider a sequence $\{T_n\}$ of operators in $\mathcal{B}[\mathcal{X}]$ and a sequence $\{S_n\}$ of operators in $\mathcal{B}[\mathcal{Y}]$. Suppose there exists $W \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$ such that $WT_n = S_nW$ for every integer n. Show that

$$S_n \xrightarrow{u} S \quad \text{implies} \quad T_n \xrightarrow{u} W^{-1}SW,$$

$$S_n \xrightarrow{s} S \quad \text{implies} \quad T_n \xrightarrow{s} W^{-1}SW.$$

In particular, if $WT = SW$ and $S^n \xrightarrow{s} P$ (or $S^n \xrightarrow{u} P$) for some $P \in \mathcal{B}[\mathcal{Y}]$, then show that $T^n \xrightarrow{s} W^{-1}PW$ (or $T^n \xrightarrow{u} W^{-1}PW$) and $W^{-1}PW$ is a projection in $\mathcal{B}[\mathcal{X}]$ (cf. Problem 4.55).

Hint: $T_n - W^{-1}S_nW = W^{-1}(S_n - S)W$.

Now let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert spaces. Recall that $T \in \mathcal{B}[\mathcal{X}]$ and $S \in \mathcal{B}[\mathcal{Y}]$ are unitarily equivalent if there exists a unitary $U \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$ such that $UT = SU$ (i.e., $T = U^{-1}SU$ — see Problem 5.9). Again, suppose there exists a unitary $U \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$ such that $UT_n = S_nU$ for every integer n. Show that

$$S_n \xrightarrow{w} S \quad \text{implies} \quad T_n \xrightarrow{w} U^{-1}SU.$$

In particular, if $UT = SU$ and $S^n \xrightarrow{w} P$ for some $P \in \mathcal{B}[\mathcal{Y}]$, then show that $T^n \xrightarrow{w} U^{-1}PU$, and that $U^{-1}PU$ is a projection in $\mathcal{B}[\mathcal{X}]$ (not necessarily orthogonal — cf. Problem 5.24).

Hint: $\langle (T_n - U^{-1}S_nU)x\, y\rangle = \langle U^{-1}(S_n - S)Ux\,; y\rangle = \langle (S_n - S)Ux\,; Uy\rangle$.

Problem 5.26. Take a $\mathcal{B}[\mathcal{H}, \mathcal{K}]$-valued sequence $\{T_n\}$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, and take $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$. Show that

$$T_n \xrightarrow{u} T \quad \text{if and only if} \quad T_n^* \xrightarrow{u} T^*,$$

$$T_n \xrightarrow{w} T \quad \text{if and only if} \quad T_n^* \xrightarrow{w} T^*.$$

The adjoint operation preserves uniform and weak convergence. But it does not preserve strong convergence. In fact, as we shall see in Problem 5.29(b,c),

$$T_n \xrightarrow{s} T \quad \text{does not imply} \quad T_n^* \xrightarrow{s} T^*.$$

However, if $T_n \xrightarrow{w} T$ and $T_n^* \xrightarrow{s} S$, then $S = T^*$. Why?

Problem 5.27. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Show that

(a) $V \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ is an isometry if and only if $V^{*n}V^n = I$ (where I is the identity on $\mathcal{H}$) for every $n \geq 1$.

(b) $U \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ is a unitary transformation if and only if $U^{*n}U^n = I$ (identity on $\mathcal{H}$) and $U^nU^{*n} = I$ (identity on $\mathcal{K}$) for every $n \geq 1$.

Hint: Propositions 5.72 and 5.73.

Let $T \in \mathcal{B}[\mathcal{H}]$ be a diagonalizable operator (Problem 5.17). Show that

(c) $T^*x = \sum_{k=1}^{\infty} \overline{\lambda}_k \langle x\,; e_k\rangle e_k$ for every $x \in \mathcal{H}$,

(d) T is unitary if and only if $|\lambda_k| = 1$ for every $k \geq 1$.

Problem 5.28. Let $\mathcal{H}_i$ and $\mathcal{K}_i$ be Hilbert spaces for $i = 1,2$ and consider the Hilbert spaces $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K}_1 \oplus \mathcal{K}_2$ of Example 5.E. Take $T_{ij} \in \mathcal{B}[\mathcal{H}_j,\mathcal{K}_i]$ for $i,j = 1,2$ and consider transformation

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \quad \text{in} \quad \mathcal{B}[\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{K}_1 \oplus \mathcal{K}_2]$$

of Problem 4.17. Show that

$$T^* = \begin{pmatrix} T_{11}^* & T_{21}^* \\ T_{12}^* & T_{22}^* \end{pmatrix} \quad \text{in} \quad \mathcal{B}[\mathcal{K}_1 \oplus \mathcal{K}_2, \mathcal{H}_1 \oplus \mathcal{H}_2].$$

In particular, if $\mathcal{K}_i = \mathcal{H}_i$ for $i = 1,2$ so that $T, T^* \in \mathcal{B}[\mathcal{H}_1 \oplus \mathcal{H}_2]$, then

$$T = T_{11} \oplus T_{22} = \begin{pmatrix} T_{11} & O \\ O & T_{22} \end{pmatrix} \quad \text{implies} \quad T^* = T_{11}^* \oplus T_{22}^* = \begin{pmatrix} T_{11}^* & O \\ O & T_{22}^* \end{pmatrix}.$$

Let $\{\mathcal{H}_k\}$ be a sequence of Hilbert spaces and let $\bigoplus_k \mathcal{H}_k$ stand for the Hilbert space $\big(\big[\bigoplus_{k=1}^{\infty} \mathcal{H}_k\big]_2, \langle\ ;\ \rangle\big)$ of Examples 5.F and 5.G. Let $\{T_k\}$ be a sequence of operators in $\mathcal{B}[\mathcal{H}_k]$ and consider their direct sum $T = \bigoplus_k T_k \in \mathcal{B}\big[\bigoplus_k \mathcal{H}_k\big]$ as in Problem 4.16. Show that $T^* = \bigoplus_k T_k^* \in \mathcal{B}\big[\bigoplus_k \mathcal{H}_k\big]$. Next consider the usual identification $\mathcal{H}_k \cong \bigoplus_{j<k}\{0\} \oplus \mathcal{H}_k \oplus \bigoplus_{j>k}\{0\}$ so that we may interpret each $\mathcal{H}_k$ as a subspace of $\bigoplus_k \mathcal{H}_k$. Show that (cf. Corollary 5.75)

$$\mathcal{H}_k \text{ reduces } T = \bigoplus_k T_k, \quad T|_{\mathcal{H}_k} = T_k, \quad \text{and} \quad (T|_{\mathcal{H}_k})^* = T^*|_{\mathcal{H}_k}, \quad \text{for each } k.$$

Now let $\{\mathcal{H}_\gamma\}$ be a family of Hilbert spaces and consider the orthogonal direct sum $\bigoplus_\gamma \mathcal{H}_\gamma$ as in Problem 5.10. Rewrite Examples 5.F and 5.G for uncountable orthogonal sums. Restate the above results for uncountable families $\{T_\gamma\}$ of operators in $\mathcal{B}[\mathcal{H}_\gamma]$ considering their direct sum $T = \bigoplus_\gamma T_\gamma \in \mathcal{B}\big[\bigoplus_\gamma \mathcal{H}_\gamma\big]$.

Problem 5.29. An operator S_+ acting on a Hilbert space $\mathcal{H}$ is a *unilateral shift* if there exists an infinite sequence $\{\mathcal{H}_k\}_{k=0}^{\infty}$ of nonzero pairwise orthogonal subspaces of $\mathcal{H}$ such that $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$ and S_+ maps each $\mathcal{H}_k$ isometrically onto $\mathcal{H}_{k+1}$. That is, $\mathcal{H}_j \perp \mathcal{H}_k$ whenever $j \neq k$, $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$, and

$$S_+ : \mathcal{H} \to \mathcal{H} \text{ is such that } S_+|_{\mathcal{H}_k} : \mathcal{H}_k \to \mathcal{H}_{k+1} \text{ is a surjective isometry.}$$

Thus each $S_+|_{\mathcal{H}_k}\colon \mathcal{H}_k \to \mathcal{H}_{k+1}$ is a unitary transformation (a surjective isometry) and so $\dim \mathcal{H}_{k+1} = \dim \mathcal{H}_k$ for every $k \geq 0$ (Theorem 5.49). Such a common dimension is the *multiplicity* of S_+. The adjoint of $S_+ \in \mathcal{B}[\mathcal{H}]$, $S_+^* \in \mathcal{B}[\mathcal{H}]$, is referred to as a *backward unilateral shift*. We shall write $\bigoplus_{k=0}^{\infty} x_k$ for $\{x_k\}_{k=0}^{\infty}$ in $\bigoplus_{k=0}^{\infty} \mathcal{H}_k$. Prove the following assertions.

(a) $S_+\colon \mathcal{H} \to \mathcal{H}$ and $S_+^*\colon \mathcal{H} \to \mathcal{H}$ are given by the formulas

$$S_+ x \;=\; 0 \oplus \textstyle\bigoplus_{k=1}^{\infty} U_k\, x_{k-1} \quad \text{and} \quad S_+^* x \;=\; \bigoplus_{k=0}^{\infty} U_{k+1}^* x_{k+1}$$

for every $x = \bigoplus_{k=0}^{\infty} x_k$ in $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$, with 0 denoting the origin of $\mathcal{H}_0$, where $\{U_{k+1}\}_{k=0}^{\infty}$ is an arbitrary sequence of unitary transformations $U_{k+1}\colon \mathcal{H}_k \to \mathcal{H}_{k+1}$, so that $S_+|_{\mathcal{H}_k} = U_{k+1}$, for each $k \geq 0$.

The operators S_+ and S_+^* are identified with the infinite matrices

$$S_+ = \begin{pmatrix} O & & & & \\ U_1 & O & & & \\ & U_2 & O & & \\ & & U_3 & O & \\ & & & & \ddots \end{pmatrix} \quad \text{and} \quad S_+^* = \begin{pmatrix} O & U_1^* & & & \\ & O & U_2^* & & \\ & & O & U_3^* & \\ & & & O & \\ & & & & \ddots \end{pmatrix}$$

of transformations, where every entry immediately below (above) the main block diagonal in the matrix of S_+ (or S_+^*) is a unitary transformation, and the remaining entries are all null transformations.

(b) S_+^* is a strongly stable coisometry. That is, the operator S_+ is an isometry and $S_+^{*n} \xrightarrow{s} O$ (so that S_+ is an isometry that is not a coisometry).

Hint: $\|S_+ x\|^2 = \|x\|^2$ and $S_+^{*n} x = \bigoplus_{k=0}^{\infty} U_{k+1}^* \cdots U_{k+n}^* x_{k+n}$ (by induction) so that $\|S_+^{*n} x\|^2 = \sum_{k=n}^{\infty} \|x_k\|^2$.

(c) $S_+^n \xrightarrow{w} O$ but $\{S_+^n\}$ does not converge strongly.

Hint: $S_+^{*n} \xrightarrow{w} O$ and $\|S_+^n x\| = \|x\|$ (i.e., $S_+^{*n} S_+^n = I$).

Let $\mathcal{K}_0$ be any Hilbert space unitarily equivalent to $\mathcal{H}_0$, and let $U_0\colon \mathcal{K}_0 \to \mathcal{H}_0$ be a unitary transformation, so that $\dim \mathcal{H}_k = \dim \mathcal{K}_0$ for all k. Consider the Hilbert space $\ell_+^2(\mathcal{K}_0) = \bigoplus_{k=0}^{\infty} \mathcal{K}_0$ of Example 5.F.

(d) $U = \bigoplus_{k=0}^{\infty} U_k \cdots U_0\colon\ \ell_+^2(\mathcal{K}_0) \to \mathcal{H}$ is a unitary transformation, and

$$U^* S_+ U \;=\; \begin{pmatrix} O & & & & \\ I & O & & & \\ & I & O & & \\ & & I & O & \\ & & & & \ddots \end{pmatrix} \quad \text{in} \quad \mathcal{B}[\ell_+^2(\mathcal{K}_0)].$$

Thus S_+ is unitarily equivalent to $U^* S_+ U$, which is a unilateral shift of multiplicity $\dim \mathcal{K}_0$. This is called the *canonical unilateral shift* on $\ell_+^2(\mathcal{K}_0)$.

Item (b) says that the adjoint of a unilateral shift is a strongly stable coisometry. It can be shown that the converse holds true, so that a unilateral shift is precisely an operator whose adjoint is a strongly stable coisometry: *S_+ is a unilateral shift if and only if S_+^* is a strongly stable coisometry*.

(e) If an isometry commutes with a unilateral shift, then their product is again a unilateral shift.

Hint: If V is an isometry and S_+ a unilateral shift, then, since S_+ is an isometry, both S_+V and VS_+ are isometries (Problem 4.41(b)). Suppose $S_+V = VS_+$ so that $\|(S_+V)^{*n}x\| = \|(S_+V)^{n*}x\| = \|(S_+^{*n}V^n)^*x\|$. Hence, since an isometry is a contraction, $\|(S_+V)^{*n}x\| = \|V^{n*}(S_+^{*n}x)\| \le \|S_+^{*n}x\|$ for every $n \ge 1$ and $x \in \mathcal{H}$. Thus, since $S_+^{*n} \xrightarrow{s} O$, we get $(S_+V)^{*n} \xrightarrow{s} O$.

We shall see later (Problem 5.31(c)) that if an isometry does not commute with a unilateral shift, then their product may not be a unilateral shift.

Problem 5.30. An operator S acting on a Hilbert space $\mathcal{H}$ is a *bilateral shift* if there exists an infinite family $\{\mathcal{H}_k\}_{k=-\infty}^{\infty}$ of nonzero pairwise orthogonal subspaces of $\mathcal{H}$ such that $\mathcal{H} = \bigoplus_{k=-\infty}^{\infty}\mathcal{H}_k$ and S maps each $\mathcal{H}_k$ isometrically onto $\mathcal{H}_{k+1}$. That is, $\mathcal{H}_j \perp \mathcal{H}_k$ whenever $j \ne k$, $\mathcal{H} = \bigoplus_{k=-\infty}^{\infty}\mathcal{H}_k$, and

$$S\colon \mathcal{H} \to \mathcal{H} \ \text{ is such that } \ S|_{\mathcal{H}_k}\colon \mathcal{H}_k \to \mathcal{H}_{k+1} \ \text{ is a surjective isometry.}$$

As in the case of a unilateral shift, each $S|_{\mathcal{H}_k}\colon \mathcal{H}_k \to \mathcal{H}_{k+1}$ is a unitary operator. Thus the subspaces $\mathcal{H}_k$ of $\mathcal{H}$ are all unitarily equivalent, and their common dimension is the *multiplicity* of S. The adjoint $S^* \in \mathcal{B}[\mathcal{H}]$ of $S \in \mathcal{B}[\mathcal{H}]$ is referred to as a *backward bilateral shift*. Prove the following assertions.

(a) $S\colon \mathcal{H} \to \mathcal{H}$ and $S^*\colon \mathcal{H} \to \mathcal{H}$ are given by the formulas

$$Sx = \bigoplus_{k=-\infty}^{\infty} U_k\, x_{k-1} \quad \text{and} \quad S^*x = \bigoplus_{k=\infty}^{\infty} U_{k+1}^* x_{k+1}$$

for every $x = \bigoplus_{k=-\infty}^{\infty} x_k$ in $\mathcal{H} = \bigoplus_{k=-\infty}^{\infty}\mathcal{H}_k$, where $\{U_k\}_{k=-\infty}^{\infty}$ is an arbitrary family of unitary transformations $U_{k+1}\colon \mathcal{H}_k \to \mathcal{H}_{k+1}$, so that $S|_{\mathcal{H}_k} = U_{k+1}$, for each integer k.

The operators S and S^* are identified with the following (doubly) infinite matrices of transformations (the inner parenthesis indicates the zero-zero entry).

$$S = \begin{pmatrix} \ddots & & & & & & \\ & O & & & & & \\ & U_{-1} & O & & & & \\ & & U_0 & (O) & & & \\ & & & U_1 & O & & \\ & & & & & \ddots & \end{pmatrix} \quad \text{and} \quad S^* = \begin{pmatrix} \ddots & & & & & & \\ & O & U_{-1}^* & & & & \\ & & O & U_0^* & & & \\ & & & (O) & U_1^* & & \\ & & & & O & & \\ & & & & & \ddots & \end{pmatrix}.$$

Again, every entry immediately below (above) the main block diagonal in the matrix of S (or S^*) is a unitary transformation, and the remaining entries are all null transformations,

(b) S is unitarily equivalent to the *canonical bilateral shift* acting on $\ell^2(\mathcal{K}_0) = \bigoplus_{k=-\infty}^{\infty} \mathcal{K}_0$,

$$\begin{pmatrix} \ddots & & & & & & \\ & O & & & & & \\ & I & O & & & & \\ & & I & (O) & & & \\ & & & I & O & & \\ & & & & & \ddots & \end{pmatrix} \quad \text{in} \quad \mathcal{B}[\ell^2(\mathcal{K}_0)],$$

where $\mathcal{K}_0$ is any Hilbert space with the property that $\dim \mathcal{K}_0$ is the multiplicity of S. (*Hint*: Problem 5.29(d).)

(c) S is a weakly stable unitary operator. In other words, S is an isometry and a coisometry (i.e., $S^{*n}S^n = S^nS^{*n} = I$) and $S^n \xrightarrow{w} O$.

Hint: Let S_0 denote the above canonical bilateral shift on $\ell^2(\mathcal{H}_0)$, which is unitarily equivalent to S. Take $x = \bigoplus_{k=-\infty}^{\infty} x_k$ in $\ell^2(\mathcal{H}_0) = \bigoplus_{k=-\infty}^{\infty} \mathcal{H}_0$. Show by induction that $S_0^n x = \bigoplus_{k=-\infty}^{\infty} x_{k-n}$. Suppose n is even. Thus

$$\langle S_0^n x\,;x\rangle \,=\, \sum_{k=-\infty}^{\infty} \langle x_{k-n}\,;x_k\rangle \,=\, \sum_{k=-\infty}^{-(n/2)-1} \langle x_k\,;x_{k+n}\rangle \,+\, \sum_{k=-n/2}^{\infty} \langle x_k\,;x_{k+n}\rangle.$$

Now apply the Schwarz inequality in $\mathcal{H}_0$ and in ℓ^2 to show that

$$\sum_{k=-\infty}^{-(n/2)-1} |\langle x_k\,;x_{k+n}\rangle| \,\le\, \sum_{k=-\infty}^{-(n/2)-1} \|x_k\|\,\|x_{k+n}\|$$

$$\le \Big(\sum_{k=-\infty}^{-(n/2)-1} \|x_k\|^2\Big)^{\frac{1}{2}} \Big(\sum_{k=-\infty}^{-(n/2)-1} \|x_{k+n}\|^2\Big)^{\frac{1}{2}} \,\le\, \Big(\sum_{k=-\infty}^{-(n/2)-1} \|x_k\|^2\Big)^{\frac{1}{2}} \|x\|.$$

Similarly,

$$\sum_{k=-n/2}^{\infty} |\langle x_k\,;x_{k+n}\rangle| \,\le\, \Big(\sum_{k=-n/2}^{\infty} \|x_{k+n}\|^2\Big)^{\frac{1}{2}} \|x\| \,=\, \Big(\sum_{j=n/2}^{\infty} \|x_j\|^2\Big)^{\frac{1}{2}} \|x\|.$$

Next verify that

$$\lim_n \sum_{k=-\infty}^{-(n/2)-1} \|x_k\|^2 \,=\, \lim_n \sum_{j=n/2}^{\infty} \|x_j\|^2 \,=\, 0.$$

A similar argument holds for n odd. Thus infer that $S_0^n \xrightarrow{w} O$ (cf. Proposition 5.67). Use Problem 5.25 to conclude that $S^n \xrightarrow{w} O$.

(d) Both $\{S^n\}$ and $\{S^{*n}\}$ do not converge strongly.

Hint: $S^{*n} \xrightarrow{w} O$ and $\|S^n x\| = \|S^{*n} x\| = \|x\|$.

Problem 5.31. According to Problem 5.29, unilateral shifts exist only on infinite-dimensional spaces. Now let $\mathcal{H}$ be a *separable* Hilbert space.

(a) $S_+ \in \mathcal{B}[\mathcal{H}]$ is a unilateral shift of *multiplicity* 1 if and only if

$$S_+ e_k \;=\; e_{k+1} \quad \text{for every} \quad k \in \mathbb{N}_0$$

for some orthonormal basis $\{e_k\}_{k=0}^{\infty}$ for $\mathcal{H}$; that is, if and only if it shifts some orthonormal basis indexed by $\mathbb{N}_0$ (or by any set in a one-to-one order-preserving correspondence with $\mathbb{N}_0$). In this case, also show that

$$S_+^* e_0 \;=\; 0 \quad \text{and} \quad S_+^* e_k \;=\; e_{k-1} \quad \text{for every} \quad k \in \mathbb{N}_0.$$

Hint: Set $\mathcal{H}_k = \operatorname{span}\{e_k\}$, so that $\dim \mathcal{H}_k = 1$, for each $k \in \mathbb{N}_0$.

(b) Verify that the operator $S_+ \in \mathcal{B}[\ell_+^2]$ of Problem 4.39 is, in fact, the canonical unilateral shift of multiplicity 1 on ℓ_+^2, which shifts the canonical orthonormal basis for ℓ_+^2. (*Hint*: $\ell_+^2 = \bigoplus_{k=0}^{\infty} \mathbb{C}$.)

(c) Take the unilateral shift $S_+ \in \mathcal{B}[\ell_+^2]$ of item (b). Exhibit a unitary operator $U \in \mathcal{B}[\ell_+^2]$ such that both products US_+ and S_+U are not unilateral shifts.

Hint: $U = W \oplus I$ in $\mathcal{B}[\ell_+^2]$ is unitary, where $W = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in $\mathcal{B}[\mathbb{C}^2]$ and I is the identity in $\mathcal{B}[\ell_+^2]$. $US_+ = 1 \oplus S_+$ is an isometry (Problem 4.41(b)), but

$$(US_+)^{*n} = 1 \oplus S_+^{*n} \xrightarrow{s} 1 \oplus O = \operatorname{diag}(1, 0, 0, 0, \ldots) \neq O,$$

and so the isometry US_+ is not a unilateral shift (see Problem 5.29). Moreover, since the unitary U also is self-adjoint, we get

$$(S_+U)^{*(n+1)} = (US_+^*)^{n+1} = U(S_+^*U)^n S_+^* = U(US_+)^{*n} S_+^* \xrightarrow{s} U(1 \oplus O)S_+^*,$$

where $U(1 \oplus O)S^* = \operatorname{diag}(0, 1, 0, 0, \ldots)$. Again, S_+U is an isometry with an adjoint that is not strongly stable, and so it is not a unilateral shift.

(d) Let $\{e_k\}_{k=-\infty}^{\infty}$ be the canonical orthonormal basis for ℓ^2. Reindex this basis as follows. Set, for every $n \in \mathbb{N}$,

$$f_n = e_{\frac{1-n}{2}} \;\text{ if } n \text{ is odd} \quad \text{and} \quad f_n = e_{\frac{n}{2}} \;\text{ if } n \text{ is even.}$$

Check that $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis for ℓ^2 and that the operator S_+ in $\mathcal{B}[\ell^2]$ given by the (doubly) infinite matrix

$$S_+ = \begin{pmatrix} & & & & & & \iddots \\ & & & & & 1 & \\ & & & & 1 & & \\ & & & (0) & & & \\ & & & 0 & 1 & & \\ & & 0 & 1 & & & \\ & \iddots & 1 & & & & \end{pmatrix}$$

is a unilateral shift on ℓ^2 that shifts the orthonormal basis $\{f_n\}_{n=1}^{\infty}$.

Problem 5.32. According to Problem 5.30, bilateral shifts exist only on infinite-dimensional spaces. Now let $\mathcal{H}$ be a *separable* Hilbert space.

(a) $S \in \mathcal{B}[\mathcal{H}]$ is a bilateral shift of *multiplicity* 1 if and only if

$$Se_k \;=\; e_{k+1} \quad \text{for every} \quad k \in \mathbb{Z}$$

for some orthonormal basis $\{e_k\}_{k=-\infty}^{\infty}$ for $\mathcal{H}$; that is, if and only if it shifts some orthonormal basis indexed by $\mathbb{Z}$ (or by any set in a one-to-one order-preserving correspondence with $\mathbb{Z}$). In this case, also show that

$$S^*e_k \;=\; e_{k-1} \quad \text{for every} \quad k \in \mathbb{Z}.$$

Hint: Set $\mathcal{H}_k = \operatorname{span}\{e_k\}$, so that $\dim \mathcal{H}_k = 1$, for each $k \in \mathbb{Z}$.

(b) Verify that the operator $S \in \mathcal{B}[\ell^2]$ of Problem 4.40 is, in fact, the canonical bilateral shift of multiplicity 1 on ℓ^2, which shifts the canonical orthonormal basis for ℓ^2. (*Hint*: $\ell^2 = \bigoplus_{k=-\infty}^{\infty} \mathbb{C}$.)

(c) Let $\{e_n\}_{n=1}^{\infty}$ be the canonical orthonormal basis for ℓ_+^2. Reindex this basis as follows. Set, for every $k \in \mathbb{Z}$,

$$f_k = e_{1-2k} \;\text{ if }\; k \leq 0 \quad \text{and} \quad f_k = e_{2k} \;\text{ if }\; n \;\; k > 0.$$

Check that $\{f_k\}_{k=\infty}^{\infty}$ is an orthonormal basis for ℓ_+^2 and that the operator S in $\mathcal{B}[\ell_+^2]$ given by the infinite matrix

$$S \;=\; \begin{pmatrix} b & A & & & \\ & B & A & & \\ & & B & A & \\ & & & B & \ddots \\ & & & & \ddots \end{pmatrix}, \;\text{ with }\; b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\; A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\; \text{and}\; B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

is a bilateral shift on ℓ_+^2 that shifts the orthonormal basis $\{f_k\}_{k=-\infty}^{\infty}$.

Problem 5.33. Consider the orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$ for the Hilbert space $L^2(\mathbb{T})$ of Example 5.L(c), where $\mathbb{T}$ denotes the unit circle about the origin of the complex plane and, for each $k \in \mathbb{Z}$, $e_k(z) = z^k$ for every $z \in \mathbb{T}$. Define a map $U\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ as follows. If $f \in L^2(\mathbb{T})$, then Uf is given by

$$(Uf)(z) \;=\; zf(z) \quad \text{for every} \quad z \in \mathbb{T}.$$

(a) Verify that $Uf \in L^2(\mathbb{T})$ for every $f \in L^2(\mathbb{T})$, and $U \in \mathcal{B}[L^2(\mathbb{T})]$.

(b) Show that U is a bilateral shift of multiplicity 1 on $L^2(\mathbb{T})$ that shifts the orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$.

(c) Prove the *Riemann–Lebesgue Lemma*:

$$\text{If} \;\; f \in L^2(\mathbb{T}), \;\; \text{then} \;\; \textstyle\int_{\mathbb{T}} z^k f(z)\, dz \to 0 \;\; \text{as} \;\; k \to \pm\infty.$$

Hint: $(U^k f)(z) = z^k f(z)$ so that $\langle U^k f\,;1\rangle = \int_{\mathbb{T}} z^k f(z)\,dz$, where $1(z) = 1$ for all $z \in \mathbb{T}$. Recall that $U^k \xrightarrow{w} O$ (cf. Problem 5.30(c)).

Problem 5.34. Let $\mathcal{H}$ be a Hilbert space and take $T, S \in \mathcal{B}[\mathcal{H}]$. Use Problem 4.20 and Corollary 5.75 to prove the following assertion.

If S commutes with both T and T^*, then $\mathcal{N}(S)$ and $\mathcal{R}(S)^-$ reduce T.

Problem 5.35. Take $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. Prove the following propositions.

(a) $\mathcal{N}(T) = \{0\} \iff \mathcal{N}(T^*T) = \{0\} \iff \mathcal{R}(T^*)^- = \mathcal{H} \iff \mathcal{R}(TT^*)^- = \mathcal{H}$.

Moreover, $\mathcal{R}(T^*) = \mathcal{H} \iff \mathcal{R}(T^*T) = \mathcal{H}$.

(a*) $\mathcal{N}(T^*) = \{0\} \iff \mathcal{N}(TT^*) = \{0\} \iff \mathcal{R}(T)^- = \mathcal{K} \iff \mathcal{R}(TT^*)^- = \mathcal{K}$.

Moreover, $\mathcal{R}(T) = \mathcal{K} \iff \mathcal{R}(TT^*) = \mathcal{K}$.

Hint: Use Propositions 5.15, 5.76, and 5.77 and recall that $\mathcal{R}(T) = \mathcal{K}$ if and only if $\mathcal{R}(T) = \mathcal{R}(T)^- = \mathcal{K}$.

(b) $\mathcal{R}(T) = \mathcal{K} \iff T^*$ has a bounded inverse on $\mathcal{R}(T^*)$.

(b*) $\mathcal{R}(T^*) = \mathcal{H} \iff T$ has a bounded inverse on $\mathcal{R}(T)$.

Hint: Corollary 4.24 and Proposition 5.77.

Problem 5.36. Consider the following assertions (setup of Problem 5.35):

(a)	$\mathcal{N}(T) = \{0\}$.	(a*)	$\mathcal{N}(T^*) = \{0\}$.
(b)	$\dim \mathcal{R}(T) = n$.	(b*)	$\dim \mathcal{R}(T^*) = m$.
(c)	$\mathcal{R}(T^*) = \mathcal{H}$.	(c*)	$\mathcal{R}(T) = \mathcal{K}$.
(d)	$T^*T \in \mathcal{G}[\mathcal{H}]$.	(d*)	$TT^* \in \mathcal{G}[\mathcal{K}]$.

If $\dim \mathcal{H} = n$, then (a), (b), (c), and (d) are pairwise equivalent. If $\dim \mathcal{K} = m$, then (a*), (b*), (c*), and (d*) are pairwise equivalent. Prove.

Problem 5.37. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and take $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$. If $y \in \mathcal{R}(T)$, then there is a solution $x \in \mathcal{H}$ to the equation $y = Tx$. It is clear that this solution is unique whenever T is injective. If, in addition, $\mathcal{R}(T)$ is closed in $\mathcal{K}$, then this unique solution is given by

$$x \;=\; (T^*T)^{-1}T^*y.$$

In other words, suppose $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T) = \mathcal{R}(T)^-$. According to Corollary 4.24, there exists $T^{-1} \in \mathcal{B}[\mathcal{R}(T), \mathcal{H}]$. Use Propositions 5.76 and 5.77 to show that there exists $(T^*T)^{-1} \in \mathcal{B}[\mathcal{R}(T^*), \mathcal{H}] = \mathcal{B}[\mathcal{H}]$ and

$$T^{-1} \;=\; (T^*T)^{-1}T^* \quad \text{on} \quad \mathcal{R}(T).$$

Problem 5.38. (Least-Squares). Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and take $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$. If $y \in \mathcal{K} \backslash \mathcal{R}(T)$, then there is no solution $x \in \mathcal{H}$ to the equation $y = Tx$. Question: Is there a vector x in $\mathcal{H}$ that minimizes $\|y - Tx\|$? Use Theorem 5.13, Proposition 5.76, and Problem 5.37 to prove the following proposition. If $\mathcal{R}(T) = \mathcal{R}(T)^-$, then for each $y \in \mathcal{K}$ there is an $x_y \in \mathcal{H}$ such that

$$\|y - Tx_y\| \;=\; \inf_{x \in \mathcal{H}} \|y - Tx\| \quad \text{and} \quad T^*Tx_y \;=\; T^*y.$$

Moreover, if T is injective, then x_y is unique and given by

$$x_y \;=\; (T^*T)^{-1}T^*y.$$

Problem 5.39. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and take $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$. If $y \in \mathcal{R}(T)$ and $\mathcal{R}(T) = \mathcal{R}(T)^-$, then show that there is an x_0 in $\mathcal{H}$ such that $y = Tx_0$ and

$$\|x_0\| \;\leq\; \|x\|$$

for all $x \in \mathcal{H}$ such that $y = Tx$. That is, if $\mathcal{R}(T) = \mathcal{R}(T)^-$, then for each $y \in \mathcal{R}(T)$ there exists a solution $x_0 \in \mathcal{H}$ to the equation $y = Tx$ with *minimum norm*. Moreover, if T^* is injective, then show that x_0 is unique and given by

$$x_0 \;=\; T^*(TT^*)^{-1}y.$$

Hint: If $\mathcal{R}(T) = \mathcal{R}(T)^-$, then $\mathcal{R}(TT^*) = \mathcal{R}(T)$ (Propositions 5.76 and 5.77). Take $y \in \mathcal{R}(T)$ so that $y = TT^*z$ for some z in $\mathcal{K}$. Set $x_0 = T^*z$ in $\mathcal{H}$, and so $y = Tx_0$. If $x \in \mathcal{H}$ is such that $y = Tx$, then $\|x_0\|^2 = \langle T^*z\,; x_0\rangle = \langle z\,; Tx\rangle = \langle T^*z\,; x\rangle = \langle x_0\,; x\rangle \leq \|x_0\|\,\|x\|$. If $\mathcal{N}(T^*) = \{0\}$, then $\mathcal{N}(TT^*) = \{0\}$ (Proposition 5.76). Since $\mathcal{R}(TT^*) = \mathcal{R}(T) = \mathcal{R}(T)^-$, there exists $(TT^*)^{-1}$ in $\mathcal{B}[\mathcal{R}(T), \mathcal{K}]$ (Corollary 4.24). Thus $z = (TT^*)^{-1}y$ is unique and so is $x_0 = T^*z$.

Problem 5.40. Show that $T \in \mathcal{B}_0[\mathcal{H}, \mathcal{K}]$ if and only if $T^* \in \mathcal{B}_0[\mathcal{K}, \mathcal{H}]$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. Moreover, $\dim \mathcal{R}(T) = \dim \mathcal{R}(T^*)$.

Hint: $\mathcal{B}_0[\mathcal{H}, \mathcal{K}]$ denotes the set of all finite-rank bounded linear transformations of $\mathcal{H}$ into $\mathcal{K}$. If $T \in \mathcal{B}_0[\mathcal{H}, \mathcal{K}]$, then $\mathcal{R}(T) = \mathcal{R}(T)^-$. (Why?) Now use Propositions 5.76 and 5.77 to show that $\mathcal{R}(T^*) = T^*(\mathcal{R}(T))$. Thus conclude: $\dim \mathcal{R}(T^*) \leq \dim \mathcal{R}(T)$ (cf. Problems 2.17 and 2.18).

Problem 5.41. Let $T \in \mathcal{B}[\mathcal{H}, \mathcal{Y}]$ be a bounded linear transformation of a Hilbert space $\mathcal{H}$ into a normed space $\mathcal{Y}$. Show that the following assertions are pairwise equivalent.

(a) T is compact (i.e., $T \in \mathcal{B}_\infty[\mathcal{H}, \mathcal{Y}]$).

(b) $Tx_n \to Tx$ in $\mathcal{Y}$ whenever $x_n \xrightarrow{w} x$ in $\mathcal{H}$.

(c) $Tx_n \to 0$ in $\mathcal{Y}$ whenever $x_n \xrightarrow{w} 0$ in $\mathcal{H}$.

Hint: Problem 4.69 for (a)⇒(b). Conversely, let $\{x_n\}$ be a bounded sequence in $\mathcal{H}$. Apply Lemma 5.69 to ensure the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges in $\mathcal{Y}$ whenever (b) holds true. Now conclude that T is compact (Theorem 4.52(d)). Hence (b)⇒(a). Trivially, (b)⇒(c). On the other hand, if $x_n \xrightarrow{w} x$ in $\mathcal{H}$, then verify that $T(x_n - x) \to 0$ in $\mathcal{Y}$ whenever (c) holds; that is, (c)⇒(b).

Problem 5.42. If $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then show that the following assertions are pairwise equivalent.

(a) T is compact (i.e., $T \in \mathcal{B}_\infty[\mathcal{H}, \mathcal{K}]$).

(b) T is the (uniform) limit in $\mathcal{B}[\mathcal{H}, \mathcal{K}]$ of a sequence of finite-rank bounded linear transformations of $\mathcal{H}$ into $\mathcal{K}$. That is, there exists a $\mathcal{B}_0[\mathcal{H}, \mathcal{K}]$-valued sequence $\{T_n\}$ such that $\|T_n - T\| \to 0$.

(c) T^* is compact (i.e., $T^* \in \mathcal{B}_\infty[\mathcal{K}, \mathcal{H}]$).

Hint: Take any $T \in \mathcal{B}_\infty[\mathcal{H}, \mathcal{K}]$ and let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for $\mathcal{R}(T)^-$. *If* $P_n : \mathcal{K} \to \mathcal{K}$ *is the orthogonal projection onto* $\bigvee\{e_k\}_{k=1}^n$, *then* $P_nT \xrightarrow{u} T$. Indeed, $\mathcal{R}(T)^-$ is separable (Proposition 4.57), and Theorem 5.52 ensures the existence of P_n. Show: $P_n \xrightarrow{s} P$, where $P : \mathcal{K} \to \mathcal{K}$ is the orthogonal projection onto $\mathcal{R}(T)^-$ (Problem 5.15). Use Problem 4.57 to verify that $P_nT \xrightarrow{u} PT = T$. Set $T_n = P_nT$ and show that each T_n lies in $\mathcal{B}_0[\mathcal{H}, \mathcal{K}]$. Hence (a)⇒(b). For the converse, see Corollary 4.55. Thus (a)⇔(b), which implies (a)⇔(c) (Proposition 5.65(d) and Problem 5.40).

Now prove the following proposition: $\mathcal{B}_0[\mathcal{H}, \mathcal{K}]$ *is dense in* $\mathcal{B}_\infty[\mathcal{H}, \mathcal{K}]$.

Problem 5.43. An operator $J \in \mathcal{B}[\mathcal{H}]$ on a Hilbert space $\mathcal{H}$ is an involution if $J^2 = I$ (cf. Problem 1.11). A *symmetry* is a unitary involution.

(a) Take $S \in \mathcal{B}[\mathcal{H}]$. Show that the following assertions are pairwise equivalent.

(i) S is a unitary involution.

(ii) S is a self-adjoint involution.

(iii) S is self-adjoint and unitary.

(iv) S is an involution such that $S^*S = SS^*$.

(b) Exhibit an involution on $\mathbb{C}^2$ that is not self-adjoint.

Hint: $J = \begin{pmatrix} \sqrt{2} & i \\ i & -\sqrt{2} \end{pmatrix}$ in $\mathcal{B}[\mathbb{C}^2]$.

(c) Exhibit a unitary on $\mathbb{C}^2$ that is not self-adjoint.

Hint: $U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ in $\mathcal{B}[\mathbb{C}^2]$ for any $\theta \in (0, \pi)$.

(d) Consider the symmetry $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in $\mathcal{B}[\mathbb{C}^2]$. Find a resolution of the identity on $\mathbb{C}^2$, say $\{P_1, P_{-1}\}$, such that $S = P_1 - P_{-1}$. (As we shall see in Chapter 6, $P_1 - P_{-1}$ is the *spectral decomposition* of S.)

Hint: $\{P_1, P_{-1}\}$ with $P_1 = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $P_{-1} = \frac{1}{2}\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ in $\mathcal{B}[\mathbb{C}^2]$ is a resolution of the identity on $\mathbb{C}^2$ (i.e., ${P_1}^2 = P_1 = {P_1}^*$, ${P_{-1}}^2 = P_{-1} = {P_{-1}}^*$, $P_1 P_{-1} = P_{-1} P_1 = O$, $P_1 + P_{-1} = I$) such that $S = P_1 - P_{-1}$.

(e) Exhibit a symmetry S and a nilpotent T (both acting on the same Hilbert space) such that ST is a nonzero idempotent. That is, exhibit $S, T \in \mathcal{B}[\mathcal{H}]$, where $S = S^* = S^{-1}$ and $T^2 = O$, such that $O \neq ST = (ST)^2$.

Hint: $T = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

(f) Exhibit an operator in $\mathcal{B}[\mathcal{H}]$ that is unitarily equivalent (through a symmetry) to its adjoint but is not self-adjoint. That is, exhibit $T \in \mathcal{B}[\mathcal{H}]$ such that $STS^* = T^* \neq T$ for some $S \in \mathcal{B}[\mathcal{H}]$ with $S = S^* = S^{-1}$.

Hint: $T = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{smallmatrix}\right)$, $\alpha \in \mathbb{C}\backslash\mathbb{R}$, or $T = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)$, $\beta \neq \alpha, \beta \in \mathbb{R}$; with $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Problem 5.44. Let $\mathcal{H}$ be a Hilbert space. Show that the set of all self-adjoint operators from $\mathcal{B}[\mathcal{H}]$ is weakly closed in $\mathcal{B}[\mathcal{H}]$.

Hint: Verify: $|\langle Tx\,;y\rangle - \langle x\,;Ty\rangle| = |\langle Tx\,;y\rangle - \langle T_n x\,;y\rangle + \langle x\,;T_n y\rangle - \langle x\,;Ty\rangle| \leq |\langle (T_n - T)x\,;y\rangle| + |\langle (T_n - T)y\,;x\rangle|$ whenever $T_n^* = T_n$.

Problem 5.45. Let S and T be self-adjoint operators in $\mathcal{B}[\mathcal{H}]$, where $\mathcal{H}$ is a Hilbert space. Prove the following results.

(a) $T + S$ is self-adjoint.

(b) αT is self-adjoint if and only if $\alpha \in \mathbb{R}$.

Therefore, if $\mathcal{H}$ is a *real* Hilbert space, then the set of all self-adjoint operators from $\mathcal{B}[\mathcal{H}]$ is a subspace of $\mathcal{B}[\mathcal{H}]$.

(c) TS is self-adjoint if and only if $TS = ST$.

(d) $p(T) = p(T)^*$ for every polynomial p with *real* coefficients.

(e) $T^{2n} \geq O$ and $\|T^{2^n}\| = \|T\|^{2^n}$ for each $n \geq 1$. (*Hint*: Proposition 5.78.)

Problem 5.46. If an operator $T \in \mathcal{B}[\mathcal{H}]$ acting on a complex Hilbert space $\mathcal{H}$ is such that $T = A + iB$, where A and B are self-adjoint operators in $\mathcal{B}[\mathcal{H}]$, then the representation $T = A + iB$ is called the *Cartesian decomposition* of T. Prove the following propositions.

(a) Every operator $T \in \mathcal{B}[\mathcal{H}]$ on a complex Hilbert space $\mathcal{H}$ has a unique Cartesian decomposition.

Hint: Set $A = \frac{1}{2}(T^* + T)$ and $B = \frac{i}{2}(T^* - T)$.

(b) $T^*T = TT^*$ if and only if $AB = BA$. In this case, $T^*T = A^2 + B^2$ and $\max\{\|A\|^2, \|B\|^2\} \leq \|T\|^2 \leq \|A^2\| + \|B^2\|$.

Problem 5.47. If $T \in \mathcal{B}[\mathcal{H}]$ is a *self-adjoint* operator acting on a *real* Hilbert space $\mathcal{H}$, then show that

$$\langle Tx\,;y\rangle \;=\; \tfrac{1}{4}\big(\langle T(x+y)\,;x+y\rangle - \langle T(x-y)\,;x-y\rangle\big)$$

for every $x, y \in \mathcal{H}$. (*Hint*: Problem 5.3(a).)

Problem 5.48. Let $\mathcal{H}$ be any (real or complex) Hilbert space.

(a) If $\{T_n\}$ is a sequence of *self-adjoint* operators, then the five assertions of Proposition 5.67 are all pairwise equivalent, even in a real Hilbert space.

Hint: If $T_n^* = T_n$ and the real sequence $\{\langle T_n x\,;x\rangle\}$ converges in $\mathbb{R}$ for every $x \in \mathcal{H}$, and if $\mathcal{H}$ is real, then use Problem 5.47 to show that $\{\langle T_n x\,;y\rangle\}$ converges in $\mathbb{R}$ for every $x, y \in \mathcal{H}$. Now apply Proposition 5.67.

(b) If $\{T_n\}$ is a sequence of *self-adjoint* operators, then the four assertions of Problem 5.5 are all pairwise equivalent, even in a real Hilbert space.

Hint: Problems 5.5 and 5.47.

Problem 5.49. The set $\mathcal{B}^+[\mathcal{H}]$ of all nonnegative operators on a Hilbert space $\mathcal{H}$ is a weakly closed convex cone in $\mathcal{B}[\mathcal{H}]$.

Hint: If $Q_n \geq O$ for every positive integer n and $Q_n \xrightarrow{w} Q$, then $Q \geq O$ since $0 \leq \langle Q_n x\,;x\rangle = \langle (Q_n - Q)x\,;x\rangle + \langle Qx\,;x\rangle$. See Problems 2.2 and 2.21.

Problem 5.50. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and take $T \in \mathcal{B}[\mathcal{H},\mathcal{K}]$. Verify that $T^*T \in \mathcal{B}^+[\mathcal{H}]$ and $TT^* \in \mathcal{B}^+[\mathcal{K}]$, and prove the following assertions.

(a) $T^*T > O$ if and only if T is injective.

(b) $T^*T \in \mathcal{G}^+[\mathcal{H}]$ if and only if $T \in \mathcal{G}[\mathcal{H},\mathcal{K}]$.

(a*) $TT^* > O$ if and only if T^* is injective.

(b*) $TT^* \in \mathcal{G}^+[\mathcal{K}]$ if and only if $T^* \in \mathcal{G}[\mathcal{K},\mathcal{H}]$.

Problem 5.51. Let $\mathcal{H}$ be a Hilbert space and take Q, R, and T in $\mathcal{B}[\mathcal{H}]$. Prove the following implications.

(a) $Q \geq O$ implies $T^*QT \geq O$.

(b) $Q \geq O$ and $R \geq O$ imply $Q + R \geq O$.

(c) $Q > O$ and $R \geq O$ imply $Q + R > O$.

(d) $Q \succ O$ and $R \geq O$ imply $Q + R \succ O$.

Problem 5.52. Let Q be an operator acting on a Hilbert space $\mathcal{H}$. Prove the following propositions.

(a) $Q \geq O$ implies $Q^n \geq O$ for every integer $n \geq 0$.

(b) $Q > O$ implies $Q^n > O$ for every integer $n \geq 0$.

(c) $Q \succ O$ implies $Q^n \succ O$ for every integer $n \geq 0$.

(d) $Q \succ O$ implies $Q^{-1} \succ O$.

(e) If p is an arbitrary polynomial with positive coefficients, then

$Q \geq O$ implies $p(Q) \geq O$,

$Q > O$ implies $p(Q) > O$,

$Q \succ O$ implies $p(Q) \succ O$.

Hints: (a), (b), and (c) are trivially verified for $n = 0, 1$. Suppose $n \geq 2$.

(a) Show that: $\langle Q^n x\,;x\rangle = \|Q^{\frac{n}{2}}x\|^2$ for every $x \in \mathcal{H}$ if n is even, and $\langle Q^n x\,;x\rangle = \langle QQ^{\frac{n-1}{2}}x\,;Q^{\frac{n-1}{2}}x\rangle$ for every $x \in \mathcal{H}$ if n is odd.

(b, c) $Q > O$ if and only if $Q \geq O$ and $\mathcal{N}(Q) \neq \{0\}$; and $Q \succ O$ if and only if $Q \geq O$ and Q is bounded below. In both cases, $Q \neq O$. Note that (i) $\langle Q^{2n}x\,;x\rangle = \|Q^n x\|^2$, and (ii) $\langle Q^{2n-1}x\,;x\rangle \geq \|Q\|^{-1}\|Q^n x\|^2$ (since, by Proposition 5.82, $\|Q^n x\|^2 = \|QQ^{n-1}x\|^2 \leq \|Q\|\langle QQ^{n-1}x\,;Q^{n-1}x\rangle$). Apply (i) to show that (b) and (c) hold for $n = 2$, and hence they hold for $n = 3$ by (ii). Conclude the proofs by induction.

(d) $\|x\|^2 = \|QQ^{-1}x\|^2 \leq \|Q\|\langle QQ^{-1}x\,;Q^{-1}x\rangle = \|Q\|\langle Q^{-1}x\,;x\rangle$. Why?

Problem 5.53. Let $\mathcal{H}$ be a Hilbert space and take $Q, R \in \mathcal{B}[\mathcal{H}]$. Prove that

(a) $O \prec Q \prec R$ implies $O \prec R^{-1} \prec Q^{-1}$,

(b) $O \prec Q \leq R$ implies $O \prec R^{-1} \leq Q^{-1}$,

(c) $O \prec Q < R$ implies $O \prec R^{-1} < Q^{-1}$.

Hints: Consider the result in Problem 5.52(d).

(a) If $O \prec Q \prec R$, then $Q^{-1} \succ O$, $R^{-1} \succ O$, and $(R-Q)^{-1} \succ O$. Observe that $Q^{-1} - R^{-1} = Q^{-1}(R-Q)R^{-1} = ((R-Q+Q)(R-Q)^{-1}Q)^{-1} = (Q+Q(R-Q)^{-1}Q)^{-1}$ and $Q + Q(R-Q)^{-1}Q \succ O$. So $Q^{-1} - R^{-1} \succ O$.

(b) If $O \prec Q \leq R$, then $Q^{-1} \succ O$, $R^{-1} \succ O$ (there is an $\alpha > 0$ such that $\alpha\|x\|^2 \leq \langle R^{-1}x\,;x\rangle$ for every $x \in \mathcal{H}$), and $O \prec Q \leq R \prec \frac{n+1}{n}R$. Note that $Q^{-1} - R^{-1} = Q^{-1} - (\frac{n+1}{n}R)^{-1} - \frac{1}{n+1}R^{-1}$ and $Q^{-1} - (\frac{n+1}{n}R)^{-1} \succ O$. Thus $\langle (Q^{-1}-R^{-1})x\,;x\rangle = \langle Q^{-1} - (\frac{n+1}{n}R)^{-1}x\,;x\rangle - \frac{1}{n+1}\langle R^{-1}x\,;x\rangle \geq -\frac{\alpha}{n+1}\|x\|^2$ for all $n \geq 1$, and so $\langle (Q^{-1}-R^{-1})x\,;x\rangle \geq 0$, for every $x \in \mathcal{H}$.

(c) If $O \prec Q < R$, then $Q^{-1} \succ O$, $R^{-1} \succ O$, and $R - Q > O$. Therefore, there is an $\alpha > 0$ such that $\alpha\|x\| \leq \|Q^{-1}x\|$ for every $x \in \mathcal{H}$, $R^{-1} \in \mathcal{G}[\mathcal{H}]$, and $\mathcal{N}(R-Q) = \{0\}$. Hence $0 < \alpha\|(R-Q)R^{-1}x\| \leq \|Q^{-1}(R-Q)R^{-1}x\| = \|(Q^{-1}-R^{-1})x\|$ for every nonzero vector x in $\mathcal{H}$, and so $\mathcal{N}(Q^{-1}-R^{-1}) = \{0\}$. Recall that $Q^{-1} - R^{-1} \geq O$ by item (b).

Problem 5.54. Show that the following equivalences hold for every T in $\mathcal{B}[\mathcal{H},\mathcal{K}]$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces (apply Corollary 5.83).

$$T^{*n}T^n \xrightarrow{s} O \iff T^{*n}T^n \xrightarrow{w} O \iff T^n \xrightarrow{s} O.$$

Now conclude that *for a self-adjoint operator the concepts of strong and weak stabilities coincide* (i.e., if $T^* = T$, then $T^n \xrightarrow{s} O \iff T^n \xrightarrow{w} O$).

Problem 5.55. Take $Q, T \in \mathcal{B}[\mathcal{H}]$, where $\mathcal{H}$ is a Hilbert space. Prove the following assertions.

(a) $-I \le T^* = T \le I$ if and only if $T^* = T$ and $\|T\| \le 1$.

Hint: Use Propositions 5.78 and 5.79 to show the "only if" part. On the other hand, use Proposition 5.79 and recall that $|\langle Tx\,;x\rangle| \le \|T\|\,\|x\|^2$.

(b) $O \le Q \le I \iff O \le Q$ and $\|Q\| \le 1 \iff Q^* = Q$ and $Q^2 \le Q$.

Hint: Equivalent characterizations for a nonnegative contraction.

Problem 5.56. Take $P, Q, T \in \mathcal{B}[\mathcal{H}]$ on a Hilbert space $\mathcal{H}$. Prove the results:

(a) If $T^* = T$ and $T^n \xrightarrow{w} P$, then P is an orthogonal projection.

Hint: Problems 5.24 and 5.44 and Proposition 5.81.

(b) If $O \le Q \le I$, then $Q^{n+1} \le Q^n$ for every integer $n \ge 0$.

Hint: Take $n \ge 1$ and $x \in \mathcal{H}$. If n is even, use Problem 5.55(b) and Proposition 5.82 to show that $\langle Q^n x\,;x\rangle = \|Q^{\frac{n}{2}}x\|^2 \le \|Q\|\langle QQ^{\frac{n-2}{2}}x\,;Q^{\frac{n-2}{2}}x\rangle \le \langle Q^{n-1}x\,;x\rangle$. If n is odd, then show that $\langle Q^n x\,;x\rangle = \langle QQ^{\frac{n-1}{2}}x\,;Q^{\frac{n-1}{2}}x\rangle \le \langle Q^{\frac{n-1}{2}}x\,;Q^{\frac{n-1}{2}}x\rangle = \langle Q^{n-1}x\,;x\rangle$.

(c) If $O \le Q \le I$, then $Q^n \xrightarrow{s} P$ and P is an orthogonal projection.

Hint: Problems 5.55(b), 4.47(a), 5.24, items (a,b), and Proposition 5.84.

Problem 5.57. This is our first problem that uses the square root of a nonnegative operator (Theorem 5.85). Take $T \in \mathcal{B}[\mathcal{H}]$ acting on a complex Hilbert space $\mathcal{H}$ and prove the following propositions.

(a) If $T \ne O$ is self-adjoint, then $U_\pm(T) = \|T\|^{-1}\big(T \pm i(\|T\|^2 I - T^2)^{\frac{1}{2}}\big)$ are unitary operators in $\mathcal{B}[\mathcal{H}]$.

Hint: $\|T\|^{-2}T^2 \le I$ so that $O \le \|T\|^2 I - T^2$ (cf. Problems 5.45 and 5.55). See Proposition 5.73.

(b) Every operator on a complex Hilbert space is a linear combination of four unitary operators.

Hint: If $O \ne T = T^*$, then show that $T = \frac{\|T\|}{2}U_+(T) + \frac{\|T\|}{2}U_-(T)$. Apply the Cartesian decomposition (Problem 5.46) if $O \ne T \ne T^*$.

Problem 5.58. If $Q \in \mathcal{B}^+[\mathcal{H}]$, where $\mathcal{H}$ is a Hilbert space, then show that (cf. Theorem 5.85 and Proposition 5.86)

(a) $\langle Qx\,;x\rangle = \|Q^{\frac{1}{2}}x\|^2 \le \|Q\|^{\frac{1}{2}}\langle Q^{\frac{1}{2}}x\,;x\rangle$ for every $x \in \mathcal{H}$,

(b) $\langle Q^{\frac{1}{2}}x\,;x\rangle \le \langle Qx\,;x\rangle^{\frac{1}{2}}\|x\|$ for every $x \in \mathcal{H}$,

(c) $Q^{\frac{1}{2}} > 0$ if and only if $Q > O$,

(d) $Q^{\frac{1}{2}} \succ 0$ if and only if $Q \succ O$.

Problem 5.59. Take $Q, R \in \mathcal{B}^+[\mathcal{H}]$ on a Hilbert space $\mathcal{H}$. Prove the following two assertions.

(a) If $Q \le R$ and $QR = RQ$, then $Q^2 \le R^2$.

(b) $Q \le R$ does not imply $Q^2 \le R^2$.

Hints: (a) $\langle RQ^{\frac{1}{2}}x\,;Q^{\frac{1}{2}}x\rangle = \langle QR^{\frac{1}{2}}x\,;R^{\frac{1}{2}}x\rangle$. (b) $Q = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$.

Remark: Applying the Spectral Theorem of Section 6.8 and the square root of Theorem 5.85, it can be shown that

$$Q^2 \le R^2 \quad \text{implies} \quad Q \le R \qquad \text{and so} \qquad Q \le R \quad \text{implies} \quad Q^{\frac{1}{2}} \le R^{\frac{1}{2}}.$$

Problem 5.60. Let Q and R be nonnegative operators acting on a Hilbert space. Use Problem 5.52 and Theorem 5.85 to prove that

$$QR = RQ \quad \text{implies} \quad Q^nR^m \ge O \text{ for every } m, n \ge 1.$$

Show that $p(Q)q(R) \ge O$ for every pair of polynomials p and q with positive coefficients whenever $Q \ge O$ and $R \ge O$ commute.

Problem 5.61. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Take any T in $\mathcal{B}[\mathcal{H}, \mathcal{K}]$ and recall that T^*T lies in $\mathcal{B}^+[\mathcal{H}]$. Set

$$|T| \,=\, (T^*T)^{\frac{1}{2}}$$

in $\mathcal{B}^+[\mathcal{H}]$ so that $|T|^2 = T^*T$. Prove the following assertions.

(a) $\|T\| = \||T|^2\|^{\frac{1}{2}} = \||T|\| = \||T|^{\frac{1}{2}}\|^2$.

(b) $\langle |T|x\,;x\rangle = \||T|^{\frac{1}{2}}x\|^2 \le \||T|x\|\|x\|$ for every $x \in \mathcal{H}$.

(c) $\|Tx\|^2 = \||T|x\|^2 \le \|T\|\langle |T|x\,;x\rangle$ for every $x \in \mathcal{H}$.

Moreover, if $\mathcal{H} = \mathcal{K}$ (i.e., if $T \in \mathcal{B}[\mathcal{H}]$), then show that

(d) $T^n \xrightarrow{s} O \iff |\,T^n| \xrightarrow{s} O \iff |T^n| \xrightarrow{w} O$,

(e) $\mathcal{B}^+[\mathcal{H}] = \{T \in \mathcal{B}[\mathcal{H}]\colon\ T = |T|\}$ (i.e., $T \ge O$ if and only if $T = |T|$).

Problem 5.62. Let Q be a nonnegative operator on a Hilbert space.

$$Q \text{ is compact if and only if } Q^{\frac{1}{2}} \text{ is compact.}$$

Hint: If $Q^{\frac{1}{2}}$ is compact, then Q is compact by Proposition 4.54. On the other hand, $\|Q^{\frac{1}{2}}x_n\|^2 = \langle Qx_n\,;x_n\rangle \le \sup_k\|x_k\|\,\|Qx_n\|$ (Problem 5.41).

Take $T \in \mathcal{B}[\mathcal{H},\mathcal{K}]$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. Also prove that

$$T \in \mathcal{B}_\infty[\mathcal{H},\mathcal{K}] \iff T^*T \in \mathcal{B}_\infty[\mathcal{H}] \iff |T| \in \mathcal{B}_\infty[\mathcal{H}] \iff |T|^{\frac{1}{2}} \in \mathcal{B}_\infty[\mathcal{H}].$$

Problem 5.63. Consider a sequence $\{Q_n\}$ of nonnegative operators on a Hilbert space $\mathcal{H}$ (i.e., $Q_n \ge O$ for every n). Prove the following propositions.

(a) $Q_n \xrightarrow{s} Q$ implies $Q_n^{1/2} \xrightarrow{s} Q^{1/2}$.

(b) If Q_n is compact for every n and $Q_n \xrightarrow{u} Q$, then $Q_n^{1/2} \xrightarrow{u} Q^{1/2}$.

Hints: $Q \ge O$ by Problem 5.49 and Propositions 5.68 and 4.48.

(a) Recall that $Q^{1/2}$ is the strong limit of a sequence $\{p_k(Q)\}$ of polynomials in Q, where the polynomials $\{p_k\}$ themselves do not depend on Q; that is, $p_k(Q) \xrightarrow{s} Q^{1/2}$ for every $Q \ge O$ (cf. proof of Theorem 5.85). First verify that $\|(Q_n^{1/2} - Q^{1/2})x\| \le \|(Q_n^{1/2} - p_k(Q_n))x\| + \|(p_k(Q_n) - p_k(Q))x\| + \|(p_k(Q) - Q^{1/2})x\|$. Now take an arbitrary $\varepsilon \ge 0$ and any $x \in \mathcal{H}$. Show that there are positive integers n_ε and k_ε such that $\|(p_{k_\varepsilon}(Q) - Q^{1/2})x\| < \varepsilon$, $\|(p_{k_\varepsilon}(Q_n) - p_{k_\varepsilon}(Q))x\| < \varepsilon$ for every $n \ge n_\varepsilon$ (since $Q_n^j \xrightarrow{s} Q^j$ for every positive integer j by Problem 4.46), and $\|(Q_{n_\varepsilon}^{1/2} - p_{k_\varepsilon}(Q_{n_\varepsilon}))x\| < \varepsilon$.

(b) Note that $Q \in \mathcal{B}_\infty[\mathcal{H}]$ by Theorem 4.53. Since $Q_n^{1/2} \xrightarrow{s} Q^{1/2}$ by part (a), we get $Q_n^{1/2}Q^{1/2} \xrightarrow{u} Q$ (Problems 5.62 and 4.57). Hence $(Q_n^{1/2} - Q^{1/2})^2 = Q_n + Q - Q_n^{1/2}Q^{1/2} - (Q_n^{1/2}Q^{1/2})^* \xrightarrow{u} O$ (Problem 5.26). But $Q_n^{1/2} - Q^{1/2}$ is self-adjoint so that $\|Q_n^{1/2} - Q^{1/2}\|^2 = \|(Q_n^{1/2} - Q^{1/2})^2\|$ (Problem 5.45).

Problem 5.64. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ and $\{f_\gamma\}_{\gamma\in\Gamma}$ be orthonormal bases for a Hilbert space $\mathcal{H}$. Take any operator $T \in \mathcal{B}[\mathcal{H}]$. Use the Parseval identity to show that

$$\sum_{\gamma\in\Gamma}\|Te_\gamma\|^2 = \sum_{\gamma\in\Gamma}\|T^*f_\gamma\|^2 = \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma}|\langle Te_\alpha\,;f_\beta\rangle|^2$$

whenever the family of nonnegative numbers $\{\|Te_\gamma\|^2\}_{\gamma\in\Gamma}$ is summable; that is, whenever $\sum_{\gamma\in\Gamma}\|Te_\gamma\|^2 < \infty$ (cf. Proposition 5.31). Apply the above result to the operator $|T|^{\frac{1}{2}} \in \mathcal{B}^+[\mathcal{H}]$ (cf. Problem 5.61) and show that

$$\sum_{\gamma\in\Gamma}\langle|T|e_\gamma\,;e_\gamma\rangle = \sum_{\gamma\in\Gamma}\langle|T|f_\gamma\,;f_\gamma\rangle$$

whenever $\sum_{\gamma\in\Gamma}\langle|T|e_\gamma\,;e_\gamma\rangle < \infty$. Outcome: If the sum $\sum_{\gamma\in\Gamma}\langle|T|e_\gamma\,;e_\gamma\rangle$ exists in $\mathbb{R}$ (i.e., if $\{\langle|T|e_\gamma\,;e_\gamma\rangle\}_{\gamma\in\Gamma}$ is summable), then it is independent of the choice

of the orthonormal basis $\{e_\gamma\}_{\gamma\in\Gamma}$ for $\mathcal{H}$. An operator $T\in\mathcal{B}[\mathcal{H}]$ is *trace-class* (or *nuclear*) if $\sum_{\gamma\in\Gamma}\langle|T|e_\gamma\,;e_\gamma\rangle<\infty$ (equivalently, if $\sum_{\gamma\in\Gamma}\||T|^{\frac{1}{2}}e_\gamma\|^2<\infty$) for some orthonormal basis $\{e_\gamma\}_{\gamma\in\Gamma}$ for $\mathcal{H}$. Let $\mathcal{B}_1[\mathcal{H}]$ denote the subset of $\mathcal{B}[\mathcal{H}]$ consisting of all trace-class operators on $\mathcal{H}$. If $T\in\mathcal{B}_1[\mathcal{H}]$, then set

$$\|T\|_1 \,=\, \sum_{\gamma\in\Gamma}\langle|T|e_\gamma\,;e_\gamma\rangle \,=\, \sum_{\gamma\in\Gamma}\||T|^{\frac{1}{2}}e_\gamma\|^2.$$

Problem 5.65. Let $T\in\mathcal{B}[\mathcal{H}]$ be an operator on a Hilbert space $\mathcal{H}$, and let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal basis for $\mathcal{H}$. If the operator $|T|^2$ is trace-class (as defined in Problem 5.64; that is, if $\sum_{\gamma\in\Gamma}\||T|e_\gamma\|^2<\infty$ or, equivalently, if $\sum_{\gamma\in\Gamma}\|Te_\gamma\|^2<\infty$ — Problem 5.61(c)), then T is a *Hilbert–Schmidt* operator. Let $\mathcal{B}_2[\mathcal{H}]$ denote the subset of $\mathcal{B}[\mathcal{H}]$ made up of all Hilbert–Schmidt operators on $\mathcal{H}$. Take $T\in\mathcal{B}_2[\mathcal{H}]$. According to Problems 5.61 and 5.64, set

$$\|T\|_2 \,=\, \|T^*T\|_1^{\frac{1}{2}} \,=\, \||T|^2\|_1^{\frac{1}{2}} \,=\, \Big(\sum_{\gamma\in\Gamma}\||T|e_\gamma\|^2\Big)^{\frac{1}{2}} \,=\, \Big(\sum_{\gamma\in\Gamma}\|Te_\gamma\|^2\Big)^{\frac{1}{2}}$$

for any orthonormal basis $\{e_\gamma\}_{\gamma\in\Gamma}$ for $\mathcal{H}$. Prove the following results.

(a) $T\in\mathcal{B}_2[\mathcal{H}] \iff |T|\in\mathcal{B}_2[\mathcal{H}] \iff |T|^2\in\mathcal{B}_1[\mathcal{H}]$. In this case,

$$\|T\|_2^2 \,=\, \||T|\|_2^2 \,=\, \||T|^2\|_1.$$

(b) $T\in\mathcal{B}_1[\mathcal{H}] \iff |T|\in\mathcal{B}_1[\mathcal{H}] \iff |T|^{\frac{1}{2}}\in\mathcal{B}_2[\mathcal{H}]$. In this case,

$$\|T\|_1 \,=\, \||T|\|_1 \,=\, \||T|^{\frac{1}{2}}\|_2^2.$$

(c) If $T\in\mathcal{B}_2[\mathcal{H}]$, then $T^*\in\mathcal{B}_2[\mathcal{H}]$ and $\|T^*\|_2=\|T\|_2$. (*Hint*: Problem 5.64.)

(d) $\|T\|\le\|T\|_2$ for every $T\in\mathcal{B}_2[\mathcal{H}]$. (*Hint*: $\|Te\|\le\|T\|_2$ if $\|e\|=1$.)

(e) If $T,S\in\mathcal{B}_2[\mathcal{H}]$, then $T+S\in\mathcal{B}_2[\mathcal{H}]$ and $\|T+S\|_2\le\|T\|_2+\|S\|_2$.

Hint: Since $\sum_{\gamma\in\Gamma}\|Te_\gamma\|\|Se_\gamma\| \le \big(\sum_{\gamma\in\Gamma}\|Te_\gamma\|^2\big)^{\frac{1}{2}}\big(\sum_{\gamma\in\Gamma}\|Se_\gamma\|^2\big)^{\frac{1}{2}} = \|T\|_2\|S\|_2$ (Schwarz inequality in ℓ^2_Γ), we get $\|T+S\|_2^2\le(\|T\|_2+\|S\|_2)^2$.

(f) $\mathcal{B}_2[\mathcal{H}]$ is a linear space and $\|\ \|_2$ is a norm on $\mathcal{B}_2[\mathcal{H}]$.

(g) ST and TS lie in $\mathcal{B}_2[\mathcal{H}]$ and $\max\{\|ST\|_2,\|TS\|_2\}\le\|S\|\,\|T\|_2$ for every S in $\mathcal{B}[\mathcal{H}]$ and every T in $\mathcal{B}_2[\mathcal{H}]$.

Hint: $\|STe_\gamma\|^2\le\|S\|^2\|Te_\gamma\|^2$ and $\|(TS)^*e_\gamma\|^2\le\|S\|^2\|T^*e_\gamma\|^2$.

(h) $\mathcal{B}_2[\mathcal{H}]$ is a two-sided ideal of $\mathcal{B}[\mathcal{H}]$.

Problem 5.66. Consider the setup of the previous problem and prove the following assertions.

(a) If $T,S\in\mathcal{B}_1[\mathcal{H}]$, then $T+S\in\mathcal{B}_1[\mathcal{H}]$ and $\|T+S\|_1\le\|T\|_1+\|S\|_1$.

Hint: Polar decompositions: $T+S = W|T+S|,\ T = W_1|T|$, and $S = W_2|S|$. Thus $|T+S| = W^*(T+S),\ |T| = W_1^*T$, and $|S| = W_2^*T$. Verify:

$$\begin{aligned}
&\sum_{\gamma\in\Gamma}\langle|T+S|e_\gamma; e_\gamma\rangle \le \sum_{\gamma\in\Gamma}|\langle Te_\gamma; W^*e_\gamma\rangle| + \sum_{\gamma\in\Gamma}|\langle Se_\gamma; W^*e_\gamma\rangle| \\
&= \sum_{\gamma\in\Gamma}|\langle|T|^{\frac12}e_\gamma\,; |T|^{\frac12}W_1^*We_\gamma\rangle| + \sum_{\gamma\in\Gamma}|\langle|S|^{\frac12}e_\gamma\,; |S|^{\frac12}W_2^*We_\gamma\rangle| \\
&\le \Big(\sum_{\gamma\in\Gamma}\||T|^{\frac12}e_\gamma\|^2\Big)^{\frac12}\Big(\sum_{\gamma\in\Gamma}\||T|^{\frac12}W_1^*We_\gamma\|^2\Big)^{\frac12} \\
&+ \Big(\sum_{\gamma\in\Gamma}\||S|^{\frac12}e_\gamma\|^2\Big)^{\frac12}\Big(\sum\nolimits_{\gamma\in\Gamma}\||S|^{\frac12}W_2^*We_\gamma\|^2\Big)^{\frac12} \\
&\le \||T|^{\frac12}\|_2^2\|W_1^*W\| + \||S|^{\frac12}\|_2^2\|W_2^*W\| \le \|T\|_1 + \|S\|_1.
\end{aligned}$$

(Problem 5.65(b,g); recall that $\|W\| = \|W_1\| = \|W_2\| = 1$.)

(b) $\mathcal{B}_1[\mathcal{H}]$ is a linear space and $\|\ \|_1$ is a norm on $\mathcal{B}_1[\mathcal{H}]$.

(c) $\mathcal{B}_1[\mathcal{H}] \subseteq \mathcal{B}_2[\mathcal{H}]$ (i.e., every trace-class operator is Hilbert–Schmidt). If $T \in \mathcal{B}_1[\mathcal{H}]$, then $\|T\|_2 \le \|T\|_1$.

Hint: Problem 5.65(a,b,g) to prove the inclusion, and Problems 5.61(c) and 5.65(b) to prove the inequality.

(d) $\mathcal{B}_2[\mathcal{H}] \subseteq \mathcal{B}_\infty[\mathcal{H}]$ (i.e., every Hilbert–Schmidt operator is compact).

Hint: Take $T \in \mathcal{B}_2[\mathcal{H}]$ so that $T^* \in \mathcal{B}_2[\mathcal{H}]$ (Proposition 5.65(c)), and hence $\sum_{\gamma\in\Gamma}\|T^*e_\gamma\|^2 < \infty$. Take an arbitrary integer $n \ge 1$. There exists a finite $N_n \subseteq \Gamma$ such that $\sum_{k\in N}\|T^*e_k\|^2 < \frac{1}{n}$ for all finite $N \subseteq \Gamma\backslash N_n$ (Theorem 5.27). Thus $\sum_{\gamma\in\Gamma\backslash N_n}\|T^*e_\gamma\|^2 < \frac{1}{n}$. Recall that $Tx = \sum_{\gamma\in\Gamma}\langle Tx\,; e_\gamma\rangle e_\gamma$ (Theorem 5.48) and define $T_n\colon \mathcal{H} \to \mathcal{H}$ by $T_nx = \sum_{k\in N_n}\langle Tx\,; e_k\rangle e_k$. Show that $\|(T-T_n)x\|^2 = \sum_{\gamma\in\Gamma\backslash N_n}|\langle Tx; e_\gamma\rangle|^2 \le \sum_{\gamma\in\Gamma\backslash N_n}\|T^*e_\gamma\|^2\|x\|^2$ and $T_n \in \mathcal{B}_0[\mathcal{H}]$. Thus $\|T_n - T\| \to 0$, and hence $T \in \mathcal{B}_\infty[\mathcal{H}]$ (Problem 5.42).

(e) $T \in \mathcal{B}_1[\mathcal{H}]$ if and only if $T = AB$ for some $A, B \in \mathcal{B}_2[\mathcal{H}]$.

Hint: Let $T = W|T| = W|T|^{\frac12}|T|^{\frac12}$ be the polar decomposition of T. If $T \in \mathcal{B}_1[\mathcal{H}]$, then use Problem 5.65(b,g). Conversely, suppose $T = AB$ with $A, B \in \mathcal{B}_2[\mathcal{H}]$. Since $|T| = W^*T$, we get $|T| = W^*AB$ with $A^*W \in \mathcal{B}_2[\mathcal{H}]$ (Problem 5.65(c,g)). Verify: $\sum_{\gamma\in\Gamma}\langle|T|e_\gamma\,; e_\gamma\rangle \le \sum_{\gamma\in\Gamma}\|Be_\gamma\|\|A^*We_\gamma\| \le \big(\sum_{\gamma\in\Gamma}\|Be_\gamma\|^2\big)^{\frac12}\big(\sum_{\gamma\in\Gamma}\|A^*We_\gamma\|^2\big)^{\frac12}$. Hence $\|T\|_1 \le \|B\|_2\|A^*W\|_2$.

(f) ST and TS lie in $\mathcal{B}_1[\mathcal{H}]$ for every T in $\mathcal{B}_1[\mathcal{H}]$ and every S in $\mathcal{B}[\mathcal{H}]$.

Hint: Apply (e). $T = AB$ for some $A, B \in \mathcal{B}_2[\mathcal{H}]$. SA and BS lie in $\mathcal{B}_2[\mathcal{H}]$, and so $ST = SAB$ and $TS = ABS$ lie in $\mathcal{B}_1[\mathcal{H}]$.

(g) $\mathcal{B}_1[\mathcal{H}]$ is a two-sided ideal of $\mathcal{B}[\mathcal{H}]$.

Problem 5.67. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary orthonormal basis for a Hilbert space $\mathcal{H}$. If $T \in \mathcal{B}_1[\mathcal{H}]$ (i.e., if T is a trace-class operator), then show that

$$\sum_{\gamma\in\Gamma} |\langle Te_\gamma\,;e_\gamma\rangle| < \infty.$$

Hint: $2|\langle Te_\gamma\,;e_\gamma\rangle| = 2|\langle ABe_\gamma\,;e_\gamma\rangle| \le 2\|Be_\gamma\|\,\|A^*e_\gamma\|$ for $A, B \in \mathcal{B}_2[\mathcal{H}]$ (Problem 5.66(e)). So $2|\langle Te_\gamma\,;e_\gamma\rangle| \le \|Be_\gamma\|^2 + \|A^*e_\gamma\|^2$. Then $\sum_{\gamma\in\Gamma}|\langle Te_\gamma\,;e_\gamma\rangle| \le \frac{1}{2}(\|A\|_2^2 + \|B\|_2^2)$ (Problem 5.65(c)).

Thus, by Corollary 5.29, $\{\langle Te_\gamma\,;e_\gamma\rangle\}_{\gamma\in\Gamma}$ is a summable family of scalars (since $\mathbb{F}$ is a Banach space). Let $\sum_{\gamma\in\Gamma}\langle Te_\gamma\,;e_\gamma\rangle$ in $\mathbb{F}$ be its sum and show that

$$\textstyle\sum_{\gamma\in\Gamma}\langle Te_\gamma\,;e_\gamma\rangle \text{ does not depend on } \{e_\gamma\}_{\gamma\in\Gamma}.$$

Hint: $\sum_{\alpha\in\Gamma}\langle Te_\alpha\,;e_\alpha\rangle = \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma}\langle Te_\alpha\,;f_\beta\rangle\langle f_\beta\,;e_\alpha\rangle$, where $\{e_\gamma\}_{\gamma\in\Gamma}$ and $\{f_\gamma\}_{\gamma\in\Gamma}$ are any orthonormal bases for $\mathcal{H}$ (Theorem 5.48(c)). Now observe that $\sum_{\beta\in\Gamma}\sum_{\alpha\in\Gamma}\langle Te_\alpha\,;f_\beta\rangle\langle f_\beta\,;e_\alpha\rangle = \sum_{\beta\in\Gamma}\langle f_\beta\,;T^*f_\beta\rangle = \sum_{\beta\in\Gamma}\langle Tf_\beta\,;f_\beta\rangle$.

If $T \in \mathcal{B}_1[\mathcal{H}]$ and $\{e_\gamma\}_{\gamma\Gamma}$ is any orthonormal basis for $\mathcal{H}$, then set

$$\mathrm{tr}(T) = \sum_{\gamma\in\Gamma}\langle Te_\gamma\,;e_\gamma\rangle \quad \text{ so that } \quad \|T\|_1 = \mathrm{tr}(|T|).$$

Hence $\mathcal{B}_1[\mathcal{H}] = \{T \in \mathcal{B}[\mathcal{H}]\colon\ \mathrm{tr}(|T|) < \infty\}$. The number $\mathrm{tr}(T)$ is called the *trace* of $T \in \mathcal{B}_1[\mathcal{H}]$ (thus the terminology "trace-class"). *Warning*: If T lies in $\mathcal{B}[\mathcal{H}]$ and $\sum_{\gamma\in\Gamma}\langle Te_\gamma\,;e_\gamma\rangle < \infty$ for some orthonormal basis $\{e_\gamma\}_{\gamma\in\Gamma}$ for $\mathcal{H}$, then it does not follow that $T \in \mathcal{B}_1[\mathcal{H}]$. However, if $\sum_{\gamma\in\Gamma}\langle|T|e_\gamma\,;e_\gamma\rangle < \infty$ for some orthonormal basis $\{e_\gamma\}_{\gamma\in\Gamma}$ for $\mathcal{H}$, then $T \in \mathcal{B}_1[\mathcal{H}]$ (Problem 5.64).

Problem 5.68. Consider the setup of the previous problem and prove the following assertions.

(a) $\mathrm{tr}\colon \mathcal{B}_1[\mathcal{H}] \to \mathbb{F}$ is a linear functional.

(b) $|\mathrm{tr}(T)| \le \|T\|_1$ for every $T \in \mathcal{B}_1[\mathcal{H}]$ (i.e., $\mathrm{tr}\colon (\mathcal{B}_1[\mathcal{H}], \|\ \|_1) \to \mathbb{F}$ is a contraction, and hence a bounded linear functional).

Hint: Let $T = W|T|$ be the polar decomposition of T. Recall that $\|W\| = 1$. If T is trace-class, then verify that $|\mathrm{tr}(T)| \le \sum_{\gamma\in\Gamma}|\langle|T|^{\frac{1}{2}}e_\gamma\,;|T|^{\frac{1}{2}}W^*e_\gamma\rangle| \le \big(\sum_{\gamma\in\Gamma}\||T|^{\frac{1}{2}}e_\gamma\|^2\big)^{\frac{1}{2}}\big(\sum_{\gamma\in\Gamma}\||T|^{\frac{1}{2}}W^*e_\gamma\|^2\big)^{\frac{1}{2}} \le \|T\|_1$ (Problem 5.65).

(c) $\mathrm{tr}(T^*) = \overline{\mathrm{tr}(T)}$ for every $T \in \mathcal{B}_1[\mathcal{H}]$.

(d) $\mathrm{tr}(TS) = \mathrm{tr}(ST)$ whenever $T \in \mathcal{B}_1[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{H}]$.

Hint: $\overline{\mathrm{tr}(TS)} = \sum_{\alpha\in\Gamma}\langle T^*e_\alpha\,;Se_\alpha\rangle = \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma}\langle T^*e_\alpha\,;f_\beta\rangle\langle f_\beta\,;Se_\alpha\rangle$ and $\overline{\mathrm{tr}(TS)} = \sum_{\beta\in\Gamma}\langle S^*f_\beta\,;Tf_\beta\rangle = \sum_{\beta\in\Gamma}\sum_{\alpha\in\Gamma}\langle S^*f_\beta\,;e_\alpha\rangle\langle e_\alpha\,;Tf_\beta\rangle$ (cf. Problem 5.66(f), item (c), and Theorem 5.48(c)).

(e) $|\mathrm{tr}(S|T|)| = |\mathrm{tr}(|T|S)| \le \|S\|\,\|T\|_1$ if $T \in \mathcal{B}_1[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{H}]$.

Hint: Use Problems 5.65(b,g) and 5.66(f), and verify that (see item (d)) $\big|\sum_{\gamma\in\Gamma}\langle S|T|e_\gamma\,;e_\gamma\rangle\big| \le \sum_{\gamma\in\Gamma}|\langle |T|^{\frac12}e_\gamma\,;|T|^{\frac12}S^*e_\gamma\rangle| \le \||T|^{\frac12}\|_2\||T|^{\frac12}S^*\|_2$.

(f) $T^*\in\mathcal{B}_1[\mathcal{H}]$ and $\|T^*\|_1=\|T\|_1$ for every $T\in\mathcal{B}_1[\mathcal{H}]$.

Hint: Let $T=W_1|T|$ and $T^*=W_2|T^*|$ be the polar decompositions of T and T^*. Since $|T^*|=W_2^*T^*=W_2^*|T|W_1^*$, T^* lies in $\mathcal{B}_1[\mathcal{H}]$ (by Problems 5.65(b) and 5.66(f)). Now show that $\|T^*\|_1=\mathrm{tr}(|T^*|)=\mathrm{tr}(W_2^*|T|W_1^*)\le \|W_1^*W_2^*\|\,\|T\|_1$ (Problem 5.65(b) and items (d) and (e)). But $\|W_1^*W_2^*\|\le \|W_1\|\,\|W_2\|=1$. Therefore, $\|T^*\|_1\le\|T\|_1$. Dually, $\|T\|_1\le\|T^*\|_1$.

(g) $\max\{\|ST\|_1,\|TS\|_1\}\le\|S\|\,\|T\|_1$ whenever $T\in\mathcal{B}_1[\mathcal{H}]$ and $S\in\mathcal{B}[\mathcal{H}]$.

Hint: Let $T=W|T|$, $ST=W_1|ST|$, and $TS=W_2|TS|$ be the polar decompositions of T, ST, and TS, respectively, and verify that $\|ST\|_1=\mathrm{tr}(|ST|)=\mathrm{tr}(W_1^*SW|T|)$ and $\|TS\|_1=\mathrm{tr}(|TS|)=\mathrm{tr}(W_2^*W|T|S)$. Use items (d) and (e) and recall that $\|W\|=\|W_1\|=\|W_2\|=1$.

(h) $\mathcal{B}_0[\mathcal{H}]\subseteq\mathcal{B}_1[\mathcal{H}]$ (i.e., every finite-rank operator is trace-class).

Hint: If $\dim\mathcal{R}(T)$ is finite, then $\dim\mathcal{N}(T^*)^\perp$ is finite (Proposition 5.76). Let $\{f_\alpha\}$ be an orthonormal basis for $N(T^*)$ and let $\{g_k\}$ be a *finite* orthonormal basis for $\mathcal{N}(T^*)^\perp$. Since $\mathcal{H}=\mathcal{N}(T^*)+\mathcal{N}(T^*)^\perp$ (Theorem 5.20), $\{e_\gamma\}=\{f_\alpha\}\cup\{g_k\}$ is an orthonormal basis for $\mathcal{H}$ (Problem 5.11). Now, either $T^*e_\gamma=0$ or $T^*e_\gamma=T^*g_k$. Show that $\sum_\gamma\langle|T^*|e_\gamma;e_\gamma\rangle=\sum_k\langle|T^*|g_k;g_k\rangle<\infty$ (e.g., see Problem 5.61(c)). Thus $T^*\in\mathcal{B}_1[\mathcal{H}]$ (Problem 5.64), and hence $T\in\mathcal{B}_1[\mathcal{H}]$ by item (f).

Problem 5.69. Let $(\mathcal{B}_1[\mathcal{H}],\|\ \|_1)$ and $(\mathcal{B}_2[\mathcal{H}],\|\ \|_2)$ be the normed spaces of Problems 5.65(f) and 5.66(b). Show that

(a) $(\mathcal{B}_1[\mathcal{H}],\|\ \|_1)$ is a Banach space.

Hint: Take a $\mathcal{B}_1[\mathcal{H}]$-valued sequence $\{T_n\}$. If $\{T_n\}$ is a Cauchy sequence in $(\mathcal{B}_1[\mathcal{H}],\|\ \|_1)$, then it is a Cauchy sequence in the Banach space $(\mathcal{B}[\mathcal{H}],\|\ \|)$ (Problems 5.65(d) and 5.66(c)), and so $T_n\xrightarrow{u}T$ for some $T\in\mathcal{B}[\mathcal{H}]$. Use Problems 5.26 and 4.46 to verify that $|T_n|^2\xrightarrow{u}|T|^2$, and so $|T_n|^{\frac12}\xrightarrow{u}|T|^{\frac12}$ (Problems 5.63(b) and 5.66(c,d)). Therefore, show that $\sum_{\gamma\in\Gamma}\||T|^{\frac12}e_\gamma\|^2\le\limsup_n\sum_{\gamma\in\Gamma}\||T_n|^{\frac12}e_\gamma\|^2\le\sup_n\|T_n\|_1<\infty$ (recall that $\{T_n\}$ is Cauchy in $(\mathcal{B}_1[\mathcal{H}],\|\ \|_1)$). Thus $T\in\mathcal{B}_1[\mathcal{H}]$. Since $T_n-T\xrightarrow{u}O$, $|T_n-T|^{\frac12}\xrightarrow{u}O$ (Problem 5.61(a)). Observe that $\|T_n-T\|_1=\sum_{\gamma\in\Gamma}\||T_n-T|^{\frac12}e_\gamma\|^2=\sum_{k\in N_\varepsilon}\||T_n-T|^{\frac12}e_k\|^2+\sup_N\sum_{k\in N}\||T_n-T|^{\frac12}e_k\|^2<\infty$ for every finite set $N_\varepsilon\subseteq\Gamma$, where the supremum is taken over all finite sets $N\subseteq\Gamma\backslash N_\varepsilon$ (Proposition 5.31). Use Theorem 5.27 to conclude that $\|T_n-T\|_1\to0$.

Consider the function $\langle\ ;\ \rangle\colon\mathcal{B}_2[\mathcal{H}]\times\mathcal{B}_2[\mathcal{H}]\to\mathbb{F}$ given by

$$\langle T\,;S\rangle\ =\ \mathrm{tr}(S^*T)$$

for every $S, T \in \mathcal{B}_2[\mathcal{H}]$. Show that $\langle\ ;\ \rangle$ is an inner product on $\mathcal{B}_2[\mathcal{H}]$ that induces the norm $\|\ \|_2$. (*Hint*: Problem 5.68(a,c).) Moreover,

(b) $(\mathcal{B}_2[\mathcal{H}], \langle\ ;\ \rangle)$ is a Hilbert space.

Recall that $\mathcal{B}_0[\mathcal{H}] \subseteq \mathcal{B}_1[\mathcal{H}] \subseteq \mathcal{B}_2[\mathcal{H}] \subseteq \mathcal{B}_\infty[\mathcal{H}]$ and that $\mathcal{B}_0[\mathcal{H}]$ is dense in the Banach space $(\mathcal{B}_\infty[\mathcal{H}], \|\ \|)$. Now show that

(c) $\mathcal{B}_0[\mathcal{H}]$ is dense in $(\mathcal{B}_1[\mathcal{H}], \|\ \|_1)$ and in $(\mathcal{B}_2[\mathcal{H}], \|\ \|_2)$.

Problem 5.70. Two normed spaces $\mathcal{X}$ and $\mathcal{Y}$ are topologically isomorphic if there exists a topological isomorphism between them (i.e., if there exists W in $\mathcal{G}[\mathcal{X}, \mathcal{Y}]$ — see Section 4.6). Two inner product spaces $\mathcal{X}$ and $\mathcal{Y}$ are unitarily equivalent if there exists a unitary transformation between them (i.e., if there exists a unitary U in $\mathcal{G}[\mathcal{X}, \mathcal{Y}]$ — see Section 5.6). *Two Hilbert spaces are topologically isomorphic if and only if they are unitarily equivalent*. That is, if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then

$$\mathcal{G}[\mathcal{H}, \mathcal{K}] \neq \varnothing \quad \text{if and only if} \quad \big\{U \in \mathcal{G}[\mathcal{H}, \mathcal{K}]\colon\ U \text{ is unitary}\big\} \neq \varnothing.$$

Hint: If $W \in \mathcal{G}[\mathcal{H}, \mathcal{K}]$, then $|W| = (W^*W)^{\frac{1}{2}} \in \mathcal{G}^+[\mathcal{H}]$ (Problems 5.50(b) and 5.58(d)). Show that $U = W|W|^{-1} \in \mathcal{G}[\mathcal{H}, \mathcal{K}]$ is unitary (Proposition 5.73) and that $U|W|$ is the polar decomposition of W (Corollary 5.90).

Problem 5.71. Let $\{T_k\}$ and $\{S_k\}$ be (equally indexed) countable collections of operators acting on Hilbert spaces $\mathcal{H}_k$ (i.e., $T_k, S_k \in \mathcal{B}[\mathcal{H}_k]$ for each k). Consider the direct sum operators $\bigoplus_k T_k \in \mathcal{B}[\bigoplus_k \mathcal{H}_k]$ and $\bigoplus_k S_k \in \mathcal{B}[\bigoplus_k \mathcal{H}_k]$ acting on the (orthogonal) direct sum space $\bigoplus_k \mathcal{H}_k$, which is a Hilbert space (as in Examples 5.F and 5.G, and Problems 4.16 and 5.28). Verify that

(a) $\mathcal{R}\big(\bigoplus_k T_k\big) = \bigoplus_k \mathcal{R}(T_k)$ and $\mathcal{N}\big(\bigoplus_k T_k\big) = \bigoplus_k \mathcal{N}(T_k)$.

(b) $\big(\bigoplus_k T_k\big)^* = \bigoplus_k T_k^*$,

(c) $p\big(\bigoplus_k T_k\big) = \bigoplus_k p(T_k)$ for every polynomial p,

(d) $\big(\bigoplus_k T_k\big) + \big(\bigoplus_k S_k\big) = \bigoplus_k (T_k + S_k)$,

(e) $\big(\bigoplus_k T_k\big)\big(\bigoplus_k S_k\big) = \bigoplus_k T_k S_k$.

Problem 5.72. Consider the setup of the previous problem. Show that

(a) $\big|\bigoplus_k T_k\big| = \bigoplus_k |T_k|$.

Now suppose the countable collections are *finite* and show that

(b) $\bigoplus_{k=1}^n T_k$ is compact, trace-class, or Hilbert Schmidt if and only if every T_k is compact, trace-class, or Hilbert Schmidt, respectively.

Hint: For the compact case use Theorem 4.52 and recall that the restriction of a compact to a linear manifold is again compact (Section 4.9). For the trace-class and Hilbert Schmidt cases use item (a) and Problem 5.11.

(c) The "if" part of (b) fails for infinite collections (but not the "only if" part).

Hint: Set $T_k = 1$ on $\mathcal{H}_k = \mathbb{C}$ (see Example 4.N).

Problem 5.73. Consider the setup of Problem 5.71.

(a) Show that a countable direct sum $\bigoplus_k T_k$ is an involution, an orthogonal projection, nonnegative, positive, self-adjoint, an isometry, a unitary operator, or a contraction, if and only if each T_k is.

(b) Show that if each T_k is invertible (or strictly positive), then every finite direct sum $\bigoplus_{k=1}^n T_k$ is invertible (or strictly positive), but not every infinite direct sum. However, the converse holds even for an infinite direct sum.

Hint: Set $T_k = \frac{1}{k}$ on $\mathcal{H}_k = \mathbb{C}$ (see Examples 4.J and 5.R).

Problem 5.74. Let $\mathrm{Lat}(\mathcal{H})$ be the lattice of all subspaces of a Hilbert space $\mathcal{H}$, and let $\mathrm{Lat}(T)$ be the lattice of all invariant subspaces for an operator T on $\mathcal{H}$ (see Problems 4.19 and 4.23). Extend the concept of diagonal operator of Problem 5.17 to operators on an arbitrary (not necessarily separable) Hilbert space: an operator $T \in \mathcal{B}[\mathcal{H}]$ is a *diagonal with respect to an orthonormal basis* $\{e_\gamma\}_{\gamma\in\Gamma}$ for $\mathcal{H}$ if there exists a bounded family of scalars $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ such that

$$Tx = \sum_{\gamma\in\Gamma} \lambda_\gamma \langle x\,; e_\gamma\rangle e_\gamma$$

for every $x \in \mathcal{H}$. Show that the following assertions are pairwise equivalent.

(a) $\mathrm{Lat}(T) = \mathrm{Lat}(\mathcal{H})$.

(b) T is a diagonal operator with respect to every orthonormal basis for $\mathcal{H}$.

(c) U^*TU is a diagonal operator for every unitary operator $U \in \mathcal{B}[\mathcal{H}]$.

(d) T is a scalar operator (i.e., $T = \lambda I$ for some $\lambda \in \mathbb{C}$).

Hint: (a) $\Rightarrow$(b): Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary orthonormal basis for $\mathcal{H}$. Take any $x \in \mathcal{H}$. Theorem 5.48 says that $x = \sum_{\gamma\in\Gamma}\langle x\,; e_\gamma\rangle e_\gamma$, and so (why?) $Tx = \sum_{\gamma\in\Gamma}\langle x\,; e_\gamma\rangle Te_\gamma$. Fix $\gamma \in \Gamma$. If $\mathrm{Lat}(T) = \mathrm{Lat}(\mathcal{H})$, then every one-dimensional subspace of $\mathcal{H}$ is T-invariant. Thus, if $\alpha \in \mathbb{C}\backslash\{0\}$, then $T(\alpha\, e_\gamma) = \lambda(\alpha\, e_\gamma)\alpha\, e_\gamma$ for some function $\lambda\colon (\mathrm{span}\,\{e_\gamma\}\backslash\{0\}) \to \mathbb{C}$. Since $\alpha\,\lambda(\alpha\, e_\gamma)e_\gamma = \lambda(\alpha\, e_\gamma)\alpha\, e_\gamma = T(\alpha\, e_\gamma) = \alpha\, Te_\gamma = \alpha\,\lambda(e_\gamma)e_\gamma$, we get $\lambda(\alpha\, e_\gamma) = \lambda(e_\gamma)$ so that λ is a constant function. That is, $\lambda(\alpha\, e_\gamma) = \lambda_\gamma$ for all $\alpha \in \mathbb{C}\backslash\{0\}$, and hence $Te_\gamma = \lambda_\gamma e_\gamma$. This implies that T is a diagonal with respect to basis $\{e_\gamma\}_{\gamma\in\Gamma}$.

(b) $\Rightarrow$(c): Take any unitary operator $U \in \mathcal{B}[\mathcal{H}]$, let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal basis for $\mathcal{H}$, and set $f_\gamma = Ue_\gamma$ for each $\gamma \in \Gamma$ so that $\{f_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal basis for $\mathcal{H}$ (Proof of Theorem 5.49). Take any $x \in \mathcal{H}$. If (b) holds, then there is a bounded family of scalars $\{\mu_\gamma\}_{\gamma\in\Gamma}$ such that $Tx = \sum_{\gamma\in\Gamma}\mu_\gamma\langle x\,; f_\gamma\rangle f_\gamma = \sum_{\gamma\in\Gamma}\mu_\gamma\langle x\,; Ue_\gamma\rangle Ue_\gamma = U(\sum_{\gamma\in\Gamma}\mu_\gamma\langle U^*x\,; e_\gamma\rangle e_\gamma) = UDU^*x$, where D is the

diagonal operator with respect to $\{e_\gamma\}_{\gamma\in\Gamma}$ given by $Dx = \sum_{\gamma\in\Gamma}\mu_\gamma\langle x\,;e_\gamma\rangle e_\gamma$. Thus $T = UDU^*$ or, equivalently, $D = U^*TU$.

(c) $\Rightarrow$(d): If (c) holds, then T is a diagonal operator with respect to an orthonormal basis $\{e_\gamma\}_{\gamma\in\Gamma}$ for some bounded family of scalars $\{\lambda_\gamma\}_{\gamma\in\Gamma}$. Suppose $\dim\mathcal{H}\geq 2$. Take any pair of (distinct) indices $\{\gamma_1,\gamma_2\}$ from Γ, split $\{e_\gamma\}_{\gamma\in\Gamma}$ into $\{e_\gamma\}_{\gamma\in\Gamma} = \{e_{\gamma_1,e_{\gamma_2}}\}\cup\{e_\gamma\}_{\gamma\in(\Gamma\setminus\{\gamma_1,\gamma_2\})}$, and decompose $\mathcal{H} = \mathcal{M}\oplus\mathcal{M}^\perp$ (Theorem 5.25) with $\mathcal{M} = \operatorname{span}\{e_{\gamma_1},e_{\gamma_2}\}$ and $\mathcal{M}^\perp = \bigvee\{e_\gamma\}_{\gamma\in(\Gamma\setminus\{\gamma_1,\gamma_2\})}$. As $\mathcal{M}$ reduces T, $T = A\oplus B$ with $A = T|_{\mathcal{M}}$ and $B = T|_{\mathcal{M}^\perp}$ (Problem 5.28). Thus A is a diagonal operator on $\mathcal{M}$ with respect to the orthonormal basis $\{e_{\gamma_1},e_{\gamma_2}\}$ for $\mathcal{M}$. Let $\{e_1,e_2\}$ be the canonical basis for $\mathbb{C}^2$. Since $\mathcal{M}\cong\mathbb{C}^2$, there is a unitary $W\colon\mathbb{C}^2\to\mathcal{M}$ such that $We_1 = e_{\gamma_1}$ and $We_2 = e_{\gamma_2}$ (Theorem 5.49), and hence $W^*AWy = W^*(\sum_{i=1}^2\lambda_{\gamma_i}\langle Wy\,;e_{\gamma_i}\rangle e_{\gamma_i}) = \sum_{i=1}^2\lambda_{\gamma_i}\langle y\,;W^*e_{\gamma_i}\rangle W^*e_{\gamma_i} = \sum_{i=1}^2\lambda_{\gamma_i}\langle y\,;e_i\rangle e_i$ for each $y\in\mathbb{C}^2$. Therefore $W^*AW = \operatorname{diag}(\lambda_{\gamma_1},\lambda_{\gamma_2})$, a diagonal operator in $\mathcal{B}[\mathbb{C}^2]$ with respect to $\{e_1,e_2\}$. Consider the unitary operator $U = \frac{\sqrt{2}}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$ in $B[\mathbb{C}^2]$. So $U^*W^*AWU = \frac{1}{2}\begin{pmatrix}\lambda_{\gamma_1}+\lambda_{\gamma_2}&\lambda_{\gamma_2}-\lambda_{\gamma_1}\\ \lambda_{\gamma_2}-\lambda_{\gamma_1}&\lambda_{\gamma_1}+\lambda_{\gamma_2}\end{pmatrix}$ in $B[\mathbb{C}^2]$. But if (c) holds, this must be a diagonal (why?), which implies that $\lambda_{\gamma_1} = \lambda_{\gamma_2}$. Since the pair of (distinct) indices $\{\gamma_1,\gamma_2\}$ from Γ was arbitrarily taken, it follows that $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ is a constant family. Hence $T = \lambda I$.

(d) $\Rightarrow$(a): Every subspace of $\mathcal{H}$ trivially is invariant for a scalar operator.

Problem 5.75. If $T\in\mathcal{B}[\mathcal{H}]$ is a contraction on a Hilbert space $\mathcal{H}$, then

$$\mathcal{U} = \big\{x\in\mathcal{H}\colon \|T^nx\| = \|T^{*n}x\| = \|x\|\big\}$$

is a reducing subspace for T. Prove. Also show that the restriction of T to $\mathcal{U}$,

$$T|_{\mathcal{U}}\colon\mathcal{U}\to\mathcal{U},\ \text{ is a unitary operator.}$$

A contraction on a nonzero Hilbert space is called *completely nonunitary* if the restriction of it to every nonzero reducing subspace is not unitary. That is, $T\in\mathcal{B}[\mathcal{H}]$ on $\mathcal{H}\neq\{0\}$ with $\|T\|\leq 1$ is completely nonunitary, if $T|_{\mathcal{M}}\in\mathcal{B}[\mathcal{M}]$ is not unitary for every subspace $\mathcal{M}\neq\{0\}$ of $\mathcal{H}$ that reduces T. Show that a contraction T is completely nonunitary if and only if $\mathcal{U} = \{0\}$. Equivalently,

> T is *not* completely nonunitary if and only if there is a nonzero vector $x\in\mathcal{H}$ such that $\|T^nx\| = \|T^{*n}x\| = \|x\|$ for every $n\geq 1$.

Also verify the following (almost tautological) assertions. Every completely nonunitary contraction on a nonzero Hilbert space is itself nonzero. A completely nonunitary contraction has a completely nonunitary adjoint.

Hints: $\mathcal{U}$ reduces T by Proposition 5.74. Use Proposition 5.73(j) to show that $T|_{\mathcal{U}}$ is unitary and that T is completely nonunitary if and only if $\mathcal{U} = \{0\}$, and therefore T is completely nonunitary if and only if T^* is.

Problem 5.76. Prove the following proposition.

A countable direct sum of contractions is completely nonunitary if and only if every direct summand is completely nonunitary.

Hint: Let $\{T_k\}$ be a nonzero contraction on each Hilbert space $\mathcal{H}_k$. Recall that a countable direct sum $\bigoplus_k T_k$ is a contraction if and only if every T_k is a contraction (Problem 5.73). Let $\mathcal{M}$ be a subspace of $\bigoplus_k \mathcal{H}_k$ that reduces $\bigoplus_k T_k$. Recall that $(\bigoplus_k T_k)^n = \bigoplus_k T_k^n$ (Problem 5.71). Verify that $\mathcal{M}$ reduces $(\bigoplus_k T_k)^n$ for every $n \geq 1$, and that this implies, for every $n \geq 1$, that

$$\Big((\bigoplus_k T_k)|_{\mathcal{M}}\Big)^n = (\bigoplus_k T_k)^n|_{\mathcal{M}} = (\bigoplus_k T_k^n)|_{\mathcal{M}}$$

and

$$\Big((\bigoplus_k T_k)|_{\mathcal{M}}\Big)^{*n} = (\bigoplus_k T_k)^{*n}|_{\mathcal{M}} = (\bigoplus_k T_k^{*n})|_{\mathcal{M}}$$

(cf. Corollary 5.75 and Problems 5.24(d) and 5.28). If $(\bigoplus_k T_k)|_{\mathcal{M}}$ is unitary, then so is $((\bigoplus_k T_k)|_{\mathcal{M}})^n$ for every $n \geq 1$, and hence, by the above identities,

$$(\bigoplus_k T_k^n)|_{\mathcal{M}}(\bigoplus_k T_k^{*n})|_{\mathcal{M}} = I = (\bigoplus_k T_k^{*n})|_{\mathcal{M}}(\bigoplus_k T_k^n)|_{\mathcal{M}}$$

for every $n \geq 1$. Thus, for an arbitrary sequence $u = \{u_k\} \in \mathcal{M} \subseteq \bigoplus_k \mathcal{H}_k$,

$$(\bigoplus_k T_k^n)(\bigoplus_k T_k^{*n})\, u = u = (\bigoplus_k T_k^{*n})(\bigoplus_k T_k^n)\, u,$$

which means that $(\bigoplus_k T_k^n T_k^{*n})\{u_k\} = \{u_k\} = (\bigoplus_k T_k^{*n} T_k^n)\{u_k\}$. Therefore,

$$T_k^n T_k^{*n} u_k = T_k^{*n} T_k^n u_k = u_k \quad \text{and so} \quad \|T_k^n u_k\| = \|T_k^{*n} u_k\| = \|u_k\|$$

(why?) for each k and every n. If each T_k is completely nonunitary, then every u_k is zero (Problem 5.75). Thus $u = 0$ so that $\mathcal{M} = \{0\}$, and $\bigoplus_k T_k$ is completely nonunitary. Conversely, if one of the T_k is not completely nonunitary, then (Problem 5.75 again) there exists a nonzero vector $x_k \in \mathcal{H}_k$ such that

$$\|T_k^n x_k\| = \|T_k^{*n} x_k\| = \|x_k\| \quad \text{for every } n \geq 1.$$

Then there is a nonzero vector $x = (0, \dots, 0, x_k, 0, \dots)$ in $\bigoplus_k \mathcal{H}_k$ such that

$$\begin{aligned}\big\|(\bigoplus_k T_k)^n x\big\| &= \big\|(\bigoplus_k T_k^n)x\big\| = \|T_k^n x_k\| = \|x_k\| = \|x\| \\ &= \|x_k\| = \|T_k^{*n} x_k\| = \big\|(\bigoplus_k T_k^{*n})x\big\| = \big\|(\bigoplus_k T_k)^{*n} x\big\|,\end{aligned}$$

for every $n \geq 1$, and hence the contraction $\bigoplus_k T_k$ is not completely nonunitary.

Problem 5.77. Let $T \in \mathcal{B}[\mathcal{H}]$ be a nonzero contraction on a Hilbert space $\mathcal{H}$. Prove the following assertions.

(a) Every strongly stable contraction is completely nonunitary.

Hint: If $\|T^n x\| \to 0$ for every x, then $\mathcal{U} = \{0\}$ (Problem 5.75).

(b) There is a completely nonunitary contraction that is not strongly stable.

Hint: A unilateral shift S_+ is an isometry (thus a contraction that is not strongly stable) such that $S_+^{*n} \xrightarrow{s} O$ (Problem 5.29), and hence S_+ is completely nonunitary (cf. Problem 5.75 and item (a) above).

(c) There is a completely nonunitary contraction that is not strongly stable and whose adjoint is also not strongly stable.

Hint: $S_+ \oplus S_+^*$ (see Problem 5.76).

Problem 5.78. Show that if a contraction is completely nonunitary, then so is every operator unitarily equivalent to it. That is, *the property of being completely nonunitary is invariant under unitary equivalence*.

Hint: An operator unitarily equivalent to a contraction is again a contraction (since unitary equivalence is norm-preserving — Problem 5.9). Let $T \in \mathcal{B}[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{K}]$ be unitarily equivalent contractions, and let $U \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ be any unitary transformation intertwining T to S. Take any nonzero reducing subspace $\mathcal{M}$ for T so that $U(\mathcal{M})$ is a nonzero reducing subspace for S by Problem 5.9(d). Since $UT|_{\mathcal{M}} = SU|_{\mathcal{M}} = S|_{U(\mathcal{M})}$, if $T|_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{M}$ is unitary, then so is $UT|_{\mathcal{M}}\colon \mathcal{M} \to U(\mathcal{M})$ (composition of invertible isometries is again an invertible isometry) and, conversely, if $UT|_{\mathcal{M}}\colon \mathcal{M} \to U(\mathcal{M})$ is unitary, then so is $T|_{\mathcal{M}} = U^*(UT|_{\mathcal{M}})\colon \mathcal{M} \to \mathcal{M}$. Therefore,

$$T|_{\mathcal{M}} \text{ is unitary if and only if } S|_{U(\mathcal{M})} \text{ is unitary.}$$

On the other hand, recall that U^* is unitary and $U^*S = TU^*$. Thus if $\mathcal{N}$ is a nonzero reducing subspace for S, then $U^*(\mathcal{N})$ is a nonzero reducing subspace for T. Again, since $U^*S|_{\mathcal{N}} = TU^*|_{\mathcal{N}} = S|_{U^*(\mathcal{N})}$, conclude that

$$S|_{\mathcal{N}} \text{ is unitary if and only if } T|_{U^*(\mathcal{N})} \text{ is unitary.}$$

Therefore, $T|_{\mathcal{M}}$ is not unitary for every nonzero T-reducing subspace $\mathcal{M}$ if and only if $S|_{\mathcal{N}}$ is not unitary for every nonzero S-reducing subspace $\mathcal{N}$.

6

The Spectral Theorem

The Spectral Theorem is a landmark in the theory of operators on Hilbert space, providing a full statement about the nature and structure of *normal* operators. Normal operators play a central role in operator theory; they will be defined in Section 6.1 below. It is customary to say that the Spectral Theorem can be applied to answer essentially all questions on normal operators. This indeed is the case as far as "essentially all" means "almost all" or "all the principal": there exist open questions on normal operators. First we consider the class of normal operators and its relatives (predecessors and successors). Next, the notion of spectrum of an operator acting on a complex Banach space is introduced. The Spectral Theorem for compact normal operators is fully investigated, yielding the concept of diagonalization. The Spectral Theorem for plain normal operators needs measure theory. We would not dare to relegate measure theory to an appendix just to support a proper proof of the Spectral Theorem for plain normal operators. Instead we assume just once, in the very last section of this book, that the reader has some familiarity with measure theory, just enough to grasp the statement of the Spectral Theorem for plain normal operators after having proved it for compact normal operators.

6.1 Normal Operators

Throughout this section $\mathcal{H}$ stands for a Hilbert space. An operator $T \in \mathcal{B}[\mathcal{H}]$ is *normal* if it commutes with its adjoint (i.e., T is normal if $T^*T = TT^*$). Here is another characterization of normal operators.

Proposition 6.1. *The following assertions are pairwise equivalent.*

(a) T *is normal* (i.e., $T^*T = TT^*$).

(b) $\|T^*x\| = \|Tx\|$ *for every* $x \in \mathcal{H}$.

(c) T^n *is normal for every positive integer* n.

(d) $\|T^{*n}x\| = \|T^nx\|$ *for every* $x \in \mathcal{H}$ *and every* $n \geq 1$.

C.S. Kubrusly, *The Elements of Operator Theory*, DOI 10.1007/978-0-8176-4998-2_6,

© Springer Science+Business Media, LLC 2011

Proof. If $T \in \mathcal{B}[\mathcal{H}]$, then $\|T^*x\|^2 - \|Tx\|^2 = \langle (TT^* - T^*T)x\,;x\rangle$ for every x in $\mathcal{H}$. Since $TT^* - T^*T$ is self-adjoint, it follows by Corollary 5.80 that $TT^* = T^*T$ if and only if $\|T^*x\| = \|Tx\|$ for every $x \in \mathcal{H}$. This shows that (a)$\Leftrightarrow$(b). Therefore, as $T^{*n} = T^{n*}$ for every $n \geq 1$ (cf. Problem 5.24), (c)$\Leftrightarrow$(d). If T^* commutes with T, then it commutes with T^n and, dually, T^n commutes with $T^{*n} = T^{n*}$. So (a)$\Rightarrow$(c). Since (d)$\Rightarrow$(b) trivially, the proposition is proved. □

Clearly, *every self-adjoint operator is normal* (i.e., $T^* = T$ implies $T^*T = TT^* = T^2$), and so are the nonnegative operators and, in particular, the orthogonal projections (cf. Proposition 5.81). It is also clear that *every unitary operator is normal* (recall from Proposition 5.73 that $U \in \mathcal{B}[\mathcal{H}]$ is unitary if and only if $U^*U = UU^* = I$). In fact, normality distinguishes the orthogonal projections among the projections, and the unitaries among the isometries.

Proposition 6.2. *$P \in \mathcal{B}[\mathcal{H}]$ is an orthogonal projection if and only if it is a normal projection.*

Proof. If P is an orthogonal projection, then it is a self-adjoint projection (Proposition 5.81), and hence a normal projection. On the other hand, if P is normal, then $\|P^*x\| = \|Px\|$ for every $x \in \mathcal{H}$ (by the previous proposition) so that $\mathcal{N}(P^*) = \mathcal{N}(P)$. If P is a projection, then $\mathcal{R}(P) = \mathcal{N}(I - P)$ so that $\mathcal{R}(P) = \mathcal{R}(P)^-$ by Proposition 4.13. Therefore, if P is a normal projection, then $\mathcal{N}(P)^\perp = \mathcal{N}(P^*)^\perp = \mathcal{R}(P)^- = \mathcal{R}(P)$ (cf. Proposition 5.76), and hence $\mathcal{R}(P) \perp \mathcal{N}(P)$. Thus P is an orthogonal projection. □

Proposition 6.3. *$U \in \mathcal{B}[\mathcal{H}]$ is unitary if and only if it is a normal isometry.*

Proof. Proposition 5.73(a,j). □

Let $T \in \mathcal{B}[\mathcal{H}]$ be an arbitrary operator on a Hilbert space $\mathcal{H}$ and set

$$D \;=\; T^*T - TT^*$$

in $\mathcal{B}[\mathcal{H}]$. Observe that $D = D^*$ (i.e., D is always self-adjoint). Moreover,

$$T \text{ is normal if and only if } D = O.$$

An operator $T \in \mathcal{B}[\mathcal{H}]$ is *quasinormal* if it commutes with T^*T; that is, if $T^*TT = TT^*T$ or, equivalently, if $(T^*T - TT^*)T = O$. Therefore,

$$T \text{ is quasinormal if and only if } DT = O.$$

It is plain that *every normal operator is quasinormal*. Also note that *every isometry is quasinormal*. Indeed, if $V \in \mathcal{B}[\mathcal{H}]$ is an isometry, then $V^*V = I$ (Proposition 5.72) so that $V^*VV - VV^*V = O$.

Proposition 6.4. *If $T = WQ$ is the polar decomposition of an operator T in $\mathcal{B}[\mathcal{H}]$, then* (a)

$$WQ = QW \text{ if and only if } T \text{ is quasinormal.}$$

In this case, $WT = TW$ *and* $QT = TQ$*. Moreover,* (b) *if* T *is normal, then* $W|_{\mathcal{N}(W)^\perp}$ *is unitary. That is, the partial isometry of the polar decomposition of any normal operator is, in fact, a "partial unitary transformation" in the following sense.* $W = UP$*, where* P *is the orthogonal projection onto* $\mathcal{N}^\perp$ *with* $\mathcal{N} = \mathcal{N}(T) = \mathcal{N}(W) = \mathcal{N}(Q)$*, and* $U{:}\mathcal{N}^\perp \to \mathcal{N}^\perp \subseteq \mathcal{H}$ *is a unitary operator for which* $\mathcal{N}^\perp$ *is* U*-invariant.*

Proof. (a) Let $T = WQ$ be the polar decomposition of T so that $Q^2 = T^*T$ (Theorem 5.89). If $WQ = QW$, then $Q^2W = QWQ = WQ^2$, and hence $TT^*T = WQQ^2 = Q^2WQ = T^*TT$ (i.e., T is quasinormal). Conversely, if $TT^*T = T^*TT$, then $TQ^2 = Q^2T$. Thus $TQ = QT$ by Theorem 5.85 (since $Q = (Q^2)^{\frac{1}{2}}$) so that $WQQ = QWQ$; that is, $(WQ - QW)Q = O$. Therefore, $\mathcal{R}(Q)^- \subseteq \mathcal{N}(WQ - QW)$ and so

$$\mathcal{N}(Q)^\perp \subseteq \mathcal{N}(WQ - QW)$$

by Proposition 5.76 (since $Q = Q^*$). Recall that $\mathcal{N}(Q) = \mathcal{N}(W)$ (Theorem 5.89). If $u \in \mathcal{N}(Q)$, then $u \in \mathcal{N}(W)$ so that $(WQ - QW)u = 0$. Hence

$$\mathcal{N}(Q) \subseteq \mathcal{N}(WQ - QW).$$

The above displayed inclusions imply $\mathcal{N}(WQ - QW) = \mathcal{H}$ (Problem 5.7(b)); that is, $WQ = QW$. Since $T = WQ$, it follows at once that W and Q commute with T whenever they commute with each other.

(b) Recall from Theorem 5.89 that the null spaces of T, W, and Q coincide. Thus (cf. Proposition 5.86) set

$$\mathcal{N} = \mathcal{N}(T) = \mathcal{N}(W) = \mathcal{N}(Q) = \mathcal{N}(Q^2).$$

According to Proposition 5.87, $W = VP$ where $V{:}\mathcal{N}^\perp \to \mathcal{H}$ is an isometry and $P{:}\mathcal{H} \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{N}^\perp$. Since $\mathcal{R}(Q)^- = \mathcal{N}(Q)^\perp = \mathcal{N}^\perp = \mathcal{R}(P)$, it follows that $PQ = Q$. Taking the adjoint and recalling that $P = P^*$ (Proposition 5.81), we get

$$PQ = QP = Q.$$

Moreover, since $V \in \mathcal{B}[\mathcal{N}^\perp, \mathcal{H}]$, its adjoint V^* lies in $\mathcal{B}[\mathcal{H}, \mathcal{N}^\perp]$. Then $\mathcal{R}(V^*) \subseteq \mathcal{N}^\perp = \mathcal{R}(P)$, which implies that $PV^* = V^*$. Hence

$$VPV^* = VV^*.$$

These identities hold for the polar decomposition of every operator $T \in \mathcal{B}[\mathcal{H}]$. Now suppose T is normal so that T is quasinormal. By part (a) we get

$$Q^2 = T^*T = TT^* = WQQW^* = Q^2WW^* = Q^2VPV^* = Q^2VV^*.$$

Therefore, $Q^2(I - VV^*) = O$ and hence

$$\mathcal{R}(I - VV^*) \subseteq \mathcal{N}(Q^2) = \mathcal{N}.$$

But if T is normal, then T commutes with T^* and trivially with itself. Therefore, $\mathcal{N} = \mathcal{N}(T)$ reduces T (Problem 5.34), and so $\mathcal{N}^\perp$ is T-invariant. Then $\mathcal{R}(T) \subseteq \mathcal{N}^\perp$ by Theorem 5.20, which implies that $\mathcal{R}(V) \subseteq \mathcal{N}^\perp$ (since $\mathcal{R}(V) = \mathcal{R}(W) = \mathcal{R}(T)^-$). In this case the isometry $V{:}\mathcal{N}^\perp \to \mathcal{H}$ is into $\mathcal{N}^\perp \subseteq \mathcal{H}$ so that both V and V^* lie in $\mathcal{B}[\mathcal{N}^\perp]$. Thus the above displayed inclusion now holds for I and VV^* in $\mathcal{B}[\mathcal{N}^\perp]$. Hence $\mathcal{R}(I - VV^*) = \{0\}$ (as $\mathcal{R}(I - VV^*) \subseteq \mathcal{N} \cap \mathcal{N}^\perp = \{0\}$), which means that $I - VV^* = O$. That is, $VV^* = I$ so that the isometry V also is a coisometry. Thus V is unitary (Proposition 5.73). □

A *part* of an operator is a restriction of it to an invariant subspace. For instance, *every unilateral shift is a part of some bilateral shift* (of the same multiplicity). This takes a little proving. In this sense, every unilateral shift has an extension that is a bilateral shift. Recall that unilateral shifts are isometries, and bilateral shifts are unitary operators (see Problems 5.29 and 5.30). The above italicized result can be extended as follows. *Every isometry is a part of a unitary operator*. This takes a little proving too. Since every isometry is quasinormal, and since every unitary operator is normal, we might expect that *every quasinormal operator is a part of a normal operator*. This actually is the case. We shall call an operator *subnormal* if it is a part of a normal operator or, equivalently, if it has a normal extension. Precisely, an operator T on a Hilbert space $\mathcal{H}$ is subnormal if there exists a Hilbert space $\mathcal{K}$ including $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $\mathcal{H}$ is N-invariant (i.e., $N(\mathcal{H}) \subseteq \mathcal{H}$) and T is the restriction of N to $\mathcal{H}$ (i.e., $T = N|_{\mathcal{H}}$). In other words, $T \in \mathcal{B}[\mathcal{H}]$ is subnormal if $\mathcal{H}$ is a subspace of a larger Hilbert space $\mathcal{K}$, so that $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ by Theorem 5.25, and

$$N = \begin{pmatrix} T & X \\ O & Y \end{pmatrix} : \mathcal{H} \oplus \mathcal{H}^\perp \to \mathcal{H} \oplus \mathcal{H}^\perp$$

is a normal operator in $\mathcal{B}[\mathcal{K}]$ for some $X \in \mathcal{B}[\mathcal{H}^\perp, \mathcal{H}]$ and some $Y \in \mathcal{B}[\mathcal{H}^\perp]$ (see Example 2.O). Recall that, writing the orthogonal direct sum decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, we are identifying $\mathcal{H} \subseteq \mathcal{K}$ with $\mathcal{H} \oplus \{0\}$ (a subspace of $\mathcal{H} \oplus \mathcal{H}^\perp$) and $\mathcal{H}^\perp \subseteq \mathcal{K}$ with $\{0\} \oplus \mathcal{H}^\perp$ (also a subspace of $\mathcal{H} \oplus \mathcal{H}^\perp$).

Proposition 6.5. *Every quasinormal operator is subnormal.*

Proof. Suppose $T \in \mathcal{B}[\mathcal{H}]$ is a quasinormal operator.

Claim. $\mathcal{N}(T)$ reduces T.

Proof. Since T is quasinormal, T^*T commutes with both T and T^*. So $\mathcal{N}(T^*T)$ reduces T (Problem 5.34). But $\mathcal{N}(T^*T) = \mathcal{N}(T)$ (Proposition 5.76). □

Thus $T = O \oplus S$ on $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{N}(T)^\perp$, with $O = T|_{\mathcal{N}(T)}{:}\mathcal{N}(T) \to \mathcal{N}(T)$ and $S = T|_{\mathcal{N}(T)^\perp}{:}\mathcal{N}(T)^\perp \to \mathcal{N}(T)^\perp$. Note that $T^*T = O \oplus S^*S$, and so

$$(O \oplus S^*S)(O \oplus S) = T^*TT = TT^*T = (O \oplus S)(O \oplus S^*S).$$

Then $O \oplus S^*SS = O \oplus SS^*S$, and hence $S^*SS = SS^*S$. That is, S is quasinormal. Since $\mathcal{N}(S) = \mathcal{N}(T|_{\mathcal{N}(T)^\perp}) = \{0\}$, it follows by Corollary 5.90 that the partial isometry of the polar decomposition of $S \in \mathcal{B}[\mathcal{N}(T)^\perp]$ is an isometry. Therefore $S = VQ$, where $V \in \mathcal{B}[\mathcal{N}(T)^\perp]$ is an isometry (so that $V^*V = I$) and $Q \in \mathcal{B}[\mathcal{N}(T)^\perp]$ is nonnegative. But $S = VQ = QV$ by Proposition 6.4, and hence $S^* = QV^* = V^*Q$. Set

$$U = \begin{pmatrix} V & I - VV^* \\ O & V^* \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix}$$

in $\mathcal{B}[\mathcal{N}(T)^\perp \oplus \mathcal{N}(T)^\perp]$. Observe that U is unitary. In fact,

$$\begin{aligned} U^*U &= \begin{pmatrix} V^* & O \\ I - VV^* & V \end{pmatrix} \begin{pmatrix} V & I - VV^* \\ O & V^* \end{pmatrix} \\ &= \begin{pmatrix} I & O \\ O & I \end{pmatrix} = \begin{pmatrix} V & I - VV^* \\ O & V^* \end{pmatrix} \begin{pmatrix} V^* & O \\ I - VV^* & V \end{pmatrix} = UU^*. \end{aligned}$$

Also note that the nonnegative operator R commutes with U:

$$\begin{aligned} UR &= \begin{pmatrix} VQ & (I - VV^*)Q \\ O & V^*Q \end{pmatrix} = \begin{pmatrix} S & Q(I - VV^*) \\ O & S^* \end{pmatrix} \\ &= \begin{pmatrix} QV & Q(I - VV^*) \\ O & QV^* \end{pmatrix} = RU. \end{aligned}$$

Now set $N = UR$ in $\mathcal{B}[\mathcal{N}(T)^\perp \oplus \mathcal{N}(T)^\perp]$. The middle operator matrix says that S is a part of N (i.e., $\mathcal{N}(T)^\perp$ is N-invariant and $S = N|_{\mathcal{N}(T)^\perp}$). Moreover,

$$N^*N = RU^*UR = R^2 = R^2UU^* = UR^2U^* = NN^*.$$

Thus N is normal. Then S is subnormal, and so is $T = O \oplus S$ since T trivially is a part of the normal operator $O \oplus N$ on $\mathcal{N}(T) \oplus \mathcal{N}(T)^\perp \oplus \mathcal{N}(T)^\perp$. □

An operator $T \in \mathcal{B}[\mathcal{H}]$ is *hyponormal* if $TT^* \le T^*T$. In other words,

$$T \text{ is hyponormal if and only if } D \ge O.$$

Recall that T^*T and TT^* are nonnegative and $D = (T^*T - TT^*)$ is self-adjoint, for every $T \in \mathcal{B}[\mathcal{H}]$.

Proposition 6.6. *$T \in \mathcal{B}[\mathcal{H}]$ is hyponormal if and only if $\|T^*x\| \le \|Tx\|$ for every $x \in \mathcal{H}$.*

Proof. $TT^* \le T^*T$ if and only if $\langle TT^*x\,;x\rangle \le \langle T^*Tx\,;x\rangle$ or, equivalently, $\|T^*x\| \le \|Tx\|$ for every $x \in \mathcal{H}$. □

An operator $T \in \mathcal{B}[\mathcal{H}]$ is *cohyponormal* if its adjoint $T^* \in \mathcal{B}[\mathcal{H}]$ is hyponormal (i.e., if $T^*T \le TT^*$ or, equivalently, if $D \le O$, which means by the above

proposition that $\|Tx\| \le \|T^*x\|$ for every $x \in \mathcal{H}$). Hence T *is normal if and only if it is both hyponormal and cohyponormal* (Propositions 6.1 and 6.6). If an operator is either hyponormal or cohyponormal, then it is called *seminormal*. Every normal operator is trivially hyponormal. The next proposition goes beyond that.

Proposition 6.7. *Every subnormal operator is hyponormal.*

Proof. If $T \in \mathcal{B}[\mathcal{H}]$ is subnormal, then $\mathcal{H}$ is a subspace of a larger Hilbert space $\mathcal{K}$ so that $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, and the operator

$$N = \begin{pmatrix} T & X \\ O & Y \end{pmatrix} : \mathcal{H} \oplus \mathcal{H}^\perp \to \mathcal{H} \oplus \mathcal{H}^\perp$$

in $\mathcal{B}[\mathcal{K}]$ is normal for some $X \in \mathcal{B}[\mathcal{H}^\perp, \mathcal{H}]$ and $Y \in \mathcal{B}[\mathcal{H}^\perp]$. Then

$$\begin{pmatrix} T^*T & T^*X \\ X^*T & X^*X + Y^*Y \end{pmatrix} = \begin{pmatrix} T^* & O \\ X^* & Y^* \end{pmatrix} \begin{pmatrix} T & X \\ O & Y \end{pmatrix} = N^*N$$

$$= NN^* = \begin{pmatrix} T & X \\ O & Y \end{pmatrix} \begin{pmatrix} T^* & O \\ X^* & Y^* \end{pmatrix} = \begin{pmatrix} TT^* + XX^* & XY^* \\ YX^* & YY^* \end{pmatrix}.$$

Therefore $T^*T = TT^* + XX^*$, and hence $T^*T - TT^* = XX^* \ge O$. □

Let $\mathcal{X}$ be a normed space, take any operator $T \in \mathcal{B}[\mathcal{X}]$, and consider the power sequence $\{T^n\}$. A trivial induction (cf. Problem 4.47(a)) shows that $\|T^n\| \le \|T\|^n$ for every $n \ge 0$.

Lemma 6.8. *If $\mathcal{X}$ is a normed space and T is an operator in $\mathcal{B}[\mathcal{X}]$, then the real-valued sequence $\{\|T^n\|^{\frac{1}{n}}\}$ converges in $\mathbb{R}$.*

Proof. Suppose $T \ne O$. The proof uses the following bit of elementary number theory. Take an arbitrary $m \in \mathbb{N}$. Every $n \in \mathbb{N}$ can be written as $n = mp_n + q_n$ for some $p_n, q_n \in \mathbb{N}_0$, where $q_n < m$. Hence

$$\|T^n\| = \|T^{mp_n + q_n}\| = \|T^{mp_n} T^{q_n}\| \le \|T^{mp_n}\| \|T^{q_n}\| \le \|T^m\|^{p_n} \|T^{q_n}\|.$$

Set $\mu = \max_{0 \le k \le m-1}\{\|T^k\|\} \ne 0$ and recall that $q_n \le m - 1$. Then

$$\|T^n\|^{\frac{1}{n}} \le \|T^m\|^{\frac{p_n}{n}} \mu^{\frac{1}{n}} = \mu^{\frac{1}{n}} \|T^m\|^{\frac{1}{m} - \frac{q_n}{mn}}.$$

Since $\mu^{\frac{1}{n}} \to 1$ and $\|T^m\|^{\frac{1}{m} - \frac{q_n}{mn}} \to \|T^m\|^{\frac{1}{m}}$ as $n \to \infty$, it follows that

$$\limsup_n \|T^n\|^{\frac{1}{n}} \le \|T^m\|^{\frac{1}{m}}$$

for every $m \in \mathbb{N}$. Thus $\limsup_n \|T^n\|^{\frac{1}{n}} \le \liminf_n \|T^n\|^{\frac{1}{n}}$ and so (Problem 3.13) $\{\|T^n\|^{\frac{1}{n}}\}$ converges in $\mathbb{R}$. □

We shall denote the limit of $\{\|T^n\|^{\frac{1}{n}}\}$ by $r(T)$:

$$r(T) = \lim_n \|T^n\|^{\frac{1}{n}}.$$

According to the above proof we get $r(T) \le \|T^n\|^{\frac{1}{n}}$ for every $n \ge 1$, and so $r(T) \le \|T\|$. Also note that $r(T^k)^{\frac{1}{k}} = (\lim_n \|(T^k)^n\|^{\frac{1}{n}})^{\frac{1}{k}} = \lim_n \|T^{kn}\|^{\frac{1}{kn}} = r(T)$ for each $k \ge 1$, because $\{\|T^{kn}\|^{\frac{1}{kn}}\}$ is a subsequence of the convergent sequence $\{\|T^n\|^{\frac{1}{n}}\}$. Thus $r(T^k) = r(T)^k$ for every positive integer k. Therefore, if $T \in \mathcal{B}[\mathcal{X}]$ is an operator on a normed space $\mathcal{X}$, then

$$r(T)^n = r(T^n) \le \|T^n\| \le \|T\|^n \quad \text{for each integer } n \ge 0.$$

Definition: If $r(T) = \|T\|$, then we say that $T \in \mathcal{B}[\mathcal{X}]$ is *normaloid*.

The next proposition gives an equivalent definition.

Proposition 6.9. *An operator $T \in \mathcal{B}[\mathcal{X}]$ on a normed space $\mathcal{X}$ is normaloid if and only if $\|T^n\| = \|T\|^n$ for every integer $n \ge 0$.*

Proof. If $r(T) = \|T\|$, then $\|T^n\| = \|T\|^n$ for every $n \ge 0$ by the above inequalities. If $\|T^n\| = \|T\|^n$ for every $n \ge 0$, then $r(T) = \lim_n \|T^n\|^{\frac{1}{n}} = \|T\|$. □

Proposition 6.10. *Every hyponormal operator is normaloid.*

Proof. Take an arbitrary operator $T \in \mathcal{B}[\mathcal{H}]$ and let n be a nonnegative integer.

Claim 1. If T is hyponormal, then $\|T^n\|^2 \le \|T^{n+1}\|\|T^{n-1}\|$ for every $n \ge 1$.

Proof. Note that, for every $T \in \mathcal{B}[\mathcal{H}]$,

$$\|T^n x\|^2 = \langle T^n x\,; T^n x\rangle = \langle T^* T^n x\,; T^{n-1} x\rangle \le \|T^* T^n x\|\|T^{n-1} x\|$$

for each integer $n \ge 1$ and every $x \in \mathcal{H}$. If T is hyponormal, then

$$\|T^* T^n x\|\|T^{n-1} x\| \le \|T^{n+1} x\|\|T^{n-1} x\| \le \|T^{n+1}\|\|T^{n-1}\|\|x\|^2$$

by Proposition 6.6, and hence

$$\|T^n x\|^2 \le \|T^{n+1}\|\|T^{n-1}\|\|x\|^2$$

for each $n \ge 1$ and every $x \in \mathcal{H}$, which ensures the claimed result. □

Claim 2. If $\|T^n\|^2 \le \|T^{n+1}\|\|T^{n-1}\|$ for every $n \ge 1$, then $\|T^n\| = \|T\|^n$ for every $n \ge 0$

Proof. $\|T^n\| = \|T\|^n$ holds trivially if $T = O$ (for all $n \ge 0$), and if $n = 0, 1$ (for all T in $\mathcal{B}[\mathcal{H}]$). Let T be a nonzero operator and suppose $\|T^n\| = \|T\|^n$ for some integer $n \ge 1$. If $\|T^n\|^2 \le \|T^{n+1}\|\|T^{n-1}\|$, then

$$\|T\|^{2n} = (\|T\|^n)^2 = \|T^n\|^2 \le \|T^{n+1}\|\|T^{n-1}\| \le \|T^{n+1}\|\|T\|^{n-1}$$

since $\|T^m\| \le \|T\|^m$ for every $m \ge 0$, and therefore (recall: $T \ne O$),

$$\|T\|^{n+1} = \|T\|^{2n}(\|T\|^{n-1})^{-1} \le \|T^{n+1}\| \le \|T\|^{n+1}.$$

Hence $\|T^{n+1}\| = \|T\|^{n+1}$, concluding the proof by induction. □

Claims 1 and 2 say that a hyponormal T is normaloid by Proposition 6.9. □

Since $\|T^{*n}\| = \|T^n\|$ for each $n \geq 0$ (cf. Problem 5.24(d)), it follows that $r(T^*) = r(T)$. Thus T is normaloid if and only if T^* is normaloid, and so every seminormal operator is normaloid. Summing up: An operator T is normal if it commutes with its adjoint, quasinormal if it commutes with T^*T, subnormal if it is a restriction of a normal operator to an invariant subspace, hyponormal if $TT^* \leq T^*T$, and normaloid if $r(T) = \|T\|$. These classes are related by proper inclusion as follows.

$$\text{Normal} \subset \text{Quasinormal} \subset \text{Subnormal} \subset \text{Hyponormal} \subset \text{Normaloid}.$$

Example 6.A. We shall verify that the above inclusions are, in fact, proper. The unilateral shift will do the whole job. First recall that a unilateral shift S_+ is an isometry but not a coisometry, and hence S_+ is a nonnormal quasinormal operator. Since S_+ is subnormal, $A = I + S_+$ is subnormal (if N is a normal extension of S_+, then $I + N$ is a normal extension of A). However, since S_+ is a nonnormal isometry, $A^*AA - AA^*A = A^*AS_+ - S_+A^*A = S_+^*S_+ - S_+S_+^* \neq O$, and therefore A is not quasinormal. Check that $B = S_+^* + 2S_+$ is hyponormal, but B^2 is not hyponormal. Since the square of every subnormal operator is again a subnormal operator, it follows that B is not subnormal. Finally, S_+^* is normaloid (by Proposition 6.9) but not hyponormal.

6.2 The Spectrum of an Operator

Let $T \in \mathcal{L}[\mathcal{D}(T), \mathcal{X}]$ be a linear transformation, where $\mathcal{X}$ is a nonzero normed space and $\mathcal{D}(T)$, the domain of T, is a linear manifold of $\mathcal{X} \neq \{0\}$. Let I be the identity on $\mathcal{X}$. The *resolvent set* $\rho(T)$ of T is the set of all scalars $\lambda \in \mathbb{F}$ for which $(\lambda I - T) \in \mathcal{L}[\mathcal{D}(T), \mathcal{X}]$ has a densely defined continuous inverse:

$$\rho(T) = \big\{\lambda \in \mathbb{F}\colon\ (\lambda I - T)^{-1} \in \mathcal{B}[\mathcal{R}(\lambda I - T), \mathcal{D}(T)] \text{ and } \mathcal{R}(\lambda I - T)^- = \mathcal{X}\big\}.$$

Henceforward all linear transformations are operators on a complex Banach space. In other words, $T \in \mathcal{B}[\mathcal{X}]$, where $\mathcal{D}(T) = \mathcal{X} \neq \{0\}$ is a complex Banach space; that is, $T\colon \mathcal{X} \to \mathcal{X}$ is a *bounded* linear transformation of a nonzero *complex Banach space* $\mathcal{X}$ into itself. In this case (i.e., in the unital complex Banach algebra $\mathcal{B}[\mathcal{X}]$), Corollary 4.24 ensures that the above-defined resolvent set $\rho(T)$ is the set of all complex numbers λ for which $(\lambda I - T) \in \mathcal{B}[\mathcal{X}]$ is invertible (i.e., has a bounded inverse on $\mathcal{X}$). Equivalently (Theorem 4.22),

$$\begin{aligned}\rho(T) &= \big\{\lambda \in \mathbb{C}\colon\ (\lambda I - T) \in \mathcal{G}[\mathcal{X}]\big\}\\ &= \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ has an inverse in } \mathcal{B}[\mathcal{X}]\}\\ &= \big\{\lambda \in \mathbb{C}\colon\ \mathcal{N}(\lambda I - T) = \{0\} \text{ and } \mathcal{R}(\lambda I - T) = \mathcal{X}\big\}.\end{aligned}$$

The complement of $\rho(T)$, denoted by $\sigma(T)$, is the *spectrum* of T:

$$\sigma(T) = \mathbb{C}\backslash\rho(T) = \big\{\lambda \in \mathbb{C}\colon\ \mathcal{N}(\lambda I - T) \neq \{0\} \text{ or } \mathcal{R}(\lambda I - T) \neq \mathcal{X}\big\}.$$

Proposition 6.11. *If $\lambda \in \rho(T)$, then $\delta = \|(\lambda I - T)^{-1}\|^{-1}$ is a positive number. The open ball $B_\delta(\lambda)$ with center at λ and radius δ is included in $\rho(T)$, and hence $\delta \le d(\lambda, \sigma(T))$.*

Proof. Let T be a *bounded* linear operator acting on a complex Banach space. Take $\lambda \in \rho(T)$. Then $(\lambda I - T) \in \mathcal{G}[\mathcal{X}]$, and so $O \ne (\lambda I - T)^{-1}$ is bounded. Thus $\delta = \|(\lambda I - T)^{-1}\|^{-1} > 0$. Let $B_\delta(0)$ be the nonempty open ball of radius δ about the origin of the complex plane $\mathbb{C}$, and take an arbitrary μ in $B_\delta(0)$. Since $|\mu| < \|(\lambda I - T)^{-1}\|^{-1}$, $\|\mu(\lambda I - T)^{-1}\| < 1$. Thus, by Problem 4.48(a), $[I - \mu(\lambda I - T)^{-1}] \in \mathcal{G}[\mathcal{X}]$, and so $(\lambda - \mu)I - T = (\lambda I - T)[I - \mu(\lambda I - T)^{-1}]$ also lies in $\mathcal{G}[\mathcal{X}]$ by Corollary 4.23. Outcome: $\lambda - \mu \in \rho(T)$, so that

$$B_\delta(\lambda) \,=\, B_\delta(0) + \lambda \,=\, \big\{\nu \in \mathbb{C}\colon\ \nu = \mu + \lambda \text{ for some } \mu \in B_\delta(0)\big\} \,\subseteq\, \rho(T),$$

which implies that $\sigma(T) = \mathbb{C}\backslash\rho(T) \subseteq \mathbb{C}\backslash B_\delta(\lambda)$. Hence $d(\lambda\,, \varsigma) = |\lambda - \varsigma| \ge \delta$ for every $\varsigma \in \sigma(T)$, and so $d(\lambda, \sigma(T)) = \inf_{\varsigma \in \sigma(T)} |\lambda - \varsigma| \ge \delta$. □

Corollary 6.12. *The resolvent set $\rho(T)$ is nonempty and open, and the spectrum $\sigma(T)$ is compact.*

Proof. If $T \in \mathcal{B}[\mathcal{X}]$ is an operator on a Banach space $\mathcal{X}$, then (since T is bounded) the von Neumann expansion (Problem 4.47) ensures that $\lambda \in \rho(T)$ whenever $\|T\| < |\lambda|$. Since $\sigma(T) = \mathbb{C}\backslash\rho(T)$, this is equivalent to

$$|\lambda| \,\le\, \|T\| \quad \text{ for every } \quad \lambda \in \sigma(T).$$

Thus $\sigma(T)$ is bounded, and so $\rho(T) \ne \varnothing$. By Proposition 6.11, $\rho(T)$ includes a nonempty open ball centered at each point in it. Thus $\rho(T)$ is open, and so $\sigma(T)$ is closed. In $\mathbb{C}$, closed and bounded means compact (Theorem 3.83). □

The *resolvent function* of $T \in \mathcal{B}[\mathcal{X}]$ is the map $R\colon \rho(T) \to \mathcal{G}[\mathcal{X}]$ defined by

$$R(\lambda) = (\lambda I - T)^{-1}$$

for every $\lambda \in \rho(T)$. Since $R(\lambda) - R(\mu) = R(\lambda)\big[R(\mu)^{-1} - R(\lambda)^{-1}\big]R(\mu)$, we get

$$R(\lambda) - R(\mu) \,=\, (\mu - \lambda)R(\lambda)R(\mu)$$

for every $\lambda, \mu \in \rho(T)$. This is the *resolvent identity*. Swapping λ and μ in the resolvent identity, it follows that $R(\lambda)R(\mu) = R(\mu)R(\lambda)$ for every $\lambda, \mu \in \rho(T)$. Also, $TR(\lambda) = R(\lambda)T$ for every $\lambda \in \rho(T)$ (since $R(\lambda)^{-1}R(\lambda) = R(\lambda)R(\lambda)^{-1}$).

To prove the next proposition we need a piece of elementary complex analysis. Let Λ be a nonempty and open subset of the complex plane $\mathbb{C}$. Take a function $f\colon \Lambda \to \mathbb{C}$ and a point $\mu \in \Lambda$. Suppose $f'(\mu)$ is a complex number with the following property. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\big|\frac{f(\lambda) - f(\mu)}{\lambda - \mu} - f'(\mu)\big| < \varepsilon$ for all λ in Λ for which $0 < |\lambda - \mu| < \delta$. If there exists such an $f'(\mu) \in \mathbb{C}$, then it is called the *derivative* of f at μ. If $f'(\mu)$ exists for every μ in Λ, then $f\colon \Lambda \to \mathbb{C}$ is *analytic* on Λ. A function $f\colon \mathbb{C} \to \mathbb{C}$ is *entire* if it is analytic on the whole complex plane $\mathbb{C}$. The *Liouville Theorem* is the result we need. It says that *every bounded entire function is constant*.

Proposition 6.13. *The spectrum $\sigma(T)$ is nonempty.*

Proof. Let $T \in \mathcal{B}[\mathcal{X}]$ be an operator on a complex Banach space $\mathcal{X}$. Take an arbitrary nonzero element φ in the dual $\mathcal{B}[\mathcal{X}]^*$ of $\mathcal{B}[\mathcal{X}]$ (i.e., an arbitrary nonzero bounded linear functional $\varphi\colon \mathcal{B}[\mathcal{X}] \to \mathbb{C}$ — note: $\mathcal{B}[\mathcal{X}] \neq \{O\}$ because $\mathcal{X} \neq \{0\}$, and so $\mathcal{B}[\mathcal{X}]^* \neq \{0\}$ by Corollary 4.64). Recall that $\rho(T) = \mathbb{C}\backslash\sigma(T)$ is nonempty and open in $\mathbb{C}$.

Claim 1. If $\sigma(T)$ is empty, then $\varphi \circ R\colon \rho(T) \to \mathbb{C}$ is bounded.

Proof. The resolvent function $R\colon \rho(T) \to \mathcal{G}[\mathcal{X}]$ is continuous (reason: scalar multiplication and addition are continuous mappings, and so is inversion by Problem 4.48(c)). Thus $\|R(\cdot)\|\colon \rho(T) \to \mathbb{R}$ is continuous (composition of continuous functions). Then $\sup_{|\lambda|\le\|T\|}\|R(\lambda)\| < \infty$ by Theorem 3.86 whenever $\sigma(T)$ is empty. On the other hand, if $\|T\| < |\lambda|$, then Problem 4.47(h) ensures that $\|R(\lambda)\| = \|(\lambda I - T)^{-1}\| \le (|\lambda| - \|T\|)^{-1}$, and therefore $\|R(\lambda)\| \to 0$ as $|\lambda| \to \infty$. Since the function $\|R(\cdot)\|\colon \rho(T) \to \mathbb{R}$ is continuous, it then follows that $\sup_{\|T\|<|\lambda|}\|R(\lambda)\| < \infty$. Hence $\sup_{\lambda\in\rho(T)}\|R(\lambda)\| < \infty$. Thus

$$\sup_{\lambda\in\rho(T)} \|(\varphi \circ R)(\lambda)\| \le \|\varphi\| \sup_{\lambda\in\rho(T)} \|R(\lambda)\| < \infty. \quad \square$$

Claim 2. $\varphi \circ R\colon \rho(T) \to \mathbb{C}$ is analytic.

Proof. If λ and μ are distinct points in $\rho(T)$, then

$$\frac{R(\lambda) - R(\mu)}{\lambda - \mu} + R(\mu)^2 = \big(R(\mu) - R(\lambda)\big)R(\mu)$$

by the resolvent identity. Set $f = \varphi \circ R\colon \rho(T) \to \mathbb{C}$, and let $f'\colon \rho(T) \to \mathbb{C}$ be defined by $f'(\lambda) = -\varphi(R(\lambda)^2)$ for each $\lambda \in \rho(T)$. Therefore,

$$\begin{aligned}\left|\frac{f(\lambda) - f(\mu)}{\lambda - \mu} - f'(\mu)\right| &= \big|\varphi\big[\big(R(\mu) - R(\lambda)\big)R(\mu)\big]\big| \\ &\le \|\varphi\|\|R(\mu)\|\|R(\mu) - R(\lambda)\|\end{aligned}$$

so that $f\colon \rho(T) \to \mathbb{C}$ is analytic because $R\colon \rho(T) \to \mathcal{G}[\mathcal{X}]$ is continuous. $\square$

If $\sigma(T) = \varnothing$ (i.e., if $\rho(T) = \mathbb{C}$), then $\varphi \circ R\colon \mathbb{C} \to \mathbb{C}$ is a bounded entire function, and so a constant function by the Liouville Theorem. But we have just seen (proof of Claim 1) that $\|R(\lambda)\| \to 0$ as $|\lambda| \to \infty$. Hence $\varphi(R(\lambda)) \to 0$ as $|\lambda| \to \infty$ (since φ is continuous). Then $\varphi \circ R = 0$ for all $\varphi \in \mathcal{B}[\mathcal{H}]^*$ so that $R = O$ (Corollary 4.64). That is, $(\lambda I - T)^{-1} = O$ for every $\lambda \in \mathbb{C}$, which is a contradiction ($O \notin \mathcal{G}[\mathcal{X}]$). Thus $\sigma(T) \neq \varnothing$. $\square$

Remark: $\sigma(T)$ is compact and nonempty, and so is its boundary $\partial\sigma(T)$. Thus $\partial\sigma(T) = \partial\rho(T) \neq \varnothing$ (see Problem 3.41).

The spectrum $\sigma(T)$ is the set of all λ in $\mathbb{C}$ such that $\lambda I - T$ fails to be invertible (i.e., fails to have a bounded inverse on $\mathcal{R}(\lambda I - T) = \mathcal{X}$). According

to the origin of such a failure, $\sigma(T)$ can be split into many disjoint parts. A classical partition comprises three parts. The set of those λ such that $\lambda I - T$ has no inverse is the *point spectrum*:

$$\sigma_P(T) \;=\; \big\{\lambda \in \mathbb{C}\colon\ \mathcal{N}(\lambda I - T) \neq \{0\}\big\}.$$

A scalar $\lambda \in \mathbb{C}$ is called an *eigenvalue* of T if there exists a nonzero vector x in $\mathcal{X}$ such that $Tx = \lambda x$; equivalently, if $\mathcal{N}(\lambda I - T) \neq \{0\}$. If $\lambda \in \mathbb{C}$ is an eigenvalue of T, then the nonzero vectors in $\mathcal{N}(\lambda I - T)$ are the *eigenvectors* of T, and $\mathcal{N}(\lambda I - T)$ is the *eigenspace* (which, in fact, is a subspace of $\mathcal{X}$ — cf. Proposition 4.13), associated with λ. The *multiplicity* of an eigenvalue is the dimension of the respective eigenspace. After this quick digression on eigenvalues and eigenvectors, note that *the point spectrum of T is precisely the set of all eigenvalues of T*. The set of those λ for which $\lambda I - T$ has a densely defined but unbounded inverse on its range is the *continuous spectrum*:

$$\sigma_C(T) \;=\; \big\{\lambda \in \mathbb{C}\colon\ \mathcal{N}(\lambda I - T) = \{0\},\ \mathcal{R}(\lambda I - T)^- = \mathcal{X} \text{ and } \mathcal{R}(\lambda I - T) \neq \mathcal{X}\big\}$$

(see Corollary 4.24 again). If $\lambda I - T$ has an inverse on its range that is not densely defined, then λ belongs to the *residual spectrum*:

$$\sigma_R(T) \;=\; \big\{\lambda \in \mathbb{C}\colon\ \mathcal{N}(\lambda I - T) = \{0\} \text{ and } \mathcal{R}(\lambda I - T)^- \neq \mathcal{X}\big\}.$$

The collection $\big\{\sigma_P(T), \sigma_C(T), \sigma_R(T)\big\}$ of subsets of $\sigma(T)$ is in fact a partition (i.e., a disjoint covering) of the spectrum: they are pairwise disjoint and

$$\sigma(T) \;=\; \sigma_P(T) \cup \sigma_C(T) \cup \sigma_R(T).$$

The following diagram refines this spectrum partition. The residual spectrum is split into two disjoint parts, $\sigma_R(T) = \sigma_{R_1}(T) \cup \sigma_{R_2}(T)$, and the point spectrum is split into four disjoint parts, $\sigma_P(T) = \bigcup_{i=1}^{4} \sigma_{P_i}(T)$. We adopt the following abbreviated notation: $T_\lambda = \lambda I - T$, $\mathcal{N}_\lambda = \mathcal{N}(T_\lambda)$, and $\mathcal{R}_\lambda = \mathcal{R}(T_\lambda)$. Recall that if T_λ is injective (i.e., if $\mathcal{N}(T_\lambda) = \{0\}$), then its linear inverse T_λ^{-1} on $\mathcal{R}_\lambda$ is continuous if and only if $\mathcal{R}_\lambda$ is closed (Corollary 4.24).

		$\mathcal{R}_\lambda^- = \mathcal{X}$		$\mathcal{R}_\lambda^- \neq \mathcal{X}$	
		$\mathcal{R}_\lambda^- = \mathcal{R}_\lambda$	$\mathcal{R}_\lambda^- \neq \mathcal{R}_\lambda$	$\mathcal{R}_\lambda^- \neq \mathcal{R}_\lambda$	$\mathcal{R}_\lambda^- = \mathcal{R}_\lambda$
$\mathcal{N}_\lambda = \{0\}$	$T_\lambda^{-1} \in \mathcal{B}[\mathcal{R}_\lambda, \mathcal{X}]$	$\rho(T)$	$\varnothing$	$\varnothing$	$\sigma_{R_1}(T)$
	$T_\lambda^{-1} \notin \mathcal{B}[\mathcal{R}_\lambda, \mathcal{X}]$	$\varnothing$	$\sigma_C(T)$	$\sigma_{R_2}(T)$	$\varnothing$
$\mathcal{N}_\lambda \neq \{0\}$		$\sigma_{P_1}(T)$	$\sigma_{P_2}(T)$	$\sigma_{P_3}(T)$	$\sigma_{P_4}(T)$

$\sigma_{AP}(T)$ (the last two rows)

$\sigma_{CP}(T)$ (the last column)

Recall that $\sigma(T) \neq \varnothing$, but any of the above disjoint parts of the spectrum may be empty (see Section 6.5). However, if $\sigma_P(T)$ is nonempty, then every set of eigenvectors associated with *distinct* eigenvalues is linearly independent.

Proposition 6.14. *Let $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ be any family of distinct eigenvalues of T. For each $\gamma\in\Gamma$ let x_γ be an eigenvector associated with λ_γ* (i.e., $x_\gamma\neq 0$ in $\mathcal{N}(\lambda_\gamma I - T)\neq\{0\}$). *The set $\{x_\gamma\}_{\gamma\in\Gamma}$ is linearly independent.*

Proof. Consider the set $\{x_\gamma\}_{\gamma\in\Gamma}$ (whose existence is ensured by the Axiom of Choice). Let $\{x_i\}_{i=1}^n$ be an arbitrary *finite* subset of $\{x_\gamma\}_{\gamma\in\Gamma}$.

Claim. $\{x_i\}_{i=1}^n$ is linearly independent.

Proof. If $n=1$, then linear independence is trivial. Suppose $\{x_i\}_{i=1}^n$ is linearly independent for some integer $n\geq 1$. If $\{x_i\}_{i=1}^{n+1}$ is not linearly independent, then $x_{n+1}=\sum_{i=1}^n\alpha_i x_i$, where $\{\alpha_i\}_{i=1}^n$ is a family of complex numbers with at least one nonzero number among them. Therefore $\lambda_{n+1}x_{n+1}=Tx_{n+1}=\sum_{i=1}^n\alpha_i Tx_i=\sum_{i=1}^n\alpha_i\lambda_i x_i$. If $\lambda_{n+1}=0$, then $\lambda_i\neq 0$ for every $i\neq n+1$ and $\sum_{i=1}^n\alpha_i\lambda_i x_i=0$ so that $\{x_i\}_{i=1}^n$ is not linearly independent, which is a contradiction. On the other hand, if $\lambda_{n+1}\neq 0$, then $x_{n+1}=\sum_{i=1}^n\alpha_i\lambda_{n+1}^{-1}\lambda_i x_i$, and hence $\sum_{i=1}^n\alpha_i(1-\lambda_{n+1}^{-1}\lambda_i)x_i=0$ so that $\{x_i\}_{i=1}^n$ is not linearly independent (because $\lambda_i\neq\lambda_{n+1}$ for every $i\neq n+1$), which is again a contradiction. This completes the proof by induction. □

Outcome: $\{x_\gamma\}_{\gamma\in\Gamma}$ is linearly independent by Proposition 2.3. □

There are some overlapping parts of the spectrum which are commonly used too. For instance, the *compression spectrum* $\sigma_{CP}(T)$ and the *approximate point spectrum* (or *approximation spectrum*) $\sigma_{AP}(T)$. These are defined by

$$\begin{aligned}\sigma_{CP}(T) &= \big\{\lambda\in\mathbb{C}\colon\ \mathcal{R}(\lambda I-T)\text{ is not dense in }\mathcal{X}\big\}\\ &= \sigma_{P_3}(T)\cup\sigma_{P_4}(T)\cup\sigma_R(T)\end{aligned}$$

and

$$\begin{aligned}\sigma_{AP}(T) &= \big\{\lambda\in\mathbb{C}\colon\ \lambda I-T\text{ is not bounded below}\big\}\\ &= \sigma_P(T)\cup\sigma_C(T)\cup\sigma_{R_2}(T) = \sigma(T)\backslash\sigma_{R_1}(T).\end{aligned}$$

Next we give an alternative definition of $\sigma_{AP}(T)$ which may come as a motivation for the term "approximate point spectrum". The elements of it are sometimes referred to as the *approximate eigenvalues* of T.

Proposition 6.15. *The following assertions are pairwise equivalent.*

(a) $\lambda\in\sigma_{AP}(T)$.

(b) *There exists an $\mathcal{X}$-valued sequence $\{x_n\}$ of unit vectors such that* $\|(\lambda I-T)x_n\|\to 0$.

(c) *For every $\varepsilon>0$ there is a unit vector $x_\varepsilon\in\mathcal{X}$ such that* $\|(\lambda I-T)x_\varepsilon\|<\varepsilon$.

Proof. It is clear that (c) implies (b). If (b) holds true, then there is no constant $\alpha>0$ such that $\alpha=\alpha\|x_n\|\leq\|(\lambda I-T)x_n\|$ for all n, and so $\lambda I-T$ is not bounded below. Hence (b) implies (a). If $\lambda I-T$ is not bounded below, then

there is no constant $\alpha > 0$ such that $\alpha\|x\| \leq \|(\lambda I - T)x\|$ for all $x \in \mathcal{X}$ or, equivalently, for every $\varepsilon > 0$ there exists $0 \neq y_\varepsilon \in \mathcal{X}$ such that $\|(\lambda I - T)y_\varepsilon\| < \varepsilon\|y_\varepsilon\|$. By setting $x_\varepsilon = \|y_\varepsilon\|^{-1}y_\varepsilon$, it follows that (a) implies (c). □

Proposition 6.16. *The approximate point spectrum is nonempty, closed in* $\mathbb{C}$*, and includes the boundary* $\partial\sigma(T)$ *of the spectrum.*

Proof. Take an arbitrary $\lambda \in \partial\sigma(T)$. Recall that $\rho(T) \neq \varnothing$ (Corollary 6.12) and $\partial\sigma(T) = \partial\rho(T) \subset \rho(T)^-$ (Problem 3.41). Hence there exists a sequence $\{\lambda_n\}$ in $\rho(T)$ such that $\lambda_n \to \lambda$ (Proposition 3.27). Since

$$(\lambda_n I - T) - (\lambda I - T) \;=\; (\lambda_n - \lambda)I$$

for every n, it follows that $(\lambda_n I - T) \to (\lambda I - T)$ in $\mathcal{B}[\mathcal{X}]$; that is, $\{(\lambda_n I - T)\}$ in $\mathcal{G}[\mathcal{X}]$ converges in $\mathcal{B}[\mathcal{X}]$ to $(\lambda I - T) \in \mathcal{B}[\mathcal{X}]\backslash\mathcal{G}[\mathcal{X}]$ (each λ_n lies in $\rho(T)$ and $\lambda \in \partial\sigma(T) \subseteq \sigma(T)$ because $\sigma(T)$ is closed). If $\sup_n\|(\lambda_n I - T)^{-1}\| < \infty$, then $(\lambda I - T) \in \mathcal{G}[\mathcal{X}]$ (cf. hint to Problem 4.48(c)), which is a contradiction. Thus

$$\sup_n \|(\lambda_n I - T)^{-1}\| \;=\; \infty.$$

For each n take y_n in $\mathcal{X}$ with $\|y_n\| = 1$ such that

$$\|(\lambda_n I - T)^{-1}\| - \tfrac{1}{n} \;\leq\; \|(\lambda_n I - T)^{-1}y_n\| \;\leq\; \|(\lambda_n I - T)^{-1}\|.$$

Then $\sup_n\|(\lambda_n I - T)^{-1}y_n\| = \infty$, and hence $\inf_n\|(\lambda_n I - T)^{-1}y_n\|^{-1} = 0$, so that there exist subsequences of $\{\lambda_n\}$ and $\{y_n\}$, say $\{\lambda_k\}$ and $\{y_k\}$, for which

$$\|(\lambda_k I - T)^{-1}y_k\|^{-1} \;\to\; 0.$$

Set $x_k = \|(\lambda_k I - T)^{-1}y_k\|^{-1}(\lambda_k I - T)^{-1}y_k$ and get a sequence $\{x_k\}$ of unit vectors in $\mathcal{X}$ such that $\|(\lambda_k I - T)x_k\| = \|(\lambda_k I - T)^{-1}y_k\|^{-1}$. Since

$$\|(\lambda I - T)x_k\| = \|(\lambda_k I - T)x_k - (\lambda_k - \lambda)x_k\| \leq \|(\lambda_k I - T)^{-1}y_k\|^{-1} + |\lambda_k - \lambda|$$

and $\lambda_k \to \lambda$, it follows that $\|(\lambda I - T)x_k\| \to 0$. Hence $\lambda \in \sigma_{AP}(T)$ according to Proposition 6.15. Therefore,

$$\partial\sigma(T) \;\subseteq\; \sigma_{AP}(T).$$

This inclusion implies that $\sigma_{AP}(T) \neq \varnothing$ (since $\sigma(T)$ is closed and nonempty). Finally, take an arbitrary $\lambda \in \mathbb{C}\backslash\sigma_{AP}(T)$ so that $\lambda I - T$ is bounded below. Thus there exists an $\alpha > 0$ for which

$$\alpha\|x\| \;\leq\; \|(\lambda I - T)x\| \;\leq\; \|(\mu I - T)x\| + \|(\lambda - \mu)x\|,$$

and so $(\alpha - |\lambda - \mu|)\|x\| \leq \|(\mu I - T)x\|$, for all $x \in \mathcal{X}$ and $\mu \in \mathbb{C}$. Then $\mu I - T$ is bounded below (i.e., $\mu \in \mathbb{C}\backslash\sigma_{AP}(T)$) for every μ sufficiently close to λ (such that $0 < \alpha - |\lambda - \mu|$). Hence $\mathbb{C}\backslash\sigma_{AP}(T)$ is open, and so $\sigma_{AP}(T)$ is closed. □

Remark: $\sigma_{R_1}(T)$ *is open in* $\mathbb{C}$. Indeed, since $\sigma_{AP}(T)$ is closed in $\mathbb{C}$ and includes $\partial\sigma(T)$, it follows that $\mathbb{C}\backslash\sigma_{R_1}(T) = \rho(T) \cup \sigma_{AP}(T) = \rho(T) \cup \partial\sigma(T) \cup \sigma_{AP}(T) = \rho(T) \cup \partial\rho(T) \cup \sigma_{AP}(T) = \rho(T)^- \cup \sigma_{AP}(T)$, which is closed in $\mathbb{C}$.

For the next proposition we assume that T lies in $\mathcal{B}[\mathcal{H}]$, where $\mathcal{H}$ is a nonzero complex *Hilbert space*. If Λ is any subset of $\mathbb{C}$, then set

$$\Lambda^* = \{\overline{\lambda} \in \mathbb{C}\colon \lambda \in \Lambda\}$$

so that $\Lambda^{**} = \Lambda$, $(\mathbb{C}\backslash\Lambda)^* = \mathbb{C}\backslash\Lambda^*$, and $(\Lambda_1 \cup \Lambda_2)^* = \Lambda_1^* \cup \Lambda_2^*$.

Proposition 6.17. *If $T^* \in \mathcal{B}[\mathcal{H}]$ is the adjoint of $T \in \mathcal{B}[\mathcal{H}]$, then*

$$\rho(T) = \rho(T^*)^*, \qquad \sigma(T) = \sigma(T^*)^*, \qquad \sigma_C(T) = \sigma_C(T^*)^*,$$

and the residual spectrum of T is given by the formula

$$\sigma_R(T) = \sigma_P(T^*)^*\backslash\sigma_P(T).$$

As for the subparts of the point and residual spectra,

$$\sigma_{P_1}(T) = \sigma_{R_1}(T^*)^*, \qquad \sigma_{P_2}(T) = \sigma_{R_2}(T^*)^*,$$
$$\sigma_{P_3}(T) = \sigma_{P_3}(T^*)^*, \qquad \sigma_{P_4}(T) = \sigma_{P_4}(T^*)^*.$$

For the compression and approximate point spectra, we get

$$\sigma_{CP}(T) = \sigma_P(T^*)^*,$$
$$\partial\sigma(T) \subseteq \sigma_{AP}(T) \cap \sigma_{AP}(T^*)^* = \sigma(T)\backslash(\sigma_{P_1}(T) \cup \sigma_{R_1}(T)).$$

Proof. Since $S \in \mathcal{G}[\mathcal{H}]$ if and only if $S^* \in \mathcal{G}[\mathcal{H}]$, we get $\rho(T) = \rho(T^*)^*$. Hence $\sigma(T)^* = (\mathbb{C}\backslash\rho(T))^* = \mathbb{C}\backslash\rho(T^*) = \sigma(T^*)$. Recall that $\mathcal{R}(S)^- = \mathcal{R}(S)$ if and only if $\mathcal{R}(S^*)^- = \mathcal{R}(S^*)$, and $\mathcal{N}(S) = \{0\}$ if and only if $\mathcal{R}(S^*)^- = \mathcal{H}$ (Proposition 5.77 and Problem 5.35). Thus $\sigma_{P_1}(T) = \sigma_{R_1}(T^*)^*$, $\sigma_{P_2}(T) = \sigma_{R_2}(T^*)^*$, $\sigma_{P_3}(T) = \sigma_{P_3}(T^*)^*$, and also $\sigma_{P_4}(T) = \sigma_{P_4}(T^*)^*$. Applying the same argument, $\sigma_C(T) = \sigma_C(T^*)^*$ and $\sigma_{CP}(T) = \sigma_P(T^*)^*$. Therefore,

$$\sigma_R(T) = \sigma_{CP}(T)\backslash\sigma_P(T) \quad \text{implies} \quad \sigma_R(T) = \sigma_P(T^*)^*\backslash\sigma_P(T).$$

Moreover, by using the above properties, observe that

$$\sigma_{AP}(T^*) = \sigma_P(T^*) \cup \sigma_C(T^*) \cup \sigma_{R_2}(T^*) = \sigma_{CP}(T)^* \cup \sigma_C(T)^* \cup \sigma_{P_2}(T)^*,$$

and so

$$\sigma_{AP}(T^*)^* = \sigma_{CP}(T) \cup \sigma_C(T) \cup \sigma_{P_2}(T).$$

Hence $\sigma_{AP}(T^*)^* \cap \sigma_{AP}(T) = \sigma(T)\backslash(\sigma_{P_1}(T) \cup \sigma_{R_1}(T))$. But $\sigma(T)$ is closed and $\sigma_{R_1}(T)$ is open (and so $\sigma_{P_1}(T) = \sigma_{R_1}(T^*)^*$) in $\mathbb{C}$. This implies that (cf. Problem 3.41(b,d)) $\sigma_{P_1}(T) \cup \sigma_{R_1}(T) \subseteq \sigma(T)^\circ$ and $\partial\sigma(T) \subseteq \sigma(T)\backslash(\sigma_{P_1}(T) \cup \sigma_{R_1}(T))$. □

Remark: We have just seen that $\sigma_{P_1}(T)$ *is open in* $\mathbb{C}$.

Corollary 6.18. *Let $\mathcal{H} \neq \{0\}$ be a complex Hilbert space. Let $\mathbb{D}$ be the open unit disk about the origin in the complex plane $\mathbb{C}$, and let $\mathbb{T} = \partial\mathbb{D}$ denote the unit circle about the origin in $\mathbb{C}$.*

(a) *If* $H \in \mathcal{B}[\mathcal{H}]$ *is hyponormal, then* $\sigma_P(H)^* \subseteq \sigma_P(H^*)$ *and* $\sigma_R(H^*) = \varnothing$.

(b) *If* $N \in \mathcal{B}[\mathcal{H}]$ *is normal, then* $\sigma_P(N^*) = \sigma_P(N)^*$ *and* $\sigma_R(N) = \varnothing$.

(c) *If* $U \in \mathcal{B}[\mathcal{H}]$ *is unitary, then* $\sigma(U) \subseteq \mathbb{T}$.

(d) *If* $A \in \mathcal{B}[\mathcal{H}]$ *is self-adjoint, then* $\sigma(A) \subset \mathbb{R}$.

(e) *If* $Q \in \mathcal{B}[\mathcal{H}]$ *is nonnegative, then* $\sigma(Q) \subset [0, \infty)$.

(f) *If* $R \in \mathcal{B}[\mathcal{H}]$ *is strictly positive, then* $\sigma(R) \subset [\alpha, \infty)$ *for some* $\alpha > 0$.

(g) *If* $P \in \mathcal{B}[\mathcal{H}]$ *is a nontrivial projection, then* $\sigma(P) = \sigma_P(P) = \{0, 1\}$.

(h) *If* $J \in \mathcal{B}[\mathcal{H}]$ *is a nontrivial involution, then* $\sigma(J) = \sigma_P(J) = \{-1, 1\}$.

Proof. Take any $T \in \mathcal{B}[\mathcal{H}]$ and any $\lambda \in \mathbb{C}$. It is readily verified that

$$(\lambda I - T)^*(\lambda I - T) - (\lambda I - T)(\lambda I - T)^* \,=\, T^*T - TT^*.$$

Hence $\lambda I - T$ is hyponormal if and only if T is hyponormal. If H is hyponormal, then $\lambda I - H$ is hyponormal and so (cf. Proposition 6.6)

$$\|(\overline{\lambda} I - H^*)x\| \,\le\, \|(\lambda I - H)x\| \quad \text{for every } x \in \mathcal{H} \text{ and every } \lambda \in \mathbb{C}.$$

If $\lambda \in \sigma_P(H)$, then $\mathcal{N}(\lambda I - H) \ne \{0\}$ so that $\mathcal{N}(\overline{\lambda} I - H^*) \ne \{0\}$ by the above inequality, and hence $\overline{\lambda} \in \sigma_P(H^*)$. Thus $\sigma_P(H) \subseteq \sigma_P(H^*)^*$. Equivalently,

$$\sigma_P(H)^* \subseteq \sigma_P(H^*) \quad \text{so that} \quad \sigma_R(H^*) \,=\, \sigma_P(H)^* \backslash \sigma_P(H^*) \,=\, \varnothing$$

(cf. Proposition 6.17). This proves (a). Since N is normal if and only if it is both hyponormal and cohyponormal, this also proves (b). That is,

$$\sigma_P(N)^* \,=\, \sigma_P(N^*) \quad \text{so that} \quad \sigma_R(N) \,=\, \varnothing.$$

Let U be a unitary operator (i.e., a normal isometry). Since U is an isometry, $\|Ux\| = \|x\|$ so that $\big|\,|\lambda| - 1\big|\|x\| = \big|\,\|\lambda x\| - \|Ux\|\,\big| \le \|(\lambda I - U)x\|$, for every x in $\mathcal{H}$. If $|\lambda| \ne 1$, then $\lambda I - U$ is bounded below so that $\lambda \in \rho(U) \cup \sigma_R(U) = \rho(U)$ since $\sigma_R(U) = \varnothing$ by (b). Thus $\lambda \notin \rho(U)$ implies $|\lambda| = 1$, proving (c):

$$\sigma(U) \,\subseteq\, \mathbb{T}.$$

If A is self-adjoint, then $\langle x\,; Ax\rangle \in \mathbb{R}$ for every $x \in \mathcal{H}$. Thus $\langle \beta x\,; (\alpha I - A)x\rangle$ is real, and hence $\operatorname{Re}\langle i\beta x\,; (\alpha I - A)x\rangle = 0$, for every $\alpha, \beta \in \mathbb{R}$ and every $x \in \mathcal{H}$. Therefore, with $\lambda = \alpha + i\beta$,

$$\begin{aligned} \|(\lambda I - A)x\|^2 \,&=\, \|i\beta x + (\alpha I - A)x\|^2 \\ &=\, |\beta|^2\|x\|^2 + 2\operatorname{Re}\langle i\beta x\,; (\alpha I - A)x\rangle + \|(\alpha I - A)x\|^2 \\ &=\, |\beta|^2\|x\|^2 + \|(\alpha I - A)x\|^2 \,\ge\, |\beta|^2\|x\|^2 \,=\, |\operatorname{Im}\lambda|^2\|x\|^2 \end{aligned}$$

for every $x \in \mathcal{H}$ and every $\lambda \in \mathbb{C}$. If $\lambda \notin \mathbb{R}$, then $\lambda I - A$ is bounded below, and so $\lambda \in \rho(A) \cup \sigma_R(A) = \rho(A)$ since $\sigma_R(A) = \varnothing$ by (b) once A is normal. Thus $\lambda \notin \rho(A)$ implies $\lambda \in \mathbb{R}$. Since $\sigma(A)$ is bounded, this shows that (d) holds:

$$\sigma(A) \subset \mathbb{R}.$$

If $Q \geq O$ and $\lambda \in \sigma(Q)$, then $\lambda \in \mathbb{R}$ by (d) since Q is self-adjoint, and hence $\|(\lambda I - Q)x\|^2 = |\lambda|^2\|x\|^2 - 2\lambda\langle Qx\,;x\rangle + \|Qx\|^2$ for each $x \in \mathcal{H}$. If $\lambda < 0$, then $\|(\lambda I - Q)x\|^2 \geq |\lambda|^2\|x\|^2$ for every $x \in \mathcal{H}$ (since $Q \geq O$), and so $\lambda I - Q$ is bounded below. Applying the same argument of the previous item, we get (e):

$$\sigma(Q) \subset [0, \infty).$$

If $R \succ O$, then $O \leq R \in \mathcal{G}[\mathcal{H}]$, and so $0 \in \rho(R)$ and $\sigma(R) \subset [0, \infty)$ by (e), and hence $\sigma(R) \subset (0, \infty)$. But $\sigma(R)$ is closed. Thus (f) holds:

$$\sigma(R) \subset [\alpha, \infty) \quad \text{for some} \quad \alpha > 0.$$

If $O \neq P = P^2 \neq I$ (i.e., if P is a nontrivial projection), then $\{0\} \neq \mathcal{R}(P) = \mathcal{N}(I - P)$ and $\{0\} \neq \mathcal{R}(I - P) = \mathcal{N}(P)$ (Section 2.9), and so $\{0, 1\} \subseteq \sigma_P(P)$. If λ is any complex number such that $0 \neq \lambda \neq 1$, then

$$(\lambda I - P)\left(\tfrac{1}{\lambda}\, I + \tfrac{1}{\lambda(\lambda-1)}\, P\right) = I = \left(\tfrac{1}{\lambda}\, I + \tfrac{1}{\lambda(\lambda-1)}\, P\right)(\lambda I - P)$$

so that $\lambda I - P$ is invertible (i.e., $(\lambda I - P) \in \mathcal{G}[\mathcal{H}]$ — Theorem 4.22), and hence $\lambda \in \rho(P)$. Thus $\sigma(P) \subseteq \{0, 1\}$, which concludes the proof of (g):

$$\sigma(P) = \sigma_P(P) = \{0, 1\}.$$

If $J^2 = I$ (i.e., an involution), then $(I - J)(-I - J) = 0 = (-I - J)(I - J)$ so that $\mathcal{R}(-I - J) \subseteq \mathcal{N}(I - J)$ and $\mathcal{R}(I - J) \subseteq \mathcal{N}(-I - J)$. If $1 \notin \sigma_P(J)$ or $-1 \notin \sigma_P(J)$, then $\mathcal{N}(I - J) = \{0\}$ or $\mathcal{N}(-I - J) = \{0\}$, which implies that $\mathcal{R}(I + J) = \{0\}$ or $\mathcal{R}(I - J) = \{0\}$, and hence $J = I$ or $J = -I$. Thus, if the involution J is nontrivial (i.e., if $J \neq \pm I$), then $\{-1, 1\} \in \sigma_P(J)$. Moreover, if λ in $\mathbb{C}$ is such that $\lambda^2 \neq 1$ (i.e., if $\lambda \neq \pm 1$), then

$$(\lambda I - J)\left(\tfrac{-\lambda}{1-\lambda^2}\, I - \tfrac{1}{1-\lambda^2}\, J\right) = I = \left(\tfrac{-\lambda}{1-\lambda^2}\, I - \tfrac{1}{1-\lambda^2}\, J\right)(\lambda I - J),$$

so that $(\lambda I - J) \in \mathcal{G}[\mathcal{H}]$, and hence $\lambda \in \rho(J)$. Thus $\sigma(J) \subseteq \{-1, 1\}$, which concludes the proof of (h):

$$\sigma(J) = \sigma_P(J) = \{-1, 1\}. \qquad \square$$

6.3 Spectral Radius

We open this section with the Spectral Mapping Theorem for polynomials. Let us just mention that there are versions of it that hold for functions other than polynomials. If Λ is any subset of $\mathbb{C}$, and $p\colon \mathbb{C} \to \mathbb{C}$ is any polynomial (in one variable) with complex coefficients, then set

$$p(\Lambda) = \big\{p(\lambda) \in \mathbb{C}\colon\ \lambda \in \Lambda\big\}.$$

Theorem 6.19. (The Spectral Mapping Theorem). *If $T \in \mathcal{B}[\mathcal{X}]$, where $\mathcal{X}$ is a complex Banach space, then*

$$\sigma(p(T)) \,=\, p(\sigma(T))$$

for every polynomial p with complex coefficients.

Proof. If p is a constant polynomial (i.e., if $p(T) = \alpha I$ for some $\alpha \in \mathbb{C}$), then the result is trivially verified (and has nothing to do with T; that is, $\sigma(\alpha I) = \alpha\sigma(I) = \{\alpha\}$ since $\rho(\alpha I) = \mathbb{C}\backslash\{\alpha\}$ for every $\alpha \in \mathbb{C}$). Thus let $p\colon \mathbb{C} \to \mathbb{C}$ be an arbitrary nonconstant polynomial with complex coefficients,

$$p(\lambda) \,=\, \sum_{i=0}^{n} \alpha_i \lambda^i, \quad \text{with } \; n \geq 1 \; \text{ and } \; \alpha_n \neq 0,$$

for every $\lambda \in \mathbb{C}$. Take an arbitrary $\mu \in \mathbb{C}$ and consider the factorization

$$\mu - p(\lambda) \,=\, \beta_n \prod_{i=1}^{n} (\lambda_i - \lambda),$$

with $\beta_n = (-1)^{n+1}\alpha_n$, where $\{\lambda_i\}_{i=1}^n$ are the roots of $\mu - p(\lambda)$. Thus

$$\mu I - p(T) \,=\, \beta_n \prod_{i=1}^{n} (\lambda_i I - T).$$

If $\mu \in \sigma(p(T))$, then there exists $\lambda_j \in \sigma(T)$ for some $j = 1, \ldots, n$. Indeed, if $\lambda_i \in \rho(T)$ for every $i = 1, \ldots, n$, then $\beta_n \prod_{i=1}^n (\lambda_i I - T) \in \mathcal{G}[\mathcal{X}]$, and therefore $\mu I - p(T) \in \mathcal{G}[\mathcal{X}]$, which means that $\mu \in \rho(p(T))$. However,

$$\mu - p(\lambda_j) \,=\, \beta_n \prod_{i=1}^{n} (\lambda_i - \lambda_j) \,=\, 0,$$

and so $p(\lambda_j) = \mu$. Then $\mu = p(\lambda_j) \in \{p(\lambda) \in \mathbb{C}\colon\, \lambda \in \sigma(T)\} = p(\sigma(T))$ because $\lambda_j \in \sigma(T)$. Hence

$$\sigma(p(T)) \,\subseteq\, p(\sigma(T)).$$

Conversely, if $\mu \in p(\sigma(T)) = \{p(\lambda) \in \mathbb{C}\colon\, \lambda \in \sigma(T)\}$, then $\mu = p(\lambda)$ for some $\lambda \in \sigma(T)$. Thus $\mu - p(\lambda) = 0$ so that $\lambda = \lambda_j$ for some $j = 1, \ldots, n$, and so

$$\begin{aligned} \mu I - p(T) \,&=\, \beta_n \prod_{i=1}^{n} (\lambda_i I - T) \\ &=\, \beta_n \, (\lambda_j I - T) \prod_{j \neq i = 1}^{n} (\lambda_i I - T) \,=\, \beta_n \prod_{j \neq i = 1}^{n} (\lambda_i I - T)\, (\lambda_j I - T) \end{aligned}$$

since $\lambda_j I - T$ commutes with $\lambda_i I - T$ for every integer i. If $\mu \in \rho(p(T))$, then $(\mu I - p(T)) \in \mathcal{G}[\mathcal{X}]$ so that

$$\begin{aligned}(\lambda_j I - T)\Big(\beta_n \prod_{j\neq i=1}^{n}(\lambda_i I - T)\big(\mu I - p(T)\big)^{-1}\Big) \\ &= \big(\mu I - p(T)\big)\big(\mu I - p(T)\big)^{-1} = I = \big(\mu I - p(T)\big)^{-1}\big(\mu I - p(T)\big) \\ &= \Big(\beta_n \big(\mu I - p(T)\big)^{-1} \prod_{j\neq i=1}^{n}(\lambda_i I - T)\Big)(\lambda_j I - T).\end{aligned}$$

This means that $\lambda_j I - T$ has a right and a left inverse, and so it is injective and surjective (Problems 1.5 and 1.6). The Inverse Mapping Theorem (Theorem 4.22) says that $(\lambda_j I - T) \in \mathcal{G}[\mathcal{X}]$, and so $\lambda = \lambda_j \in \rho(T)$. This contradicts the fact that $\lambda \in \sigma(T)$. Conclusion: $\mu \notin \rho(p(T))$; that is, $\mu \in \sigma(p(T))$. Hence

$$p(\sigma(T)) \subseteq \sigma(p(T)). \qquad \square$$

Remarks: Here are some useful properties of the spectrum. By the previous theorem, $\mu \in \sigma(T)^n = \{\lambda^n \in \mathbb{C}\colon\ \lambda \in \sigma(T)\}$ if and only if $\mu \in \sigma(T^n)$. Thus

$$\sigma(T^n) = \sigma(T)^n \quad \text{for every} \quad n \geq 0.$$

Moreover, $\mu \in \alpha\sigma(T) = \{\alpha\lambda \in \mathbb{C}\colon\ \lambda \in \sigma(T)\}$ if and only if $\mu \in \sigma(\alpha T)$. So

$$\sigma(\alpha T) = \alpha\sigma(T) \quad \text{for every} \quad \alpha \in \mathbb{C}.$$

The next identity is not a particular case of the Spectral Mapping Theorem for polynomials (as was the case for the above two results). If $T \in \mathcal{G}[\mathcal{X}]$, then

$$\sigma(T^{-1}) = \sigma(T)^{-1}.$$

That is, $\mu \in \sigma(T)^{-1} = \{\lambda^{-1} \in \mathbb{C}\colon\ 0 \neq \lambda \in \sigma(T)\}$ if and only if $\mu \in \sigma(T^{-1})$. Indeed, if $T \in \mathcal{G}[\mathcal{X}]$ (so that $0 \in \rho(T)$) and $\mu \neq 0$, then $-\mu T^{-1}(\mu^{-1} I - T) = \mu I - T^{-1}$, and so $\mu^{-1} \in \rho(T)$ if and only if $\mu \in \rho(T^{-1})$; which means that $\mu \in \sigma(T^{-1})$ if and only if $\mu^{-1} \in \sigma(T)$. Also notice that, for every $S, T \in \mathcal{B}[\mathcal{X}]$,

$$\sigma(ST)\backslash\{0\} = \sigma(TS)\backslash\{0\},$$

In fact, Problem 2.32 says that $I - ST$ is invertible if and only if $I - TS$ is, or equivalently, $\lambda - ST$ is invertible if and only if $\lambda - TS$ is whenever $\lambda \neq 0$, and so $\rho(ST)\backslash\{0\} = \rho(TS)\backslash\{0\}$. Now let $\mathcal{H}$ be a complex Hilbert space. Recall from Proposition 6.17 that, if $T \in \mathcal{B}[\mathcal{H}]$, then

$$\sigma(T^*) = \sigma(T)^*.$$

If $Q \in \mathcal{B}[\mathcal{H}]$ is a nonnegative operator, then it has a unique nonnegative square root $Q^{\frac{1}{2}} \in \mathcal{B}[\mathcal{H}]$ by Theorem 5.85, and $\sigma(Q) \subseteq [0, \infty)$ by Corollary 6.18. Thus Theorem 6.19 ensures that $\sigma(Q^{\frac{1}{2}})^2 = \sigma((Q^{\frac{1}{2}})^2) = \sigma(Q)$. Therefore,

$$\sigma(Q^{\frac{1}{2}}) = \sigma(Q)^{\frac{1}{2}}.$$

The *spectral radius* of an operator $T \in \mathcal{B}[\mathcal{X}]$ is the number

$$r_\sigma(T) \;=\; \sup_{\lambda\in\sigma(T)} |\lambda| \;=\; \max_{\lambda\in\sigma(T)} |\lambda|.$$

The first identity defines the spectral radius $r_\sigma(T)$, and the second follows by Theorem 3.86 (since $\sigma(T) \neq \varnothing$ is compact in $\mathbb{C}$ and $|\ |\colon \mathbb{C} \to \mathbb{R}$ is continuous).

Corollary 6.20. *$r_\sigma(T^n) = r_\sigma(T)^n$ for every $n \geq 0$.*

Proof. Take an arbitrary integer $n \geq 0$. Since $\sigma(T^n) = \sigma(T)^n$, it follows that $\mu \in \sigma(T^n)$ if and only if $\mu = \lambda^n$ for some $\lambda \in \sigma(T)$. Hence $\sup_{\mu\in\sigma(T^n)}|\mu| = \sup_{\lambda\in\sigma(T)}|\lambda^n| = \sup_{\lambda\in\sigma(T)}|\lambda|^n = (\sup_{\lambda\in\sigma(T)}|\lambda|)^n$. □

Remarks: Recall that $\lambda \in \sigma(T)$ only if $|\lambda| \leq \|T\|$ (cf. proof of Corollary 6.12), and so $r_\sigma(T) \leq \|T\|$. Therefore, according to Corollary 6.20,

$$r_\sigma(T)^n \;=\; r_\sigma(T^n) \;\leq\; \|T^n\| \;\leq\; \|T\|^n \quad \text{for each integer } n \geq 0.$$

Thus $r_\sigma(T) \leq 1$ *whenever T is power bounded*. Indeed, if $\sup_n\|T^n\| < \infty$, then $r_\sigma(T)^n \leq \sup_n\|T^n\| < \infty$ for all $n \geq 0$ so that $r_\sigma(T) \leq 1$. That is,

$$\sup_n \|T^n\| \;<\; \infty \quad \text{implies} \quad r_\sigma(T) \;\leq\; 1.$$

Also note that the spectral radius of a nonzero operator may be null. Indeed, the above inequalities ensure that $r_\sigma(T) = 0$ for every nonzero nilpotent operator T (i.e., whenever $T^n = O$ for some integer $n \geq 2$). An operator $T \in \mathcal{B}[\mathcal{X}]$ is *quasinilpotent* if $r_\sigma(T) = 0$. Thus *every nilpotent operator is quasinilpotent*. Observe that $\sigma(T) = \sigma_P(T) = \{0\}$ if T is nilpotent. In fact, if $T^{n-1} \neq O$ and $T^n = O$, then $T(T^{n-1}x) = 0$ for every $x \in \mathcal{X}$, so that $\{0\} \neq \mathcal{R}(T^{n-1}) \subseteq \mathcal{N}(T)$, and hence $\lambda = 0$ is an eigenvalue of T. Since $\sigma_P(T)$ may be empty for a quasinilpotent operator (as we shall see in Examples 6.F and 6.G of Section 6.5), it follows that the inclusion below is proper:

$$\text{Nilpotent} \quad \subset \quad \text{Quasinilpotent.}$$

The next proposition is the *Gelfand–Beurling formula* for the spectral radius. The proof of it requires another piece of elementary complex analysis, namely, *every analytic function has a power series representation*. Precisely, if $f\colon \Lambda \to \mathbb{C}$ is analytic and the annulus $B_{\alpha,\beta}(\mu) = \{\lambda \in \mathbb{C}\colon\ 0 \leq \alpha < |\lambda - \mu| < \beta\}$ lies in the open set $\Lambda \subseteq \mathbb{C}$, then f has a *unique Laurent expansion* about the point μ, viz., $f(\lambda) = \sum_{k=-\infty}^{\infty}\gamma_k(\lambda - \mu)^k$ for every $\lambda \in B_{\alpha,\beta}(\mu)$.

Proposition 6.21. *$r_\sigma(T) = \lim_n \|T^n\|^{\frac{1}{n}}$.*

Proof. Since $r_\sigma(T)^n \leq \|T^n\|$ for every positive integer n,

$$r_\sigma(T) \;\leq\; \lim_n \|T^n\|^{\frac{1}{n}}.$$

(Reason: The limit of the sequence $\{\|T^n\|^{\frac{1}{n}}\}$ exists for every $T \in \mathcal{B}[\mathcal{X}]$, according to Lemma 6.8.) Now recall the von Neumann expansion for the resolvent function $R\colon \rho(T) \to \mathcal{G}[\mathcal{X}]$:

$$R(\lambda) \;=\; (\lambda I - T)^{-1} \;=\; \lambda^{-1} \sum_{k=0}^{\infty} T^k \lambda^{-k}$$

for every $\lambda \in \rho(T)$ such that $\|T\| < |\lambda|$, where the above series converges in the (uniform) topology of $\mathcal{B}[\mathcal{X}]$ (cf. Problem 4.47). Take an arbitrary bounded linear functional $\varphi\colon \mathcal{B}[\mathcal{X}] \to \mathbb{C}$ in $\mathcal{B}[\mathcal{X}]^*$. Since φ is continuous,

$$\varphi(R(\lambda)) \;=\; \lambda^{-1} \sum_{k=0}^{\infty} \varphi(T^k) \lambda^{-k}$$

for every $\lambda \in \rho(T)$ such that $\|T\| < |\lambda|$.

Claim. The displayed identity holds whenever $r_\sigma(T) < |\lambda|$.

Proof. $\lambda^{-1} \sum_{k=0}^{\infty} \varphi(T^k) \lambda^{-k}$ is a Laurent expansion of $\varphi(R(\lambda))$ about the origin for every $\lambda \in \rho(T)$ such that $\|T\| < |\lambda|$. But $\varphi \circ R$ is analytic on $\rho(T)$ (cf. Claim 2 in Proposition 6.13) so that $\varphi(R(\lambda))$ has a unique Laurent expansion about the origin for every $\lambda \in \rho(T)$, and therefore for every $\lambda \in \mathbb{C}$ such that $r_\sigma(T) < |\lambda|$. Then $\varphi(R(\lambda)) = \lambda^{-1} \sum_{k=0}^{\infty} \varphi(T^k) \lambda^{-k}$, which holds for every $\lambda \in \mathbb{C}$ such that $r_\sigma(T) \le \|T\| < |\lambda|$, must be the Laurent expansion about the origin for every $\lambda \in \mathbb{C}$ such that $r_\sigma(T) < |\lambda|$. □

Therefore, if $r_\sigma(T) < |\lambda|$, then $\varphi((\lambda^{-1}T)^k) = \varphi(T^k)\lambda^{-k} \to 0$ (since the above series converges — see Problem 4.7) for every $\varphi \in \mathcal{B}[\mathcal{X}]^*$. But this implies that $\{(\lambda^{-1}T)^k\}$ is bounded in the (uniform) topology of $\mathcal{B}[\mathcal{X}]$ (Problem 4.67(d)). That is, $\lambda^{-1}T$ is power bounded. Hence $|\lambda|^{-n}\|T^n\| \le \sup_k \|(\lambda^{-1}T)^k\| < \infty$ so that, for every $n \ge 1$,

$$|\lambda|^{-1} \|T^n\|^{\frac{1}{n}} \;\le\; \big(\sup_k \|(\lambda^{-1}T)^k\|\big)^{\frac{1}{n}}$$

if $r_\sigma(T) < |\lambda|$. Then $|\lambda|^{-1} \lim_n \|T^n\|^{\frac{1}{n}} \le 1$, and so $\lim_n \|T^n\|^{\frac{1}{n}} \le |\lambda|$, for every $\lambda \in \mathbb{C}$ such that $r_\sigma(T) < |\lambda|$. Thus $\lim_n \|T^n\|^{\frac{1}{n}} \le r_\sigma(T) + \varepsilon$ for all $\varepsilon > 0$. Hence

$$\lim_n \|T^n\|^{\frac{1}{n}} \;\le\; r_\sigma(T). \qquad \square$$

What Proposition 6.21 says is that $r_\sigma(T) = r(T)$, where $r(T)$ is the limit of the numerical sequence $\{\|T^n\|^{\frac{1}{n}}\}$ (whose existence was proved in Lemma 6.8). We shall then adopt one and the same notation (the simplest, of course) for both of them: the limit of $\{\|T^n\|^{\frac{1}{n}}\}$ and the spectral radius. Thus, from now on, the spectral radius of an operator $T \in \mathcal{B}[\mathcal{X}]$ on a complex Banach space $\mathcal{X}$ will be denoted by $r(T)$:

$$r(T) \;=\; \sup_{\lambda \in \sigma(T)} |\lambda| \;=\; \max_{\lambda \in \sigma(T)} |\lambda| \;=\; \lim_n \|T^n\|^{\frac{1}{n}}.$$

Remarks: Thus *a normaloid operator on a complex Banach space is precisely an operator whose norm coincides with the spectral radius*. Recall that in a complex Hilbert space $\mathcal{H}$ every normal operator is normaloid, and so is every nonnegative operator. Since T^*T is always nonnegative, it follows that

$$r(T^*T) \,=\, r(TT^*) \,=\, \|T^*T\| \,=\, \|TT^*\| \,=\, \|T\|^2 \,=\, \|T^*\|^2$$

for every $T \in \mathcal{B}[\mathcal{H}]$ (cf. Proposition 5.65), which is an especially useful formula for computing the norm of operators on a Hilbert space. Also note that an operator T on a Banach space *is normaloid if and only if there exists* $\lambda \in \sigma(T)$ *such that* $|\lambda| = \|T\|$. However, for operators on a Hilbert space such a λ can never be in the residual spectrum. Explicitly, for every operator $T \in \mathcal{B}[\mathcal{H}]$,

$$\sigma_R(T) \,\subseteq\, \big\{\lambda \in \mathbb{C}\colon\ |\lambda| < \|T\|\big\}.$$

(Indeed, if $\lambda \in \sigma_R(T) = \sigma_P(T^*)^* \backslash \sigma_P(T)$, then there exists $0 \neq x \in \mathcal{H}$ such that $0 < \|Tx - \lambda x\|^2 \,=\, \|Tx\|^2 - 2\mathrm{Re}\,\langle \overline{\lambda}x\,; T^*x\rangle + |\lambda|^2\|x\|^2 \,=\, \|Tx\|^2 - |\lambda|^2\|x\|^2$, and so $|\lambda| < \|T\|$.) Moreover, as a consequence of the preceding results (see also the remarks that succeed Corollary 6.20), for all operators $S, T \in \mathcal{B}[\mathcal{X}]$,

$$r(\alpha T) \,=\, |\alpha| r(T) \quad \text{ for every } \quad \alpha \in \mathbb{C},$$

and

$$r(ST) \,=\, r(TS).$$

If $T \in \mathcal{B}[\mathcal{H}]$, where $\mathcal{H}$ is a complex Hilbert space $\mathcal{H}$, then

$$r(T^*) \,=\, r(T)$$

and, if $Q \in \mathcal{B}[\mathcal{H}]$ is a nonnegative operator, then

$$r(Q^{\frac{1}{2}}) \,=\, r(Q)^{\frac{1}{2}}.$$

An important application of the Gelfand–Beurling formula ensures that *an operator T is uniformly stable if and only if* $r(T) < 1$. In fact, there exists in the current literature a large collection of equivalent conditions for uniform stability. We shall consider below just a few of them.

Proposition 6.22. *Let $T \in \mathcal{B}[\mathcal{X}]$ be an operator on a complex Banach space $\mathcal{X}$. The following assertions are pairwise equivalent.*

(a) $T^n \xrightarrow{u} O$.

(b) $r(T) < 1$.

(c) $\|T^n\| \leq \beta\alpha^n$ *for every* $n \geq 0$, *for some* $\beta \geq 1$ *and some* $\alpha \in (0,1)$.

(d) $\sum_{n=0}^{\infty} \|T^n\|^p < \infty$ *for an arbitrary* $p > 0$.

(e) $\sum_{n=0}^{\infty} \|T^n x\|^p < \infty$ *for all* $x \in \mathcal{X}$, *for an arbitrary* $p > 0$.

Proof. Since $r(T)^n = r(T^n) \leq \|T^n\|$ for every $n \geq 0$, it follows that (a)⇒(b). Suppose $r(T) < 1$ and take any $\alpha \in (r(T), 1)$. The Gelfand–Beurling formula says that $\lim_n \|T^n\|^{\frac{1}{n}} = r(T)$. Thus there is an integer $n_\alpha \geq 1$ such that $\|T^n\| \leq \alpha^n$ for every $n \geq n_\alpha$, and so (b)⇒(c) with $\beta = \max_{0 \leq n \leq n_\alpha} \|T^n\| \alpha^{-n_\alpha}$. It is trivially verified that (c)⇒(d)⇒(e). If (e) holds, then $\sup_n \|T^n x\| < \infty$ for every $x \in \mathcal{X}$, and hence $\sup_n \|T^n\| < \infty$ by the Banach–Steinhaus Theorem (Theorem 4.43). Moreover, for $m \geq 1$ and $p > 0$ arbitrary,

$$\|m^{\frac{1}{p}} T^m x\|^p = \sum_{n=0}^{m-1} \|T^{m-n} T^n x\|^p \leq \big(\sup_n \|T^n\|\big)^p \sum_{n=0}^{\infty} \|T^n x\|^p.$$

Thus $\sup_m \|m^{\frac{1}{p}} T^m x\| < \infty$ for every $x \in \mathcal{X}$ whenever (e) holds true. Since $m^{\frac{1}{p}} T^m \in \mathcal{B}[\mathcal{X}]$ for each $m \geq 1$, it follows that $\sup_m \|m^{\frac{1}{p}} T^m\| < \infty$ by using the Banach–Steinhaus Theorem again. Hence

$$0 \leq \|T^n\| \leq n^{-\frac{1}{p}} \sup_m \|m^{\frac{1}{p}} T^m\|$$

for every $n \geq 1$, so that $\|T^n\| \to 0$ as $n \to \infty$. Therefore, (e)⇒(a). □

Remark: If an operator is similar to a strict contraction, then, by the above proposition, it is uniformly stable. Indeed, let $\mathcal{X}$ and $\mathcal{Y}$ be complex Banach spaces and take any $M \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$. Since similarity preserves the spectrum (and so the spectral radius — see Problem 6.10), it follows that

$$r(T) = r(MTM^{-1}) \leq \|MTM^{-1}\|.$$

Hence, if $T \in \mathcal{B}[\mathcal{X}]$ is similar to a strict contraction (i.e., if $\|MTM^{-1}\| < 1$ for some $M \in \mathcal{G}[\mathcal{X}, \mathcal{Y}]$), then $r(T) < 1$ or, equivalently, $T^n \xrightarrow{u} O$. There are several different ways to verify the converse. One of them uses a result from model theory for Hilbert space operators (see the references at the end of this chapter), yielding another formula for the spectral radius, which reads as follows. If $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces, and if $T \in \mathcal{B}[\mathcal{H}]$, then

$$r(T) = \inf_{M \in \mathcal{G}[\mathcal{H}, \mathcal{K}]} \|MTM^{-1}\|.$$

Thus $r(T) < 1$ if and only if $\|MTM^{-1}\| < 1$ for some $M \in \mathcal{G}[\mathcal{H}, \mathcal{K}]$. Equivalently (cf. Proposition 6.22), *a Hilbert space operator is uniformly stable if and only if it is similar to a strict contraction.*

The next result extends the von Neumann expansion of Problem 4.47. Recall that an infinite series $\sum_{k=0}^{\infty} (\frac{T}{\lambda})^k$ is said to converge uniformly or strongly if the sequence of partial sums $\big\{\sum_{k=0}^{n} (\frac{T}{\lambda})^k\big\}_{n=0}^{\infty}$ converges uniformly or strongly, and $\sum_{k=0}^{\infty} (\frac{T}{\lambda})^k \in \mathcal{B}[\mathcal{X}]$ denotes its uniform or strong limit, respectively.

Corollary 6.23. *Let $\mathcal{X}$ be a complex Banach space. Take any operator T in $\mathcal{B}[\mathcal{X}]$ and any nonzero complex number λ.*

(a) $r(T) < |\lambda|$ *if and only if* $\sum_{k=0}^{\infty}(\frac{T}{\lambda})^k$ *converges uniformly. In this case,* λ *lies in* $\rho(T)$ *and* $(\lambda I - T)^{-1} = \frac{1}{\lambda}\sum_{k=0}^{\infty}(\frac{T}{\lambda})^k$.

(b) *If* $r(T) = |\lambda|$ *and* $\sum_{k=0}^{\infty}(\frac{T}{\lambda})^k$ *converges strongly, then* λ *lies in* $\rho(T)$ *and* $(\lambda I - T)^{-1} = \frac{1}{\lambda}\sum_{k=0}^{\infty}(\frac{T}{\lambda})^k$.

(c) *If* $|\lambda| < r(T)$, *then* $\sum_{k=0}^{\infty}(\frac{T}{\lambda})^k$ *does not converge strongly.*

Proof. If $\sum_{k=0}^{\infty}(\frac{T}{\lambda})^k$ converges uniformly, then $(\frac{T}{\lambda})^n \xrightarrow{u} O$ (cf. Problem 4.7), and hence $|\lambda|^{-1}r(T) = r(\frac{T}{\lambda}) < 1$ by Proposition 6.22. Conversely, if $r(T) < |\lambda|$, then $\lambda \in \rho(T)$ so that $(\lambda I - T) \in \mathcal{G}[\mathcal{X}]$, and $r(\frac{T}{\lambda}) = |\lambda|^{-1}r(T) < 1$. Therefore, $\{(\frac{T}{\lambda})^n\}$ is an absolutely summable sequence in $\mathcal{B}[\mathcal{X}]$ by Proposition 6.22. Now follow the steps of Problem 4.47 to conclude all the properties of item (a). If $\sum_{k=0}^{\infty}(\frac{T}{\lambda})^k$ converges strongly, then $(\frac{T}{\lambda})^n x \to 0$ in $\mathcal{X}$ for every $x \in \mathcal{X}$ (Problem 4.7 again) so that $\sup_n \|(\frac{T}{\lambda})^n x\| < \infty$ for every $x \in \mathcal{X}$. Then $\sup_n \|(\frac{T}{\lambda})^n\| < \infty$ by the Banach–Steinhaus Theorem (i.e., the operator $\frac{T}{\lambda}$ is power bounded), and hence $|\lambda|^{-1}r(T) = r(\frac{T}{\lambda}) \le 1$. This proves assertion (c). Moreover,

$$(\lambda I - T)\tfrac{1}{\lambda}\sum_{k=0}^{n}\big(\tfrac{T}{\lambda}\big)^k \;=\; \tfrac{1}{\lambda}\sum_{k=0}^{n}\big(\tfrac{T}{\lambda}\big)^k(\lambda I - T) \;=\; I - \big(\tfrac{T}{\lambda}\big)^{n+1} \xrightarrow{s} I.$$

Thus $(\lambda I - T)^{-1} = \frac{1}{\lambda}\sum_{k=0}^{\infty}(\frac{T}{\lambda})^k$, where $\sum_{k=0}^{\infty}(\frac{T}{\lambda})^k \in \mathcal{B}[\mathcal{X}]$ is the strong limit of the sequence $\{\sum_{k=0}^{n}(\frac{T}{\lambda})^k\}_{n=0}^{\infty}$, which concludes the proof of (b). □

6.4 Numerical Radius

The *numerical range* of an operator T acting on a complex Hilbert space $\mathcal{H} \neq \{0\}$ is the (nonempty) set

$$W(T) \;=\; \big\{\lambda \in \mathbb{C}\colon\ \lambda = \langle Tx\,;x\rangle \text{ for some } \|x\| = 1\big\}.$$

It can be shown that $W(T)$ is always convex in $\mathbb{C}$ and, clearly,

$$W(T^*) \;=\; W(T)^*.$$

Proposition 6.24. $\sigma_P(T) \cup \sigma_R(T) \subseteq W(T)$ *and* $\sigma(T) \subseteq W(T)^-$.

Proof. Take $T \in \mathcal{B}[\mathcal{H}]$, where $\mathcal{H} \neq \{0\}$ is a complex Hilbert space.

(a) If $\lambda \in \sigma_P(T)$, then there exists a unit vector x in $\mathcal{H}$ (i.e., there exists $x \in \mathcal{H}$ with $\|x\| = 1$) such that $Tx = \lambda x$. Thus $\langle Tx\,;x\rangle = \lambda\|x\|^2 = \lambda$, which means that $\lambda \in W(T)$. If $\lambda \in \sigma_R(T)$, then $\overline{\lambda} \in \sigma_P(T^*)$ by Proposition 6.17, and hence $\overline{\lambda} \in W(T^*)$. Therefore, $\lambda \in W(T)$.

(b) If $\lambda \in \sigma_{AP}(T)$, then there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\|(\lambda I - T)x_n\| \to 0$ by Proposition 6.15. Thus

$$0 \;\le\; |\lambda - \langle Tx_n\,;x_n\rangle| \;=\; |\langle(\lambda I - T)x_n\,;x_n\rangle| \;\le\; \|(\lambda I - T)x_n\| \;\to\; 0,$$

so that $\langle Tx_n\, x_n\rangle \to \lambda$. Since each $\langle Tx_n\,;x_n\rangle$ lies in $W(T)$, it follows by the Closed Set Theorem (Theorem 3.30) that $\lambda \in W(T)^-$. Hence

$$\sigma_{AP}(T) \;\subseteq\; W(T)^-,$$

and therefore $\sigma(T) = \sigma_R(T) \cup \sigma_{AP}(T) \subseteq W(T)^-$ according to item (a). □

The *numerical radius* of $T \in \mathcal{B}[\mathcal{H}]$ is the number

$$w(T) \;=\; \sup_{\lambda\in W(T)} |\lambda| \;=\; \sup_{\|x\|=1} |\langle Tx\,;x\rangle|.$$

Note that

$$w(T^*) \;=\; w(T) \quad \text{and} \quad w(T^*T) \;=\; \|T\|^2.$$

Unlike the spectral radius, the numerical radius is a norm on $\mathcal{B}[\mathcal{H}]$. That is, $0 \le w(T)$ for every $T \in \mathcal{B}[\mathcal{H}]$ and $0 < w(T)$ if $T \neq O$, $w(\alpha T) = |\alpha| w(T)$, and $w(T+S) \le w(T) + w(S)$ for every $\alpha \in \mathbb{C}$ and every $S, T \in \mathcal{B}[\mathcal{H}]$. *Warning*: The numerical radius is a norm on $\mathcal{B}[\mathcal{H}]$ that does not have the operator norm property, which means that the inequality $w(ST) \le w(S)w(T)$ is *not* true for all operators $S, T \in \mathcal{B}[\mathcal{H}]$. However, the *power inequality* holds:

$$w(T^n) \;\le\; w(T)^n$$

for all $T \in \mathcal{B}[\mathcal{H}]$ and every positive integer n — the proof is tricky. Nevertheless, the numerical radius is a norm equivalent to the (induced uniform) operator norm of $\mathcal{B}[\mathcal{H}]$ and dominates the spectral radius, as follows.

Proposition 6.25. $0 \le r(T) \le w(T) \le \|T\| \le 2w(T)$.

Proof. Since $\sigma(T) \subseteq W(T)^-$, we get $r(T) \le w(T)$. Moreover,

$$w(T) \;=\; \sup_{\|x\|=1} |\langle Tx\,;x\rangle| \;\le\; \sup_{\|x\|=1} \|Tx\| \;=\; \|T\|.$$

Now use Problem 5.3, and recall that

$$|\langle Tz\,;z\rangle| \;\le\; \sup_{\|u\|=1} |\langle Tu\,;u\rangle| \|z\|^2 \;=\; w(T)\|z\|^2$$

for every $z \in \mathcal{H}$ (because $\langle Tz\,;z\rangle = \langle T\frac{z}{\|z\|}\,;\frac{z}{\|z\|}\rangle\|z\|^2$ for every nonzero $z \in \mathcal{H}$), and apply the parallelogram law, to get

$$\begin{aligned}
|\langle Tx\,;y\rangle| \;&\le\; \tfrac{1}{4}\big(|\langle T(x+y)\,;(x+y)\rangle| + |\langle T(x-y)\,;(x-y)\rangle| \\
&\quad + |\langle T(x+iy)\,;(x+iy)\rangle| + |\langle T(x-iy)\,;(x-iy)\rangle|\big) \\
&\le\; \tfrac{1}{4}w(T)\big(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2\big) \\
&=\; w(T)\big(\|x\|^2 + \|y\|^2\big) \;\le\; 2w(T)
\end{aligned}$$

whenever $\|x\| = \|y\| = 1$. Therefore, according to Corollary 5.71,

$$\|T\| \;=\; \sup_{\|x\|=\|y\|=1} |\langle Tx\,;y\rangle| \;\le\; 2w(T). \qquad \square$$

An operator $T \in \mathcal{B}[\mathcal{H}]$ is *spectraloid* if $r(T) = w(T)$. The next result is a straightforward application of the previous proposition.

Corollary 6.26. *Every normaloid operator is spectraloid.*

Indeed, $r(T) = \|T\|$ implies $r(T) = w(T)$ by Proposition 6.25. However, Proposition 6.25 also ensures that $r(T) = \|T\|$ implies $w(T) = \|T\|$, so that $w(T) = \|T\|$ is a property of every normaloid operator on $\mathcal{H}$. What emerges as a nice surprise is that this property can be viewed as a third definition of a normaloid operator on a complex Hilbert space.

Proposition 6.27. *$T \in \mathcal{B}[\mathcal{H}]$ is normaloid if and only if $w(T) = \|T\|$.*

Proof. The easy half of the proof was presented above. Suppose $T \neq O$ (avoiding trivialities). $W(T)^-$ is compact in $\mathbb{C}$ (since it is clearly bounded). Thus

$$\max_{\lambda \in W(T)^-} |\lambda| \;=\; \sup_{\lambda \in W(T)^-} |\lambda| \;=\; \sup_{\lambda \in W(T)} |\lambda| \;=\; w(T),$$

and so there exists a $\lambda \in W(T)^-$ such that $\lambda = w(T)$. If $w(T) = \|T\|$, then $\lambda = \|T\|$. Since $W(T)$ is always nonempty, it follows by Proposition 3.32 that there exists a sequence $\{\lambda_n\}$ in $W(T)$ that converges to λ. In other words, there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ ($\|x_n\| = 1$ for each n) such that $\lambda_n = \langle Tx_n \,; x_n \rangle \to \lambda$, where $|\lambda| = \|T\| \neq 0$. Set $S = \lambda^{-1}T \in \mathcal{B}[\mathcal{H}]$ so that

$$\langle Sx_n \,; x_n \rangle \;\to\; 1.$$

Claim. $\|Sx_n\| \to 1$ and $\mathrm{Re}\,\langle Sx_n \,; x_n \rangle \to 1$.

Proof. $|\langle Sx_n \,; x_n \rangle| \le \|Sx_n\| \le \|S\| = 1$ for each n. But $\langle Sx_n \,; x_n \rangle \to 1$ implies that $|\langle Sx_n \,; x_n \rangle| \to 1$ (and hence $\|Sx_n\| \to 1$) and also that $\mathrm{Re}\,\langle Sx_n \,; x_n \rangle \to 1$. Both arguments follow by continuity. □

Then $\|(I - S)x_n\|^2 = \|Sx_n - x_n\|^2 = \|Sx_n\|^2 - 2\mathrm{Re}\,\langle Sx_n \,; x_n \rangle + \|x_n\|^2 \to 0$ so that $1 \in \sigma_{AP}(S) \subseteq \sigma(S)$ (cf. Proposition 6.15). Hence $r(S) \ge 1$ and $r(T) = r(\lambda S) = |\lambda| r(S) \ge |\lambda| = \|T\|$, which implies $r(T) = \|T\|$ (since $r(T) \le \|T\|$ for every operator T). □

Therefore, the class of normaloid operators on $\mathcal{H}$ coincides with the class of all operators $T \in \mathcal{B}[\mathcal{H}]$ for which

$$\|T\| \;=\; \sup_{\|x\|=1} |\langle Tx \,; x \rangle|.$$

This includes the normal operators and, in particular, the self-adjoint operators (see Proposition 5.78). This includes the isometries too. In fact, every isometry is quasinormal, and hence normaloid. Thus

$$r(V) \;=\; w(V) \;=\; \|V\| \;=\; 1 \quad \text{whenever} \quad V \in \mathcal{B}[\mathcal{H}] \text{ is an isometry.}$$

(The above identity can be directly verified by Propositions 6.21 and 6.25, once $\|V^n\| = 1$ for every positive integer n — cf. Proposition 4.37.)

Remark: Since an operator T is normaloid if (and only if) $r(T) = \|T\|$, it follows that *the unique normaloid quasinilpotent is the null operator*. In other words, if T normaloid and $r(T) = 0$ (i.e., $\sigma(T) = \{0\}$), then $T = O$. In particular, *the unique normal* (*or hyponormal*) *quasinilpotent is the null operator*. More is true. In fact, *the unique spectraloid quasinilpotent is the null operator*. Proof: If $w(T) = r(T) = 0$, then $T = O$ by Proposition 6.25.

Corollary 6.28. *If there exists $\lambda \in W(T)$ such that $|\lambda| = \|T\|$, then T is normaloid and $\lambda \in \sigma_P(T)$. In other words, if there exists a unit vector x such that $\|T\| = |\langle Tx\,;x\rangle|$, then $r(T) = w(T) = \|T\|$ and $\langle Tx\,;x\rangle \in \sigma_P(T)$.*

Proof. If $\lambda \in W(T)$ is such that $|\lambda| = \|T\|$, then $w(T) = \|T\|$ (see Proposition 6.25) so that T is normaloid by Proposition 6.27. Moreover, since $\lambda = \langle Tx\,;x\rangle$ for some unit vector x, it follows that $\|T\| = |\lambda| = |\langle Tx\,;x\rangle| \le \|Tx\|\|x\| \le \|T\|$, and hence $|\langle Tx\,;x\rangle| = \|Tx\|\|x\|$. Then $Tx = \alpha x$ for some $\alpha \in \mathbb{C}$ (cf. Problem 5.2) so that $\alpha \in \sigma_P(T)$. But $\alpha = \alpha\|x\|^2 = \langle \alpha x\,;x\rangle = \langle Tx\,;x\rangle = \lambda$. □

Remark: Using the inequality $\|T^n\| \le \|T\|^n$, which holds for every operator T, we have shown in Proposition 6.9 that T is normaloid if and only if $\|T^n\| = \|T\|^n$ for every $n \ge 0$. Using the inequality $w(T^n) \le w(T)^n$, which also holds for every operator T, we can show that T *is spectraloid if and only if* $w(T^n) = w(T)^n$ *for every* $n \ge 0$. Indeed, by Corollary 6.20 and Propositions 6.25,

$$r(T)^n \;=\; r(T^n) \;\le\; w(T^n) \;\le\; w(T)^n \quad \text{for every} \quad n \ge 0.$$

Hence $r(T) = w(T)$ implies $w(T^n) = w(T)^n$. Conversely, since

$$w(T^n)^{\frac{1}{n}} \;\le\; \|T^n\|^{\frac{1}{n}} \;\to\; r(T) \;\le\; w(T),$$

if follows that $w(T^n) = w(T)^n$ implies $r(T) = w(T)$.

6.5 Examples of Spectra

Every closed and bounded subset of the complex plane (i.e., every compact subset of $\mathbb{C}$) is the spectrum of some operator.

Example 6.B. Take $T \in \mathcal{B}[\mathcal{X}]$ on a *finite-dimensional complex normed space* $\mathcal{X}$. Thus $\mathcal{X}$ and its linear manifolds are all Banach spaces (Corollaries 4.28 and 4.29). Moreover, $\mathcal{N}(\lambda I - T) = \{0\}$ if and only if $(\lambda I - T) \in \mathcal{G}[\mathcal{X}]$ (cf. Problem 4.38(c)). That is, $\mathcal{N}(\lambda I - T) = \{0\}$ if and only if $\lambda \in \rho(T)$, and hence

$$\sigma_C(T) \;=\; \sigma_R(T) \;=\; \varnothing.$$

Furthermore, since $\mathcal{R}(\lambda I - T)$ is a subspace of $\mathcal{X}$ for every $\lambda \in \mathbb{C}$, it also follows that $\sigma_{P_2}(T) = \sigma_{P_3}(T) = \varnothing$ (see diagram of Section 6.2). Finally, if $\mathcal{N}(\lambda I - T) \ne \{0\}$, then $\mathcal{R}(\lambda I - T) \ne \mathcal{X}$ whenever $\mathcal{X}$ is finite dimensional (cf. Problems 2.6(a) and 2.17), and so $\sigma_{P_1}(T) = \varnothing$. Therefore,

$$\sigma(T) \;=\; \sigma_P(T) \;=\; \sigma_{P_4}(T).$$

Example 6.C. Let $T \in \mathcal{B}[\mathcal{H}]$ be a *diagonalizable operator* on a complex (separable infinite-dimensional) Hilbert space $\mathcal{H}$. That is, according to Problem 5.17, there exists an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ and a sequence $\{\lambda_k\}_{k=1}^{\infty}$ in ℓ_+^∞ such that, for every $x \in \mathcal{H}$,

$$Tx \;=\; \sum_{k=1}^{\infty} \lambda_k \langle x\,; e_k\rangle e_k.$$

Take an arbitrary $\lambda \in \mathbb{C}$ and note that $(\lambda I - T) \in \mathcal{B}[\mathcal{H}]$ is again a diagonalizable operator. Indeed, $(\lambda I - T)x = \sum_{k=1}^{\infty}(\lambda - \lambda_k)\langle x\,; e_k\rangle e_k$ for every $x \in \mathcal{H}$. Since $\mathcal{N}(\lambda I - T) = \{0\}$ if and only if $\lambda \ne \lambda_k$ for every $k \ge 1$ (i.e., there exists $(\lambda I - T)^{-1} \in \mathcal{L}[\mathcal{R}(\lambda I - T), \mathcal{H}]$ if and only if $\lambda - \lambda_k \ne 0$ for every $k \ge 1$ — cf. Problem 5.17), it follows that

$$\sigma_P(T) \;=\; \big\{\lambda \in \mathbb{C}\colon\ \lambda = \lambda_k \text{ for some } k \ge 1\big\}.$$

Similarly, since $T^* \in \mathcal{B}[\mathcal{H}]$ also is a diagonalizable operator, given by $T^*x = \sum_{k=1}^{\infty}\overline{\lambda}_k\langle x\,; e_k\rangle e_k$ for every $x \in \mathcal{H}$ (e.g., see Problem 5.27(c)), we get

$$\sigma_P(T^*) \;=\; \big\{\lambda \in \mathbb{C}\colon\ \lambda = \overline{\lambda}_k \text{ for some } k \ge 1\big\}.$$

Then

$$\sigma_R(T) \;=\; \sigma_P(T^*)^* \backslash \sigma_P(T) \;=\; \varnothing.$$

Moreover, $\lambda \in \rho(T)$ if and only if $\lambda I - T$ lies in $\mathcal{G}[\mathcal{H}]$; equivalently, if and only if $\inf_k |\lambda - \lambda_k| > 0$ (Problem 5.17). Thus

$$\sigma(T) \;=\; \sigma_P(T) \cup \sigma_C(T) \;=\; \big\{\lambda \in \mathbb{C}\colon\ \inf_k |\lambda - \lambda_k| = 0\big\},$$

and hence $\sigma(T)\backslash\sigma_P(T)$ is the set of all cluster points of the sequence $\{\lambda_k\}_{k=1}^{\infty}$ (i.e., the set of all accumulation points of the set $\{\lambda_k\}_{k=1}^{\infty}$):

$$\sigma_C(T) \;=\; \big\{\lambda \in \mathbb{C}\colon\ \inf_k |\lambda - \lambda_k| = 0 \ \text{ and } \ \lambda \ne \lambda_k \text{ for every } k \ge 1\big\}.$$

Note that $\sigma_{P_1}(T) = \sigma_{P_2}(T) = \varnothing$ (reason: T^* is a diagonalizable operator so that $\sigma_R(T^*) = \varnothing$ — see Proposition 6.17). If $\lambda_j \in \sigma_P(T)$ also is an accumulation point of $\sigma_P(T)$, then it lies in $\sigma_{P_3}(T)$; otherwise (i.e., if it is an isolated point of $\sigma_P(T)$), it lies in $\sigma_{P_4}(T)$. Indeed, consider a new set $\{\lambda_k\}'$ without this point λ_j and the associated diagonalizable operator T' so that $\lambda_j \in \sigma_C(T')$, and hence $\mathcal{R}(\lambda_j I - T')$ is not closed, which means that $\mathcal{R}(\lambda_j I - T)$ is not closed. If $\{\lambda_k\}$ is a constant sequence, say $\lambda_k = \mu$ for all k, then $T = \mu I$ is a *scalar operator* and, in this case,

$$\sigma(\mu I) \;=\; \sigma_P(\mu I) \;=\; \sigma_{P_4}(\mu I) \;=\; \{\mu\}.$$

Recall that $\mathbb{C}$ (with its usual metric) is a separable metric space (Example 3.P). Thus it includes a countable dense subset, and so does every compact

subset Σ of $\mathbb{C}$. Let Λ be any countable dense subset of Σ, and let $\{\lambda_k\}_{k=1}^{\infty}$ be an enumeration of it (if Σ is finite, then set $\lambda_k = 0$ for all $k > \#\Sigma$). Observe that $\sup_k|\lambda_k| < \infty$ as Σ is bounded. Consider a diagonalizable operator T in $\mathcal{B}[\mathcal{H}]$ such that $Tx = \sum_{k=1}^{\infty}\lambda_k\langle x\,;e_k\rangle e_k$ for every $x \in \mathcal{H}$. As we have just seen,

$$\sigma(T) \,=\, \Lambda^- \,=\, \Sigma.$$

That is, $\sigma(T)$ is the set of all points of adherence of $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$, which means the closure of Λ. This confirms the statement that introduced this section. Precisely, *every closed and bounded subset of the complex plane is the spectrum of some diagonalizable operator on* $\mathcal{H}$.

Example 6.D. Let $\mathbb{D}$ and $\mathbb{T} = \partial\mathbb{D}$ denote the open unit disk and the unit circle in the complex plane centered at the origin, respectively. In this example we shall characterize each part of the spectrum of a *unilateral shift* of arbitrary multiplicity. Let S_+ be a unilateral shift acting on a (complex) Hilbert space $\mathcal{H}$, and let $\{\mathcal{H}_k\}_{k=0}^{\infty}$ be the underlying sequence of orthogonal subspaces of $\mathcal{H} = \bigoplus_{k=0}^{\infty}\mathcal{H}_k$ (Problem 5.29). Recall that

$$S_+x \,=\, 0 \oplus \textstyle\bigoplus_{k=1}^{\infty} U_k\, x_{k-1} \quad \text{ and } \quad S_+^*x \,=\, \bigoplus_{k=0}^{\infty} U_{k+1}^* x_{k+1}$$

for every $x = \bigoplus_{k=0}^{\infty}x_k$ in $\mathcal{H} = \bigoplus_{k=0}^{\infty}\mathcal{H}_k$, with 0 denoting the origin of $\mathcal{H}_0$, where $\{U_{k+1}\}_{k=0}^{\infty}$ is an arbitrary sequence of unitary transformations of $\mathcal{H}_k$ onto $\mathcal{H}_{k+1}$, $U_{k+1}\colon\mathcal{H}_k \to \mathcal{H}_{k+1}$. Since a unilateral shift is an isometry, we get

$$r(S_+) \,=\, 1.$$

Take any $x = \bigoplus_{k=0}^{\infty}x_k \in \mathcal{H}$ and an arbitrary $\lambda \in \mathbb{C}$. If $x \in \mathcal{N}(\lambda I - S_+)$, then $\lambda x_0 \oplus \bigoplus_{k=1}^{\infty}\lambda x_k \,=\, 0 \oplus \bigoplus_{k=1}^{\infty}U_k x_{k-1}$. Hence $\lambda x_0 = 0$ and, for every $k \geq 0$, $\lambda x_{k+1} \,=\, U_{k+1}x_k$. If $\lambda = 0$, then $x = 0$. If $\lambda \neq 0$, then $x_0 = 0$ and $x_{k+1} = \lambda^{-1}U_{k+1}x_k$, so that $\|x_0\| = 0$ and $\|x_{k+1}\| = |\lambda|^{-1}\|x_k\|$, for each $k \geq 0$. Thus $\|x_k\| = |\lambda|^{-k}\|x_0\| = 0$ for every $k \geq 0$. Hence $x = 0$, and so $\mathcal{N}(\lambda I - S_+) = \{0\}$ for all $\lambda \in \mathbb{C}$. That is,

$$\sigma_P(S_+) \,=\, \varnothing.$$

Now take any $x_0 \neq 0$ in $\mathcal{H}_0$ and any $\lambda \in \mathbb{D}$. Consider the sequence $\{x_k\}_{k=0}^{\infty}$, with each x_k in $\mathcal{H}_k$, recursively defined by $x_{k+1} = \lambda U_{k+1}x_k$, so that $\|x_{k+1}\| = |\lambda|\|x_k\|$ for every $k \geq 0$. Then $\|x_k\| = |\lambda|^k\|x_0\|$ for every $k \geq 1$, and hence $\sum_{k=0}^{\infty}\|x_k\|^2 \,=\, \|x_0\|^2(1 + \sum_{k=1}^{\infty}|\lambda|^{2k}) < \infty$, which implies that the nonzero vector $x = \bigoplus_{k=0}^{\infty}x_k$ lies in $\bigoplus_{k=0}^{\infty}\mathcal{H}_k = \mathcal{H}$. Moreover, since $\lambda x_k = U_{k+1}^*x_{k+1}$ for each $k \geq 0$, it follows that $\lambda x = S_+^*x$, and so $0 \neq x \in \mathcal{N}(\lambda I - S_+^*)$. Therefore, $\mathcal{N}(\lambda I - S_+^*) \neq \{0\}$ for all $\lambda \in \mathbb{D}$. Equivalently, $\mathbb{D} \subseteq \sigma_P(S_+^*)$. On the other hand, if $\lambda \in \sigma_P(S_+^*)$, then there exists $0 \neq x = \bigoplus_{k=0}^{\infty}x_k \in \bigoplus_{k=0}^{\infty}\mathcal{H}_k = \mathcal{H}$ such that $S_+^*x = \lambda x$. Hence $U_{k+1}^*x_{k+1} \,=\, \lambda x_k$ so that $\|x_{k+1}\| = |\lambda|\|x_k\|$ for each $k \geq 0$, and so $\|x_k\| = |\lambda|^k\|x_0\|$ for every $k \geq 1$. Thus $x_0 \neq 0$ (because $x \neq 0$) and $(1 + \sum_{k=1}^{\infty}|\lambda|^{2k})\|x_0\|^2 \,=\, \sum_{k=0}^{\infty}\|x_k\|^2 \,=\, \|x\|^2 < \infty$, which implies that $|\lambda| < 1$ (i.e., $\lambda \in \mathbb{D}$). So we may conclude that $\sigma_P(S_+^*) \subseteq \mathbb{D}$. Then

$$\sigma_P(S_+^*) = \mathbb{D}.$$

But the spectrum of any operator T on $\mathcal{H}$ is a closed set included in the disk $\{\lambda \in \mathbb{C}\colon |\lambda| \le r(T)\}$, which is the disjoint union of $\sigma_P(T)$, $\sigma_R(T)$, and $\sigma_C(T)$, where $\sigma_R(T) = \sigma_P(T^*)^* \backslash \sigma_P(T)$ (Proposition 6.17). Hence

$$\sigma_P(S_+) = \sigma_R(S_+^*) = \varnothing, \quad \sigma_R(S_+) = \sigma_P(S_+^*) = \mathbb{D}, \quad \sigma_C(S_+) = \sigma_C(S_+^*) = \mathbb{T}.$$

Example 6.E. The spectrum of a *bilateral shift* is simpler than that of a unilateral shift, since bilateral shifts are unitary (i.e., besides being isometries they are normal too). Let S be a bilateral shift of arbitrary multiplicity acting on a (complex) Hilbert space $\mathcal{H}$, and let $\{\mathcal{H}_k\}_{k=-\infty}^{\infty}$ be the underlying family of orthogonal subspaces of $\mathcal{H} = \bigoplus_{k=-\infty}^{\infty} \mathcal{H}_k$ (Problem 5.30) so that

$$Sx = \bigoplus_{k=-\infty}^{\infty} U_k\, x_{k-1} \quad \text{and} \quad S^*x = \bigoplus_{k=-\infty}^{\infty} U_{k+1}^* x_{k+1}$$

for every $x = \bigoplus_{k=-\infty}^{\infty} x_k$ in $\mathcal{H} = \bigoplus_{k=-\infty}^{\infty} \mathcal{H}_k$, where $\{U_k\}_{k=-\infty}^{\infty}$ is an arbitrary family of unitary transformations $U_{k+1}\colon \mathcal{H}_k \to \mathcal{H}_{k+1}$. Suppose there exists $\lambda \in \mathbb{T} \cap \rho(S)$ so that $\mathcal{R}(\lambda I - S) = \mathcal{H}$ and $|\lambda| = 1$. Take any $y_0 \ne 0$ in $\mathcal{H}_0$ and set $y_k = 0 \in \mathcal{H}_k$ for each $k \ne 0$. Now consider the vector $y = \bigoplus_{k=-\infty}^{\infty} y_k$ in $\mathcal{H} = \mathcal{R}(\lambda I - S)$ and let $x = \bigoplus_{k=-\infty}^{\infty} x_k \in \mathcal{H}$ be any inverse image of y under $\lambda I - S$; that is, $(\lambda I - S)x = y$. Since $y_0 \ne 0$ it follows that $y \ne 0$, and hence $x \ne 0$. On the other hand, since $y_k = 0$ for every $k \ne 0$, it also follows that $\lambda x_k = U_k x_{k-1} + y_k = U_k x_{k-1}$. Hence $\|x_k\| = \|x_{k-1}\|$ for every $k \ne 0$. Thus $\|x_j\| = \|x_{-1}\|$ for every $j \le -1$ and $\|x_j\| = \|x_0\|$ for every $j \ge 0$, and so $x = 0$ (since $\|x\|^2 = \sum_{k=-\infty}^{\infty} \|x_k\|^2 = \sum_{j=-\infty}^{-1} \|x_j\|^2 + \sum_{j=0}^{\infty} \|x_j\|^2 < \infty$). Thus the existence of a complex number λ in $\mathbb{T} \cap \rho(S)$ leads to a contradiction. Conclusion: $\mathbb{T} \cap \rho(S) = \varnothing$. That is, $\mathbb{T} \subseteq \sigma(S)$. Since S is unitary, it follows that $\sigma(S) \subseteq \mathbb{T}$ (according to Corollary 6.18(c)). Outcome:

$$\sigma(S) = \mathbb{T}.$$

Now take any pair $\{\lambda, x\}$ with λ in $\sigma(S)$ and $x = \bigoplus_{k=-\infty}^{\infty} x_k$ in $\mathcal{H}$. If x is in $\mathcal{N}(\lambda I - S)$, then $\bigoplus_{k=-\infty}^{\infty} \lambda x_k = \bigoplus_{k=-\infty}^{\infty} U_k x_{k-1}$ and so $\lambda x_k = U_k x_{k-1}$ for each k. Since $|\lambda| = 1$ (because $\sigma(S) = \mathbb{T}$), $\|x_k\| = \|x_{k-1}\|$ for each k. Hence $x = 0$ (since $\|x\|^2 = \sum_{k=-\infty}^{\infty} \|x_k\|^2$ is finite). Thus $\mathcal{N}(\lambda I - S) = \{0\}$ for all $\lambda \in \sigma(S)$. That is,

$$\sigma_P(S) = \varnothing.$$

But S is normal, so that $\sigma_R(S) = \varnothing$ (cf. Corollary 6.18(b)). Recalling that $\sigma(S^*) = \sigma(S)^*$ and $\sigma_C(S^*) = \sigma_C(S)^*$ (Proposition 6.17), we get

$$\sigma(S) = \sigma(S^*) = \sigma_C(S^*) = \sigma_C(S) = \mathbb{T}.$$

Consider a weighted sum of projections $D = \sum_k \alpha_k P_k$ on $\ell_+^2(\mathcal{H})$ or on $\ell^2(\mathcal{H})$, where $\{\alpha_k\}$ is a bounded family of scalars and $\mathcal{R}(P_k) \cong \mathcal{H}$ for all k. This is identified with an orthogonal direct sum of scalar operators $D = \bigoplus_k \alpha_k I$

(Problem 5.16), and is referred to as a *diagonal operator* on $\ell^2_+(\mathcal{H})$ or on $\ell^2(\mathcal{H})$, respectively. A *weighted shift* is the product of a shift and a diagonal operator. Such a definition implicitly assumes that the shift (unilateral or bilateral, of any multiplicity) acts on the direct sum of countably infinite copies of a single Hilbert space $\mathcal{H}$. Explicitly, a *unilateral weighted shift* on $\ell^2_+(\mathcal{H})$ is the product of a unilateral shift on $\ell^2_+(\mathcal{H})$ and a diagonal operator on $\ell^2_+(\mathcal{H})$. Similarly, a *bilateral weighted shift* on $\ell^2(\mathcal{H})$ is the product of a bilateral shift on $\ell^2(\mathcal{H})$ and a diagonal operator on $\ell^2(\mathcal{H})$. Diagonal operators acting on $\ell^2_+(\mathcal{H})$ and on $\ell^2(\mathcal{H})$, $D_+ = \bigoplus_{k=0}^{\infty} \alpha_k I$ and $D = \bigoplus_{k=-\infty}^{\infty} \alpha_k I$, where I is the identity on $\mathcal{H}$, are denoted by $D_+ = \operatorname{diag}(\{\alpha_k\}_{k=0}^{\infty})$ and $D = \operatorname{diag}(\{\alpha_k\}_{k=-\infty}^{\infty})$, respectively. Likewise, weighted shifts acting on $\ell^2_+(\mathcal{H})$ and on $\ell^2(\mathcal{H})$, $T_+ = S_+ D_+$ and $T = SD$, will be denoted by $T_+ = \operatorname{shift}(\{\alpha_k\}_{k=0}^{\infty})$ and $T = \operatorname{shift}(\{\alpha_k\}_{k=-\infty}^{\infty})$, respectively, whenever S_+ is the *canonical unilateral shift* on $\ell^2_+(\mathcal{H})$ and S is the *canonical bilateral shift* on $\ell^2(\mathcal{H})$ (see Problems 5.29 and 5.30).

Example 6.F. Let $\{\alpha_k\}_{k=0}^{\infty}$ be a bounded sequence in $\mathbb{C}$ such that

$$\alpha_k \neq 0 \text{ for every } k \geq 0 \quad \text{and} \quad \alpha_k \to 0 \text{ as } k \to \infty.$$

Consider the *unilateral weighted shift* $T_+ = \operatorname{shift}(\{\alpha_k\}_{k=0}^{\infty})$ on $\ell^2_+(\mathcal{H})$, where $\mathcal{H} \neq \{0\}$ is a complex Hilbert space. The operators T_+ and T_+^* are given by

$$T_+ x = S_+ D_+ x = 0 \oplus \bigoplus_{k=1}^{\infty} \alpha_{k-1} x_{k-1} \quad \text{and} \quad T_+^* x = D_+^* S_+^* x = \bigoplus_{k=0}^{\infty} \overline{\alpha}_k x_{k+1}$$

for every $x = \bigoplus_{k=0}^{\infty} x_k$ in $\ell^2_+(\mathcal{H}) = \bigoplus_{k=0}^{\infty} \mathcal{H}$, with 0 denoting the origin of $\mathcal{H}$. Applying the same argument used in Example 6.D to show that $\sigma(S_+) = \varnothing$, we get $\mathcal{N}(\lambda I - T_+) = \{0\}$ for all $\lambda \in \mathbb{C}$. Indeed, if $x = \bigoplus_{k=0}^{\infty} x_k$ lies in $\mathcal{N}(\lambda I - T_+)$, then $\lambda x_0 \oplus \bigoplus_{k=1}^{\infty} \lambda x_k = 0 \oplus \bigoplus_{k=1}^{\infty} \alpha_{k-1} x_{k-1}$ so that $\lambda x_0 = 0$ and $\lambda x_{k+1} = \alpha_k x_k$ for every $k \geq 0$. Thus $x = 0$ if $\lambda = 0$ (since $\alpha_k \neq 0$) and, if $\lambda \neq 0$, then $x_0 = 0$ and $\|x_{k+1}\| \leq \sup_k |\alpha_k| |\lambda|^{-1} \|x_k\|$ for every $k \geq 0$, which implies that $x = 0$. Thus

$$\sigma_P(T_+) = \varnothing.$$

Note that the vector $x = \bigoplus_{k=0}^{\infty} x_k$, with $0 \neq x_0 \in \mathcal{H}$ and $x_k = 0 \in \mathcal{H}$ for every $k \geq 1$, is in $\ell^2_+(\mathcal{H})$ but not in $\mathcal{R}(T_+)^- \subseteq \{0\} \oplus \bigoplus_{k=1}^{\infty} \mathcal{H}$. So $\mathcal{R}(T_+)^- \neq \ell^2_+(\mathcal{H})$, and hence $0 \in \sigma_{CP}(T_+)$. Since $\sigma_P(T_+) = \varnothing$, it follows that $0 \in \sigma_R(T_+)$. Then

$$\{0\} \subseteq \sigma_R(T_+).$$

However, if $\lambda \neq 0$, then $\mathcal{R}(\lambda I - T_+) = \ell^2_+(\mathcal{H})$. In fact, suppose $\lambda \neq 0$ and take any $y = \bigoplus_{k=0}^{\infty} y_k$ in $\ell^2_+(\mathcal{H})$. Set $x_0 = \lambda^{-1} y_0$ and, for each $k \geq 0$, $x_{k+1} = \lambda^{-1}(\alpha_k x_k + y_{k+1})$. Since $\alpha_k \to 0$, there exists a positive integer k_λ such that $\alpha = |\lambda|^{-1} \sup_{k \geq k_\lambda} |\alpha_k| \leq \frac{1}{2}$. Then $\|\alpha_{k+1} x_{k+1}\| \leq \alpha(\|\alpha_k x_k\| + \|y_{k+1}\|)$, so that $\|\alpha_{k+1} x_{k+1}\|^2 \leq \alpha^2(\|\alpha_k x_k\| + \|y_{k+1}\|)^2 \leq 2\alpha^2(\|\alpha_k x_k\|^2 + \|y_{k+1}\|^2)$, for each $k \geq k_\lambda$. Thus $\sum_{k=k_\lambda}^{\infty} \|\alpha_{k+1} x_{k+1}\|^2 \leq \frac{1}{2} \sum_{k=k_\lambda}^{\infty} \|\alpha_k x_k\|^2 + \frac{1}{2} \|y\|^2$, which implies that $\sum_{k=0}^{\infty} \|\alpha_k x_k\|^2 < \infty$, and hence $|\lambda|^2 (\sum_{k=0}^{\infty} \|x_{k+1}\|^2) \leq \sum_{k=0}^{\infty} (\|\alpha_k x_k\| + \|y_{k+1}\|)^2 \leq 2\big(\sum_{k=0}^{\infty} \|\alpha_k x_k\|^2 + \|y\|^2\big) < \infty$. Then $x = \bigoplus_{k=0}^{\infty} x_k$ lies in $\ell^2_+(\mathcal{H})$.

But $(\lambda I - T_+)x = \lambda x_0 \oplus \bigoplus_{k=1}^{\infty} \lambda x_k - \alpha_{k-1}x_{k-1} = y$, and so $y \in \mathcal{R}(\lambda I - T_+)$. Outcome: $\mathcal{R}(\lambda I - T_+) = \ell_+^2(\mathcal{H})$. Since $\mathcal{N}(\lambda I - T_+) = \{0\}$ for all $\lambda \in \mathbb{C}$, it then follows that $\lambda \in \rho(T_+)$ for every nonzero $\lambda \in \mathbb{C}$, and so $\sigma(T_+) = \sigma_R(T_+) = \{0\}$. Moreover, as $\sigma_{R_1}(T)$ is always an open set,

$$\sigma(T_+) \,=\, \sigma_R(T_+) \,=\, \sigma_{R_2}(T_+) \,=\, \{0\},$$

and hence

$$\sigma(T_+^*) \,=\, \sigma_P(T_+^*) \,=\, \sigma_{P_2}(T_+^*) \,=\, \{0\}.$$

This is our first instance of a quasinilpotent operator ($r(T_+) = 0$) that is not nilpotent ($\sigma_P(T_+) = \varnothing$). The next example exhibits another one. It is worth noticing that $\sigma(\mu I - T_+) = \{\mu - \lambda \in \mathbb{C}\colon\, \lambda \in \sigma(T_+)\} = \{\mu\}$ by the Spectral Mapping Theorem, and so $\sigma(\overline{\mu} I - T_+^*) = \{\overline{\mu}\}$. Moreover, if x is an eigenvector of T_+^*, then $T_+^* x = 0$ so that $(\overline{\mu} I - T_+^*)x = \overline{\mu} x$; that is, $\overline{\mu} \in \sigma_P(\overline{\mu} I - T_+^*)$. Thus

$$\sigma(\mu I - T_+) = \sigma_R(\mu I - T_+) = \{\mu\} \quad \text{and} \quad \sigma(\overline{\mu} I - T_+^*) = \sigma_P(\overline{\mu} I - T_+^*) = \{\overline{\mu}\}.$$

Example 6.G. Let $\{\alpha_k\}_{k=-\infty}^{\infty}$ be a bounded family in $\mathbb{C}$ such that

$$\alpha_k \neq 0 \;\text{ for every }\; k \in \mathbb{Z} \quad \text{and} \quad \alpha_k \to 0 \;\text{ as }\; |k| \to \infty,$$

and consider the *bilateral weighted shift* $T = \text{shift}(\{\alpha_k\}_{k=-\infty}^{\infty})$ on $\ell^2(\mathcal{H})$, where $\mathcal{H} \neq \{0\}$ is a complex Hilbert space. T and T^* are given by

$$T \,=\, SDx \,=\, \textstyle\bigoplus_{k=-\infty}^{\infty} \alpha_{k-1} x_{k-1} \quad \text{and} \quad T^* \,=\, D^* S^* x \,=\, \bigoplus_{k=-\infty}^{\infty} \overline{\alpha}_k x_{k+1}$$

for every $x = \bigoplus_{k=-\infty}^{\infty} x_k$ in $\ell^2(\mathcal{H}) = \bigoplus_{k=-\infty}^{\infty} \mathcal{H}$. Take an arbitrary $\lambda \in \mathbb{C}$. If $x = \bigoplus_{k=-\infty}^{\infty} x_k \in \mathcal{N}(\lambda I - T)$, then $\bigoplus_{k=-\infty}^{\infty} \lambda x_k - \alpha_{k-1} x_{k-1} = 0$, and hence $\lambda x_{k+1} = \alpha_k x_k$ for every $k \in \mathbb{Z}$. If $\lambda = 0$, then $x = 0$. Otherwise, if $\lambda \neq 0$, then $\|x_{k+1}\| \le |\lambda|^{-1} \sup_k |\alpha_k| \|x_k\|$ for every $k \in \mathbb{Z}$. But $\lim_{k \to -\infty} \|x_k\| = 0$ (since $\|x\|^2 = \sum_{k=-\infty}^{\infty} \|x_k\|^2 < \infty$) so that $x = 0$. Thus $\mathcal{N}(\lambda I - T) = \{0\}$ for all $\lambda \in \mathbb{C}$. That is,

$$\sigma_P(T) \,=\, \varnothing.$$

Take any vector $y = \bigoplus_{k=-\infty}^{\infty} y_k$ in $\ell^2(\mathcal{H})$ and any scalar $\lambda \neq 0$ in $\mathbb{C}$. Since $\alpha_k \to 0$ as $|k| \to \infty$, it follows that there exists a positive integer k_λ and a finite set $K_\lambda = \{k \in \mathbb{Z}\colon\, -k_\lambda \le k \le k_\lambda\}$ such that $\alpha = \sup_{k \in \mathbb{Z} \setminus K_\lambda} |\frac{\alpha_k}{\lambda}| \le \frac{1}{2}$. Thus $\sum_{j=-\infty}^{k} |\frac{\alpha_k}{\lambda}| \cdots |\frac{\alpha_j}{\lambda}| \|\frac{y_j}{\lambda}\| \le \sup_k \|\frac{y_k}{\lambda}\| \big(2 \sum_{j=0}^{\infty} \alpha^j + \#K_\lambda \sup_k |\frac{\alpha_k}{\lambda}|\big) < \infty$ for all $k \in \mathbb{Z}$, which implies that the infinite series $\sum_{j=-\infty}^{k} \frac{\alpha_k}{\lambda} \cdots \frac{\alpha_j}{\lambda} \frac{y_j}{\lambda}$ is absolutely convergent (and so convergent) in $\mathcal{H}$ for every $k \in \mathbb{Z}$. Thus, for each $k \in \mathbb{Z}$, set $x_k = \sum_{j=-\infty}^{k-1} \frac{\alpha_{k-1}}{\lambda} \cdots \frac{\alpha_j}{\lambda} \frac{y_j}{\lambda} + \frac{y_k}{\lambda}$ in $\mathcal{H}$ so that $x_{k+1} = \frac{\alpha_k}{\lambda} x_k + \frac{y_{k+1}}{\lambda}$. If $k \in \mathbb{Z} \setminus K_\lambda$, $\|\alpha_k x_k\| \le \alpha(\|\alpha_{k-1} x_{k-1}\| + \|y_k\|)$. So $\|\alpha_k x_k\|^2 \le 2\alpha^2(\|\alpha_{k-1} x_{k-1}\|^2 + \|y_k\|^2)$, and hence $\sum_{k \in \mathbb{Z} \setminus K_\lambda} \|\alpha_k x_k\|^2 \le \frac{1}{2}\big(\sum_{k \in \mathbb{Z} \setminus K_\lambda} \|\alpha_{k-1} x_{k-1}\|^2 + \|y\|^2\big)$. Therefore, $\sum_{k=-\infty}^{\infty} \|\alpha_k x_k\|^2 < \infty$. Moreover, since $\lambda x_{k+1} = \alpha_k x_k + y_{k+1}$ for each $k \in \mathbb{Z}$, it then follows that $|\lambda|^2 \big(\sum_{k=-\infty}^{\infty} \|x_{k+1}\|^2\big) \le \sum_{k=-\infty}^{\infty} (\|\alpha_k x_k\| + \|y_{k+1}\|)^2 \le$

$2\big(\sum_{k=-\infty}^{\infty}\|\alpha_k x_k\|^2 + \|y\|^2\big) < \infty$. Then $x = \bigoplus_{k=-\infty}^{\infty} x_k$ lies in $\ell^2(\mathcal{H})$. But $(\lambda I - T)x = \bigoplus_{k=-\infty}^{\infty} \lambda x_k - \alpha_{k-1}x_{k-1} = y$, and so $y \in \mathcal{R}(\lambda I - T)$. Outcome: $\mathcal{R}(\lambda I - T) = \ell^2(\mathcal{H})$. Since $\mathcal{N}(\lambda I - T) = \{0\}$ for all $\lambda \in \mathbb{C}$, every $0 \neq \lambda \in \mathbb{C}$ lies in $\rho(T)$. Conclusion:

$$\sigma(T) \,=\, \{0\}.$$

However, if $x \in \mathcal{N}(T^*)$, then $\overline{\alpha}_k x_{k+1} = 0$ so that $x_{k+1} = 0$ (since $\alpha_k \neq 0$) for every $k \in \mathbb{Z}$ and hence $x = 0$. That is, $\mathcal{N}(T^*) = \{0\}$ or, equivalently (Problem 5.35), $\mathcal{R}(T)^- = \ell^2(\mathcal{H})$. This implies that $0 \notin \sigma_R(T)$. Since $\sigma_P(T) = \varnothing$, we get

$$\sigma(T) \,=\, \sigma_C(T) \,=\, \sigma_C(T^*) \,=\, \sigma(T^*) \,=\, \{0\}.$$

Note: As in the previous example, by the Spectral Mapping Theorem we get

$$\sigma(\mu I - T) = \sigma_C(\mu I - T) = \{\mu\} \quad \text{and} \quad \sigma(\overline{\mu} I - T^*) = \sigma_C(\overline{\mu} I - T^*) \,=\, \{\overline{\mu}\}.$$

Example 6.H (Part 1). Let $F \in \mathcal{B}[\mathcal{H}]$ be an operator on a complex Hilbert space $\mathcal{H} \neq \{0\}$. Consider the operator $T \in \mathcal{B}[\ell^2_+(\mathcal{H})]$ defined by

$$Tx \,=\, 0 \oplus \textstyle\bigoplus_{k=1}^{\infty} F x_{k-1} \quad \text{ so that } \quad T^*x \,=\, \bigoplus_{k=0}^{\infty} F^* x_{k+1}$$

for every $x = \bigoplus_{k=0}^{\infty} x_k$ in $\ell^2_+(\mathcal{H}) = \bigoplus_{k=0}^{\infty} \mathcal{H}$, where 0 is the origin of $\mathcal{H}$. These can be identified with infinite matrices of operators, namely,

$$T \,=\, \begin{pmatrix} O & & & & \\ F & O & & & \\ & F & O & & \\ & & F & O & \\ & & & & \ddots \end{pmatrix} \quad \text{and} \quad T^* = \begin{pmatrix} O & F^* & & & \\ & O & F^* & & \\ & & O & F^* & \\ & & & O & \\ & & & & \ddots \end{pmatrix},$$

where the entries just below (above) the main block diagonal in the matrix of T (T^*) are copies of F (F^*), and the remaining entries are all null operators. It is readily verified by induction that $T^n x = \bigoplus_{k=0}^{n-1} 0 \oplus \bigoplus_{k=n}^{\infty} F^n x_{k-n}$, and hence $\|T^n x\|^2 = \sum_{k=0}^{\infty} \|F^n x_k\|^2$ so that $\|T^n x\| \leq \|F^n\| \, \|x\|$ for all x in $\ell^2_+(\mathcal{H})$, which implies that $\|T^n\| \leq \|F^n\|$, for each $n \geq 1$. On the other hand, take any $y_0 \neq 0$ in $\mathcal{H}$, set $y_k = 0 \in \mathcal{H}$ for all $k \geq 1$, consider the vector $y = \bigoplus_{k=0}^{\infty} y_k$ in $\ell^2_+(\mathcal{H})$ so that $\|y\| = \|y_0\| \neq 0$, and observe that $\|F^n\| = \sup_{\|y_0\|=1} \|F^n y_0\| = \sup_{\|y\|=1} \|T^n y\| \leq \sup_{\|x\|=1} \|T^n x\| = \|T^n\|$ for each $n \geq 1$. Thus

$$\|T^n\| \,=\, \|F^n\| \quad \text{ for every } \quad n \geq 1,$$

and so (Gelfand–Beurling formula — Proposition 6.21),

$$r(T) \,=\, r(F).$$

Moreover, since $y \neq 0$ and $T^*y = 0$, it follows that $0 \in \sigma_P(T^*)$. Thus

$$\{0\} \,\subseteq\, \sigma_P(T^*),$$

and hence $\{0\} \subseteq \sigma(T)$. Now take an arbitrary $\lambda \in \rho(T)$ so that $\lambda \neq 0$ and $\mathcal{R}(\lambda I - T) = \ell^2_+(\mathcal{H})$. Since $y = y_0 \oplus \bigoplus_{k=1}^{\infty} 0$ lies in $\ell^2_+(\mathcal{H})$ for every $y_0 \in \mathcal{H}$, it follows that $y \in \mathcal{R}(\lambda I - T)$. That is, $y = (\lambda I - T)x$ for some $x = \bigoplus_{k=0}^{\infty} x_k$ in $\ell^2_+(\mathcal{H})$ and so $y_0 \oplus \bigoplus_{k=1}^{\infty} 0 = \lambda x_0 \oplus \bigoplus_{k=1}^{\infty}(\lambda x_k - F x_{k-1})$. Thus $x_0 = \lambda^{-1} y_0$ and $x_{k+1} = \lambda^{-1} F x_k$ for every $k \geq 0$ and so $x_k = (\lambda^{-1}F)^k x_0 = \lambda^{-1}(\lambda^{-1}F)^k y_0$. Therefore, $\|x\|^2 = \sum_{k=0}^{\infty} \|x_k\|^2 = |\lambda|^{-2} \sum_{k=0}^{\infty} \|(\lambda^{-1}F)^k y_0\|^2 < \infty$ for every y_0 in $\mathcal{H}$ (since x lies in $\ell^2_+(\mathcal{H})$ for every $y_0 \in \mathcal{H}$). Hence $r(\lambda^{-1}F) < 1$ by Proposition 6.22. Conclusion: if $\lambda \in \rho(T)$, then $r(F) < |\lambda|$. Equivalently, if $|\lambda| \leq r(F)$, then $\lambda \in \sigma(T)$; that is, $\{\lambda \in \mathbb{C}\colon\ |\lambda| \leq r(F)\} \subseteq \sigma(T)$. But the reverse inclusion, $\sigma(T) \subseteq \{\lambda \in \mathbb{C}\colon\ |\lambda| \leq r(F)\}$, holds because $r(F) = r(T)$. Moreover, since $\sigma(T^*) = \sigma(T)^*$ for every operator T, and since $\mathbb{D}^- \cdot \sigma(F) = \{\lambda \in \mathbb{C}\colon\ |\lambda| \leq r(F)\}$ (where the product of two numerical sets is the set consisting of all products with factors in each set, and where $\mathbb{D}^-$ denotes the closed unit disk about the origin), it follows that

$$\sigma(T^*) \,=\, \sigma(T) \,=\, \mathbb{D}^- \cdot \sigma(F) \,=\, \big\{\lambda \in \mathbb{C}\colon\ |\lambda| \leq r(F)\big\}.$$

Now recall that $\lambda \in \sigma_P(T)$ if and only if $Tx = \lambda x$ (i.e., if and only if $\lambda x_0 = 0$ and $\lambda x_{k+1} = F x_k$ for every $k \geq 0$) for some nonzero $x = \bigoplus_{k=0}^{\infty} x_k$ in $\ell^2_+(\mathcal{H})$. If $0 \in \sigma_P(T)$, then $F x_k = 0$ for all $k \geq 0$ for some nonzero $x = \bigoplus_{k=0}^{\infty} x_k$ in $\ell^2_+(\mathcal{H})$ so that $0 \in \sigma_P(F)$. Conversely, if $0 \in \sigma_P(F)$, then there exists an $x_0 \neq 0$ in $\mathcal{H}$ such that $F x_0 = 0$. Thus set $x = \bigoplus_{k=0}^{\infty} (k+1)^{-1} x_0$, which is a nonzero vector in $\ell^2_+(\mathcal{H})$ such that $Tx = 0 \oplus \bigoplus_{k=1}^{\infty} k^{-1} F x_0 = 0$, and so $0 \in \sigma_P(T)$. Outcome: $0 \in \sigma_P(T)$ if and only if $0 \in \sigma_P(F)$. Moreover, if $\lambda \neq 0$ lies in $\sigma_P(T)$, then $x_0 = 0$ and $x_{k+1} = \lambda^{-1} F x_k$ for every $k \geq 0$ so that $x = 0$, which is a contradiction. Thus if $\lambda \neq 0$, then $\lambda \notin \sigma_P(T)$. Summing up:

$$\sigma_P(T) \,=\, \begin{cases} \{0\}, & 0 \in \sigma_P(F), \\ \varnothing, & 0 \notin \sigma_P(F). \end{cases}$$

Since $\sigma_R(T^*) = \sigma_P(T)^* \backslash \sigma_P(T^*)$, and since $\sigma_P(T)^* \subseteq \{0\} \subseteq \sigma_P(T^*)$, we get

$$\sigma_R(T^*) \,=\, \varnothing,$$

and hence

$$\sigma_C(T^*) \,=\, \big\{\lambda \in \mathbb{C}\colon\ |\lambda| \leq r(F)\big\} \backslash \sigma_P(T^*).$$

If $\sigma_P(T^*) \neq \{0\}$, then there exists $0 \neq \lambda \in \sigma_P(T^*)$, which means that $T^* x = \lambda x$ for some nonzero $x = \bigoplus_{k=0}^{\infty} x_k$ in $\ell^2_+(\mathcal{H})$. Thus there exists $0 \neq x_j \in \mathcal{H}$ such that $F^* x_{k+1} = \lambda x_k$ for every $k \geq j$, and so a trivial induction shows that $F^{*k} x_{j+k} = \lambda^k x_j$ for every $k \geq 0$. Hence $x_j \in \bigcap_{k=0}^{\infty} \mathcal{R}(F^{*k})$ because $\lambda \neq 0$, and therefore $\bigcap_{k=0}^{\infty} \mathcal{R}(F^{*k}) \neq \{0\}$. Conclusion:

$$\bigcap\nolimits_{k=0}^{\infty} \mathcal{R}(F^{*k}) \,=\, \{0\} \quad \text{implies} \quad \sigma_P(T^*) \,=\, \{0\},$$

and, in this case,

$$\sigma_C(T^*) \,=\, \big\{\lambda \in \mathbb{C}\colon\ |\lambda| \leq r(F)\big\} \backslash \{0\}.$$

In particular, if $F = S_+^*$ on $\mathcal{H} = \ell_+^2(\mathcal{K})$ for any nonzero complex Hilbert space $\mathcal{K}$, then $r(F) = r(S_+^*) = 1$ (according to Example 6.D) and $\mathcal{R}(F^{*k}) = \mathcal{R}(S_+^k) = \bigoplus_{j=0}^{k}\{0\} \oplus \bigoplus_{j=k+1}^{\infty}\mathcal{K} \subseteq \ell_+^2(\mathcal{K})$ so that $\bigcap_{k=0}^{\infty}\mathcal{R}(F^{*k}) = \{0\}$. Thus,

$$\sigma_P(T^*) = \sigma_P(T) = \{0\}, \quad \sigma_R(T^*) = \sigma_R(T) = \varnothing, \quad \sigma_C(T^*) = \sigma_C(T) = \mathbb{D}^-\backslash\{0\}.$$

Summing up: A backward unilateral shift of unilateral shifts (i.e., T^* with $F^* = S_+$, which is usually denoted by $T^* = S_+^* \otimes S_+$) and a unilateral shift of backward unilateral shifts (i.e., T with $F = S_+^*$, which is usually denoted by $T = S_+ \otimes S_+^*$) have a continuous spectrum equal to the punctured disk $\mathbb{D}^-\backslash\{0\}$. This was our first example of operators for which the continuous spectrum has nonempty interior.

Example 6.H (Part 2). This is a bilateral version of Part 1. Take $F \in \mathcal{B}[\mathcal{H}]$ on a complex Hilbert space $\mathcal{H} \neq \{0\}$, and consider $T \in \mathcal{B}[\ell^2(\mathcal{H})]$ defined by

$$Tx = \bigoplus_{k=-\infty}^{\infty} Fx_{k-1} \quad \text{so that} \quad T^*x = \bigoplus_{k=-\infty}^{\infty} F^*x_{k+1}$$

for every $x = \bigoplus_{k=-\infty}^{\infty} x_k$ in $\ell^2(\mathcal{H}) = \bigoplus_{k=-\infty}^{\infty}\mathcal{H}$. These can be identified with (doubly) infinite matrices of operators (the inner parenthesis indicates the zero-zero entry), namely,

$$T = \begin{pmatrix} \ddots & & & & & & \\ & O & & & & & \\ & F & O & & & & \\ & & F & (O) & & & \\ & & & F & O & & \\ & & & & & \ddots & \end{pmatrix} \quad \text{and} \quad T^* = \begin{pmatrix} \ddots & & & & & & \\ & O & F^* & & & & \\ & & O & F^* & & & \\ & & & (O) & F^* & & \\ & & & & O & & \\ & & & & & \ddots & \end{pmatrix},$$

where the entries just below (above) the main block diagonal in the matrix of T (or T^*) are copies of F (F^*), and the remaining entries are all null operators. Using the same argument of Part 1, it is easy to show that

$$r(T) = r(F).$$

If $\mathcal{N}(F) \neq \{0\}$, then there exists $0 \neq x_0 \in \mathcal{H}$ for which $Fx_0 = 0$. In this case, set $x = \bigoplus_{k=-\infty}^{\infty} x_k \neq 0$ in $\ell^2(\mathcal{H})$ with $x_k = 0$ for every $k \in \mathbb{Z}\backslash\{0\}$ so that $Tx = 0$. Thus $\mathcal{N}(T) \neq \{0\}$. Conversely, if $\mathcal{N}(T) \neq \{0\}$, then there exists an $x = \bigoplus_{k=-\infty}^{\infty} x_k \neq 0$ in $\ell^2(\mathcal{H})$ (so that $x_j \neq 0$ for some $j \in \mathbb{Z}$) for which $Tx = \bigoplus_{k=-\infty}^{\infty} Fx_{k-1} = 0$, and hence $Fx_k = 0$ for all $k \in \mathbb{Z}$; in particular, $Fx_j = 0$ so that $\mathcal{N}(F) \neq \{0\}$. Therefore,

$$0 \in \sigma_P(T) \quad \text{if and only if} \quad 0 \in \sigma_P(F).$$

Similarly (same argument), $0 \in \sigma_P(F^*)$ if and only if $0 \in \sigma_P(T^*)$, and so, recalling that $\sigma_R(F) = \sigma_P(F^*)^*\backslash\sigma_P(F)$ and $\sigma_R(T) = \sigma_P(T^*)^*\backslash\sigma_P(T)$,

$$0 \in \sigma_R(T) \quad \text{if and only if} \quad 0 \in \sigma_R(F).$$

We have seen that $\mathcal{N}(F) = \{0\}$ if and only if $\mathcal{N}(T) = \{0\}$. Dually (and similarly), $\mathcal{N}(F^*) = \{0\}$ if and only if $\mathcal{N}(T^*) = \{0\}$, which means that $\mathcal{R}(F)^- = \mathcal{H}$ if and only if $\mathcal{R}(T)^- = \ell^2(\mathcal{H})$ (Problem 5.35). Moreover, it is plain that $\mathcal{R}(F) \neq \mathcal{H}$ if and only if $\mathcal{R}(T) \neq \ell^2(\mathcal{H})$. Thus (cf. diagram of Section 6.2),

$$0 \in \sigma_C(T) \quad \text{if and only if} \quad 0 \in \sigma_C(F).$$

Next we prove the following assertion.

> If $0 \neq \lambda \in \sigma_P(T)$ and $\mathcal{N}(F) = \{0\}$, then there exists
> $0 \neq x_0 \in \mathcal{R}(F) \subseteq \mathcal{H}$ such that $\sum_{k=0}^{\infty} \|(\lambda^{-1}F)^{\pm k} x_0\|^2 < \infty$.

Indeed, $\lambda \in \sigma_P(T)$ if and only if $Tx = \lambda x$ for some nonzero $x = \bigoplus_{k=-\infty}^{\infty} x_k$ in $\ell^2(\mathcal{H})$. Suppose $\lambda \neq 0$ and $\mathcal{N}(F) = \{0\}$ so that $\lambda \in \sigma_P(T)$ if and only if $x_{k+1} = \lambda^{-1} F x_k$, with $0 \neq x_k \in \mathcal{R}(F)$, for every $k \in \mathbb{Z}$. Thus $x_{\pm k} = (\lambda^{-1}F)^{\pm k} x_0$ for every $k \geq 0$, and so $\|x\|^2 = \sum_{k=-\infty}^{\infty} \|x_k\|^2 = \sum_{k=0}^{\infty} \|(\lambda^{-1}F)^{\pm k} x_0\|^2 < \infty$. Now set $\mathcal{H} = \ell^2_+$ and let F be a diagonal operator on ℓ^2_+,

$$F \,=\, \operatorname{diag}(\{\lambda_j\}) \,\in\, \mathcal{B}[\ell^2_+],$$

where the countably infinite set $\{\lambda_j\}$ consists of an enumeration of all rational numbers in $(0, 1)$. Observe that $\sigma(F) = [0, 1]$ (and so $r(F) = 1$), with $\sigma_P(F) = \{\lambda_j\}$ (in particular, $\mathcal{N}(F) = \{0\}$), $\sigma_R(T) = \varnothing$, and $\sigma_C(F) = [0, 1] \backslash \{\lambda_j\}$ (see Example 6.C). With this F we proceed to show that

$$\sigma_P(T) \,=\, \varnothing.$$

Suppose there exists a nonzero λ in $\sigma_P(T)$. If $|\lambda| \neq |\lambda_j|$ for every j, then $0 \neq |\lambda^{-1}\lambda_j| \neq 1$ (because $0 < |\lambda_j| < 1$) for every j. Since $x_0 \neq 0$, it follows by Problem 5.18(e) that $\lim_k \|(\lambda^{-1}F)^k x_0\|^2 = \infty$ or $\lim_k \|(\lambda^{-1}F)^{-k} x_0\|^2 = \infty$ because $\lambda^{-1}F = \operatorname{diag}(\{\lambda^{-1}\lambda_j\})$ is a diagonal operator with an inverse on its range. Thus $\sum_{k=0}^{\infty} \|(\lambda^{-1}F)^{\pm k} x_0\|^2 = \infty$, which is a contradiction. On the other hand, if $|\lambda| = |\lambda_{j'}|$ for some j' then, with $x_0 = \{\xi_j\}$, we get $\|(\lambda^{-1}F)^k x_0\|^2 = \sum_{j=1}^{\infty} |\lambda^{-1}\lambda_j|^{2k} |\xi_j|^2 \to 0$ as $k \to \infty$ only if $\xi_j = 0$ for every index j such that $|\lambda_{j'}| \leq |\lambda_j|$ (i.e., such that $|\lambda_{j'}^{-1}\lambda_j| = |\lambda^{-1}\lambda_j| \geq 1$). Also, $\|(\lambda^{-1}F)^{-k} x\|^2 = \sum_{j=1}^{\infty} |\lambda^{-1}\lambda_j|^{-2k} |\xi_j|^2 = \sum_{j=1}^{\infty} |\lambda \lambda_j^{-1}|^{2k} |\xi_j|^2 \to 0$ as $k \to \infty$ only if $\xi_j = 0$ for every j such that $|\lambda_{j'}| \geq |\lambda_j|$ (i.e., such that $|\lambda_{j'} \lambda_j^{-1}| = |\lambda \lambda_j^{-1}| \geq 1$). Thus $x_0 = 0$, which is again a contradiction. Hence, $\sigma_P(T) \backslash \{0\} = \varnothing$. However, $0 \in \sigma_C(T)$ because $0 \in \sigma_C(F)$, which concludes the proof of $\sigma_P(T) = \varnothing$. Moreover, as F is a normal operator, T also is a normal operator, and so

$$\sigma_R(T) \,=\, \varnothing.$$

Therefore, $\sigma_C(T) = \sigma(T)$. Actually, $\sigma(T)$ is the annulus about the origin that includes $\sigma(F)$ but no circle entirely included in $\rho(F)$. In other words, with $\mathbb{T} = \partial\mathbb{D}$ denoting the closed unit circle about the origin, it can be shown that

$$\sigma(T) \,=\, \mathbb{T} \cdot \sigma(F).$$

This follows by an important result of Brown and Pearcy (1966) which says that *the spectrum of a tensor product is the product of the spectra*. Since $\sigma(F) = [0,1]$, it follows that $\mathbb{T} \cdot \sigma(F) = \mathbb{D}^-$, and hence

$$\sigma(T) \,=\, \sigma_C(T) \,=\, \mathbb{D}^-.$$

Summing up: A bilateral shift of operators F (which is usually denoted by $T = S \otimes F$) has only a continuous spectrum, which is equal to the (closed) disk $\mathbb{D}^-$, if F is a diagonal operator whose diagonal consists of an enumeration of all rational numbers in $(0,1)$.

6.6 The Spectrum of a Compact Operator

The spectral theory of compact operators is an essential feature for the Spectral Theorem for compact normal operators of the next section. Normal operators were defined on a Hilbert space, and therefore we assume throughout this section that the compact operators act on a complex Hilbert space $\mathcal{H} \neq \{0\}$, although the spectral theory of compact operators can also be developed on a complex Banach space. Recall that $\mathcal{B}_\infty[\mathcal{X}, \mathcal{Y}]$ stands for the collection of all compact linear transformations of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$, and so $\mathcal{B}_\infty[\mathcal{H}]$ denotes the class of all compact operators on $\mathcal{H}$ (Section 4.9).

Proposition 6.29. *If $T \in \mathcal{B}_\infty[\mathcal{H}]$ and λ is a nonzero complex number, then $\mathcal{R}(\lambda I - T)$ is a subspace of $\mathcal{H}$.*

Proof. Take an arbitrary compact transformation $K \in \mathcal{B}_\infty[\mathcal{M}, \mathcal{X}]$ of a subspace $\mathcal{M}$ of a complex Banach space $\mathcal{X} \neq \{0\}$ into $\mathcal{X}$. Let I be the identity on $\mathcal{M}$, let λ be any nonzero complex number, and consider the transformation $(\lambda I - K) \in \mathcal{B}[\mathcal{M}, \mathcal{X}]$.

Claim. If $\mathcal{N}(\lambda I - K) = \{0\}$, then $\mathcal{R}(\lambda I - K)$ is closed in $\mathcal{X}$.

Proof. If $\mathcal{N}(\lambda I - K) = \{0\}$ and $\mathcal{R}(\lambda I - K)$ is not closed in $\mathcal{X} \neq \{0\}$, then $\lambda I - K$ is not bounded below (see Corollary 4.24). This means that for every $\varepsilon > 0$ there exists $0 \neq x_\varepsilon \in \mathcal{M}$ such that $\|(\lambda I - K)x_\varepsilon\| < \varepsilon \|x_\varepsilon\|$. Therefore, $\inf_{\|x\|=1} \|(\lambda I - K)x\| = 0$ and so there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{M}$ for which $\|(\lambda I - K)x_n\| \to 0$. Since K is compact and $\{x_n\}$ is bounded, it follows by Theorem 4.52 that $\{Kx_n\}$ has a convergent subsequence, say $\{Kx_k\}$, so that $Kx_k \to y \in \mathcal{X}$. However,

$$\|\lambda x_k - y\| \,=\, \|\lambda x_k - Kx_k + Kx_k - y\| \,\leq\, \|(\lambda I - K)x_k\| + \|Kx_k - y\| \,\to\, 0.$$

Then $\{\lambda x_k\}$ also converges in $\mathcal{X}$ to y, and hence $y \in \mathcal{M}$ (since $\mathcal{M}$ is closed in $\mathcal{X}$ — Theorem 3.30). Moreover, $y \neq 0$ (since $0 \neq |\lambda| = \|\lambda x_k\| \to \|y\|$) and, as K is continuous, $Ky = K \lim_k \lambda x_k = \lambda \lim_k Kx_k = \lambda y$ so that $y \in \mathcal{N}(\lambda I - K)$. Therefore $\mathcal{N}(\lambda I - K) \neq \{0\}$, which is a contradiction. □

Now take any $T \in \mathcal{B}[\mathcal{H}]$. Recall that $(\lambda I - T)|_{\mathcal{N}(\lambda I - T)^\perp}$ in $\mathcal{B}[\mathcal{N}(\lambda I - T)^\perp, \mathcal{H}]$ is injective (i.e., $\mathcal{N}((\lambda I - T)|_{\mathcal{N}(\lambda I - T)^\perp}) = \{0\}$ — see the remark that follows Proposition 5.12) and coincides with $\lambda I - T|_{\mathcal{N}(\lambda I - T)^\perp}$ on $\mathcal{N}(\lambda I - T)^\perp$. If T is compact, then so is $T|_{\mathcal{N}(\lambda I - T)^\perp} \in \mathcal{B}[\mathcal{N}(\lambda I - T)^\perp, \mathcal{H}]$ (reason: $\mathcal{N}(\lambda I - T)^\perp$ is a subspace of $\mathcal{H}$, and the restriction of a compact linear transformation to a linear manifold is a compact linear transformation — see Section 4.9). Since $\mathcal{H} \neq 0$, we get by the above claim that $(\lambda I - T)|_{\mathcal{N}(\lambda I - T)^\perp} = \lambda I - T|_{\mathcal{N}(\lambda I - T)^\perp}$ has a closed range for all $\lambda \neq 0$. But $\mathcal{R}((\lambda I - T)|_{\mathcal{N}(\lambda I - T)^\perp}) = \mathcal{R}(\lambda I - T)$, as is readily verified. □

Proposition 6.30. *If $T \in \mathcal{B}_\infty[\mathcal{H}]$ and λ is a nonzero complex number, then $\mathcal{R}(\lambda I - T) = \mathcal{H}$ whenever $\mathcal{N}(\lambda I - T) = \{0\}$.*

Proof. Take any $\lambda \neq 0$ in $\mathbb{C}$ and any $T \in \mathcal{B}_\infty[\mathcal{H}]$. Suppose $\mathcal{N}(\lambda I - T) = \{0\}$ and $\mathcal{R}(\lambda I - T) \neq \mathcal{H}$ (recall: $\mathcal{H} \neq \{0\}$), and consider the sequence $\{\mathcal{M}_n\}_{n=0}^\infty$ of linear manifolds of $\mathcal{H}$ recursively defined by

$$\mathcal{M}_{n+1} \,=\, (\lambda I - T)(\mathcal{M}_n) \quad \text{for every} \quad n \geq 0 \quad \text{with} \quad \mathcal{M}_0 = \mathcal{H}.$$

It can be verified by induction that

$$\mathcal{M}_{n+1} \,\subseteq\, \mathcal{M}_n \quad \text{for every} \quad n \geq 0.$$

Indeed, $\mathcal{M}_1 = \mathcal{R}(\lambda I - T) \subseteq \mathcal{H} = \mathcal{M}_0$ and, if the above inclusion holds for some $n \geq 0$, then $\mathcal{M}_{n+2} = (\lambda I - T)(\mathcal{M}_{n+1}) \subseteq (\lambda I - T)(\mathcal{M}_n) = \mathcal{M}_{n+1}$, which concludes the induction. The previous proposition ensures that $\mathcal{R}(\lambda I - T)$ is a subspace of $\mathcal{H}$, and so $(\lambda I - T) \in \mathcal{G}[\mathcal{H}, \mathcal{R}(\lambda I - T)]$ by Corollary 4.24. Hence (another induction plus Theorem 3.24),

$$\{\mathcal{M}_n\}_{n=0}^\infty \text{ is a decreasing sequence of subspaces of } \mathcal{H}.$$

Moreover, if $\mathcal{M}_{n+1} = \mathcal{M}_n$ for some n, then there exists an integer $k \geq 1$ such that $\mathcal{M}_{k+1} = \mathcal{M}_k \neq \mathcal{M}_{k-1}$ (for $\mathcal{M}_0 = \mathcal{H} \neq \mathcal{R}(\lambda I - T) = \mathcal{M}_1$). But this leads to a contradiction: if $\mathcal{M}_{k+1} = \mathcal{M}_k$, then $(\lambda I - T)(\mathcal{M}_k) = \mathcal{M}_k$ so that $\mathcal{M}_k = (\lambda I - T)^{-1}(\mathcal{M}_k) = \mathcal{M}_{k-1}$. Outcome: $\mathcal{M}_{n+1}$ is properly included in $\mathcal{M}_n$ for each n; that is,

$$\mathcal{M}_{n+1} \,\subset\, \mathcal{M}_n \quad \text{for every} \quad n \geq 0.$$

Hence $\mathcal{M}_{n+1}$ is a proper subspace of $\mathcal{M}_n$ (Problem 3.38). By Lemma 4.33, for each $n \geq 0$ there is an $x_n \in \mathcal{M}_n$ with $\|x_n\| = 1$ such that $\frac{1}{2} < d(x_n, \mathcal{M}_{n+1})$. Recall that $\lambda \neq 0$, take any pair of integers $0 \leq m < n$, and set

$$x \,=\, x_n + \lambda^{-1}\big((\lambda I - T)x_m - (\lambda I - T)x_n\big)$$

so that $Tx_n - Tx_m = \lambda(x - x_m)$. Since x lies in $\mathcal{M}_{m+1}$,

$$\|Tx_n - Tx_m\| \,=\, |\lambda|\|x - x_m\| \,>\, \tfrac{1}{2}|\lambda|.$$

Thus the sequence $\{Tx_n\}$ has no convergent subsequence (every subsequence of $\{Tx_n\}$ is not a Cauchy sequence). Since $\{x_n\}$ is bounded, this ensures that T is not compact (Theorem 4.52), which is a contradiction. Conclusion: If $T \in \mathcal{B}_\infty[\mathcal{H}]$ and $\mathcal{N}(\lambda I - T) = \{0\}$ for $\lambda \neq 0$, then $\mathcal{R}(\lambda I - T) = \mathcal{H}$. □

Corollary 6.31. *If $T \in \mathcal{B}_\infty[\mathcal{H}]$, then $0 \neq \lambda \in \rho(T) \cup \sigma_{P_4}(T)$, so that*

$$\sigma(T)\backslash\{0\} = \sigma_P(T)\backslash\{0\} = \sigma_{P_4}(T)\backslash\{0\}.$$

Proof. Take $0 \neq \lambda \in \mathbb{C}$. Since $\mathcal{H} \neq \{0\}$, Propositions 6.29 and 6.30 ensure that $\lambda \in \rho(T) \cup \sigma_{P_1}(T) \cup \sigma_{P_4}(T) \cup \sigma_{R_1}(T)$ and also that $\lambda \in \rho(T) \cup \sigma_P(T)$ (see the diagram of Section 6.2). Then $\lambda \in \rho(T) \cup \sigma_{P_1}(T) \cup \sigma_{P_4}(T)$, which implies that $\overline{\lambda} \in \rho(T)^* \cup \sigma_{P_1}(T)^* \cup \sigma_{P_4}(T)^* = \rho(T^*) \cup \sigma_{R_1}(T^*) \cup \sigma_{P_4}(T^*)$ (by Proposition 6.17). But $T^* \in \mathcal{B}_\infty[\mathcal{H}]$ whenever $T \in \mathcal{B}_\infty[\mathcal{H}]$ (according to Problem 5.42) so that $\overline{\lambda} \in \rho(T^*) \cup \sigma_{P_1}(T^*) \cup \sigma_{P_4}(T^*)$, and hence $\overline{\lambda} \in \rho(T^*) \cup \sigma_{P_4}(T^*)$. That is, $\lambda \in \rho(T) \cup \sigma_{P_4}(T)$ whenever $\lambda \neq 0$. □

Example 6.I. If $T \in \mathcal{B}_0[\mathcal{H}]$ (i.e., T is a finite-rank operator on $\mathcal{H}$), then

$$\sigma(T) = \sigma_P(T) = \sigma_{P_4}(T) \text{ is finite.}$$

Indeed, if $\dim \mathcal{H} < \infty$, then $\sigma(T) = \sigma_P(T) = \sigma_{P_4}(T)$ (Example 6.B). Suppose $\dim \mathcal{H} = \infty$. Since $\mathcal{B}_0[\mathcal{H}] \subseteq \mathcal{B}_\infty[\mathcal{H}]$, it follows that $0 \neq \lambda \in \rho(T) \cup \sigma_{P_4}(T)$ by Corollary 6.31. Moreover, since $\dim \mathcal{R}(T) < \infty$ and $\dim \mathcal{H} = \infty$, it also follows that $\mathcal{R}(T)^- = \mathcal{R}(T) \neq \mathcal{H}$ and $\mathcal{N}(T) \neq \{0\}$ (because $\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim \mathcal{H}$ according to Problem 2.17). Then $0 \in \sigma_{P_4}(T)$ (cf. diagram of Section 6.2). Hence $\sigma(T) = \sigma_P(T) = \sigma_{P_4}(T)$. If $\sigma_P(T)$ is infinite, then there exists an infinite set of linearly independent eigenvectors of T (Proposition 6.14). Since every eigenvector of T lies in $\mathcal{R}(T)$, this implies that $\dim \mathcal{R}(T) = \infty$ (Theorem 2.5), which is a contradiction. Conclusion: $\sigma_P(T)$ must be finite. In particular, this shows that the spectrum in Example 6.B is, clearly, finite.

Example 6.J. Let us glance at the spectra of some compact operators.

(a) The operator $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on $\mathbb{C}^2$ is obviously compact. Its spectrum is given by (cf. Example 6.B and 6.I)

$$\sigma(A) = \sigma_P(A) = \sigma_{P_4}(A) = \{0, 1\}.$$

(b) The diagonal operator $D = \text{diag}\{\lambda_k\}_{k=0}^\infty \in \mathcal{B}[\ell_+^2]$ with $\lambda_k \to 0$ is compact (Example 4.N). By Example 6.C, $\sigma_{P_4}(D) = \{\lambda_k\}_{k=0}^\infty\backslash\{0\}$ and

$$\sigma(D) = \sigma_{P_4}(D) \cup \begin{cases} \sigma_C(D), & \lambda_k \neq 0 \text{ for all } k \geq 0 \quad (\text{with } \sigma_C(D) = \{0\}), \\ \sigma_{P_3}(D), & \lambda_k = 0 \text{ for some } k \geq 0 \quad (\text{with } \sigma_{P_3}(D) = \{0\}). \end{cases}$$

(c) The unilateral weighted shift $T_+ = \text{shift}(\{\alpha_k\}_{k=0}^\infty)$ acting on ℓ_+^2 of Example 6.F is compact ($T_+ = S_+ D_+$ and D_+ is compact) and (Example 6.F)

$$\sigma(T_+) = \sigma_R(T_+) = \sigma_{R_2}(T_+) = \{0\}.$$

Moreover, T_+^* also is compact (Problem 5.42) and (Example 6.F)

$$\sigma(T_+^*) = \sigma_P(T_+^*) = \sigma_{P_2}(T_+^*) = \{0\}.$$

(d) Finally, the bilateral weighted shift $T = \text{shift}(\{\alpha_k\}_{k=-\infty}^{\infty})$ acting on ℓ^2 of Example 6.G is compact (the same argument as above) and (Example 6.G)

$$\sigma(T) = \sigma_C(T) = \{0\}.$$

Corollary 6.32. *If an operator T on $\mathcal{H}$ is compact and normaloid, then $\sigma_P(T) \neq \varnothing$ and there exists $\lambda \in \sigma_P(T)$ such that $|\lambda| = \|T\|$.*

Proof. Recall that $\mathcal{H} \neq \{0\}$. If T is normaloid (i.e., $r(T) = \|T\|$), then $\sigma(T) = \{0\}$ only if $T = O$. If $T = O$ and $\mathcal{H} \neq \{0\}$, then $0 \in \sigma_P(T)$ and $\|T\| = 0$. If $T \neq O$, then $\sigma(T) \neq \{0\}$ and $\|T\| = r(T) = \max_{\lambda \in \sigma(T)} |\lambda|$, so that there exists λ in $\sigma(T)$ such that $|\lambda| = \|T\|$. Moreover, if T is compact and $\sigma(T) \neq \{0\}$, then $\varnothing \neq \sigma(T) \backslash \{0\} \subseteq \sigma_P(T)$ by Corollary 6.31, and hence $r(T) = \max_{\lambda \in \sigma(T)} |\lambda| = \max_{\lambda \in \sigma_P(T)} |\lambda| = \|T\|$. Thus there exists $\lambda \in \sigma_P(T)$ such that $|\lambda| = \|T\|$. □

Proposition 6.33. *If $T \in \mathcal{B}_\infty[\mathcal{H}]$ and $\{\lambda_n\}$ is an infinite sequence of distinct elements in $\sigma(T)$, then $\lambda_n \to 0$.*

Proof. Take any $T \in \mathcal{B}[\mathcal{H}]$ and let $\{\lambda_n\}$ be an infinite sequence of distinct elements in $\sigma(T)$. If $\lambda_{n'} = 0$ for some n', then the subsequence $\{\lambda_k\}$ of $\{\lambda_n\}$ consisting of all points of $\{\lambda_n\}$ except $\lambda_{n'}$ is a sequence of distinct nonzero elements in $\sigma(T)$. Since $\lambda_k \to 0$ implies $\lambda_n \to 0$, there is no loss of generality in assuming that $\{\lambda_n\}$ is a sequence of distinct *nonzero* elements in $\sigma(T)$ indexed by $\mathbb{N}$. Moreover, if T is compact and $0 \neq \lambda_n \in \sigma(T)$, then Corollary 6.31 says that $\lambda_n \in \sigma_P(T)$ for every $n \geq 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of eigenvectors associated with $\{\lambda_n\}_{n=1}^{\infty}$ (i.e., $Tx_n = \lambda_n x_n$ with $x_n \neq 0$ for every $n \geq 1$), which is a sequence of linearly independent vectors by Proposition 6.14. Set

$$\mathcal{M}_n = \text{span}\,\{x_i\}_{i=1}^{n} \quad \text{for each} \quad n \geq 1$$

so that each $\mathcal{M}_n$ is a subspace of $\mathcal{H}$ with $\dim \mathcal{M}_n = n$, and

$$\mathcal{M}_n \subset \mathcal{M}_{n+1} \quad \text{for every} \quad n \geq 1.$$

Actually, each $\mathcal{M}_n$ is properly included in $\mathcal{M}_{n+1}$ since $\{x_i\}_{i=1}^{n+1}$ is linearly independent, and so $x_{n+1} \in \mathcal{M}_{n+1} \backslash \mathcal{M}_n$. From now on the proof is similar to that of Proposition 6.30. Since each $\mathcal{M}_n$ is a proper subspace of $\mathcal{M}_{n+1}$, for every $n \geq 1$ there exists $y_{n+1} \in \mathcal{M}_{n+1}$ with $\|y_{n+1}\| = 1$ such that $\frac{1}{2} < d(y_{n+1}, \mathcal{M}_n)$ by Lemma 4.33. Write $y_{n+1} = \sum_{i=1}^{n+1} \alpha_i x_i$ in $\mathcal{M}_{n+1}$ so that

$$(\lambda_{n+1} I - T) y_{n+1} = \sum_{i=1}^{n+1} \alpha_i (\lambda_{n+1} - \lambda_i) x_i = \sum_{i=1}^{n} \alpha_i (\lambda_{n+1} - \lambda_i) x_i \in \mathcal{M}_n.$$

Recall that $\lambda_n \neq 0$ for all n, take any pair of integers $1 \leq m < n$, and set

$$y \;=\; y_m - \lambda_m^{-1}(\lambda_m I - T)y_m + \lambda_n^{-1}(\lambda_n I - T)y_n$$

so that $T(\lambda_m^{-1}y_m) - T(\lambda_n^{-1}y_n) = y - y_n$. Since y lies in $\mathcal{M}_{n-1}$,

$$\|T(\lambda_m^{-1}y_m) - T(\lambda_n^{-1}y_n)\| \;=\; \|y - y_n\| \;>\; \tfrac{1}{2},$$

which implies that the sequence $\{T(\lambda_n^{-1}y_n)\}$ has no convergent subsequence. If T is compact, then Theorem 4.52 ensures that $\{\lambda_n^{-1}y_n\}$ has no bounded subsequence. That is, $\sup_k|\lambda_k|^{-1} = \sup_k\|\lambda_k^{-1}y_k\| = \infty$, and so $\inf_k|\lambda_k| = 0$ for every subsequence $\{\lambda_k\}$ of $\{\lambda_n\}$. Thus $\lambda_n \to 0$. □

Corollary 6.34. *Take any compact operator $T \in \mathcal{B}_\infty[\mathcal{H}]$.*

(a) *0 is the only possible accumulation point of $\sigma(T)$.*

(b) *If $\lambda \in \sigma(T)\backslash\{0\}$, then λ is an isolated point of $\sigma(T)$.*

(c) *$\sigma(T)\backslash\{0\}$ is a discrete subset of $\mathbb{C}$.*

(d) *$\sigma(T)$ is countable.*

Proof. If $\lambda \neq 0$, then the previous proposition says that there is no sequence of distinct points in $\sigma(T)$ that converges to λ. Thus $\lambda \neq 0$ is not an accumulation point of $\sigma(T)$ by Proposition 3.28. Therefore, if $\lambda \in \sigma(T)\backslash\{0\}$, then it is not an accumulation point of $\sigma(T)$, which means (by definition) that it is an isolated point of $\sigma(T)$. Hence $\sigma(T)\backslash\{0\}$ consists entirely of isolated points, which means (by definition again) that it is a discrete subset of $\mathbb{C}$. But $\mathbb{C}$ is separable, and every discrete subset of a separable metric space is countable (this is a consequence of Theorem 3.35 and Corollary 3.36; see the observations that follow Proposition 3.37). Then $\sigma(T)\backslash\{0\}$ is countable and so is $\sigma(T)$. □

The point $\lambda = 0$ may be anywhere (i.e., zero may be in any part of the spectrum or in the resolvent set of a compact operator). Precisely, if $T \in \mathcal{B}_\infty[\mathcal{H}]$, then $\lambda = 0$ may lie in $\sigma_P(T)$, $\sigma_R(T)$, $\sigma_C(T)$, or $\rho(T)$ (see Example 6.J). However, if $0 \in \rho(T)$, then $\mathcal{H}$ must be finite dimensional. Indeed, if $0 \in \rho(T)$, then $T^{-1} \in \mathcal{B}[\mathcal{H}]$, and so $I = T^{-1}T$ is compact by Proposition 4.54, which implies that $\mathcal{H}$ is finite dimensional (Corollary 4.34). We show next that the eigenspaces associated with nonzero eigenvalues of a compact operator are also finite dimensional.

Proposition 6.35. *If $T \in \mathcal{B}_\infty[\mathcal{H}]$ and λ is a nonzero complex number, then* $\dim\mathcal{N}(\lambda I - T) = \dim\mathcal{N}(\overline{\lambda}I - T^*) < \infty$.

Proof. Take any $\lambda \neq 0$ in $\mathbb{C}$ and any $T \in \mathcal{B}_\infty[\mathcal{H}]$. If $\dim\mathcal{N}(\lambda I - T) = 0$, then $\mathcal{N}(\lambda I - T) = \{0\}$ so that $\overline{\lambda} \in \rho(T^*)$ by Proposition 6.17. Thus $\mathcal{N}(\overline{\lambda}I - T^*) = \{0\}$; equivalently, $\dim\mathcal{N}(\overline{\lambda}I - T^*) = 0$. Dually, since $T \in \mathcal{B}_\infty[\mathcal{H}]$ if and only if $T^* \in \mathcal{B}_\infty[\mathcal{H}]$ (Problem 5.42), it follows that $\dim\mathcal{N}(\overline{\lambda}I - T^*) = 0$ implies $\dim\mathcal{N}(\lambda I - T) = 0$. That is,

$$\dim \mathcal{N}(\lambda I - T) = 0 \quad \text{if and only if} \quad \dim \mathcal{N}(\overline{\lambda} I - T^*) = 0.$$

Now suppose $\dim \mathcal{N}(\lambda I - T) \neq 0$, and so $\dim \mathcal{N}(\overline{\lambda} I - T^*) \neq 0$. Observe that $\mathcal{N}(\lambda I - T) \neq \{0\}$ is an invariant subspace for T (if $Tx = \lambda x$, then $T(Tx) = \lambda(Tx)$), and also that $T|_{\mathcal{N}(\lambda I - T)} = \lambda I$ of $\mathcal{N}(\lambda I - T)$ into itself. If T is compact, then $T|_{\mathcal{N}(\lambda I - T)}$ is compact (Section 4.9), and so is $\lambda I \neq O$ on $\mathcal{N}(\lambda I - T) \neq \{0\}$. But $\lambda I \neq O$ is not compact in an infinite-dimensional normed space (by Corollary 4.34) so that $\dim \mathcal{N}(\lambda I - T) < \infty$. Dually, as T^* is compact, $\dim \mathcal{N}(\overline{\lambda} I - T^*) < \infty$. Therefore, there exist positive integers m and n such that

$$\dim \mathcal{N}(\lambda I - T) = m \quad \text{and} \quad \dim \mathcal{N}(\overline{\lambda} I - T^*) = n.$$

Let $\{e_i\}_{i=1}^m$ and $\{f_i\}_{i=1}^n$ be orthonormal bases for the Hilbert spaces $\mathcal{N}(\lambda I - T)$ and $\mathcal{N}(\overline{\lambda} I - T^*)$, respectively. Set $k = \min\{m, n\} \geq 1$ and consider the mappings $S: \mathcal{H} \to \mathcal{H}$ and $S^*: \mathcal{H} \to \mathcal{H}$ defined by

$$Sx = \sum_{i=1}^{k} \langle x\,; e_i \rangle f_i \quad \text{and} \quad S^* x = \sum_{i=1}^{k} \overline{\langle x\,; f_i \rangle} e_i$$

for every $x \in \mathcal{H}$. It is clear that S and S^* lie in $\mathcal{B}[\mathcal{H}]$, and also that S^* is the adjoint of S; that is, $\langle Sx\,; y \rangle = \langle x\,; S^* y \rangle$ for every $x, y \in \mathcal{H}$. Actually,

$$\mathcal{R}(S) \subseteq \bigvee \{f_i\}_{i=1}^k \subseteq \mathcal{N}(\overline{\lambda} I - T^*) \quad \text{and} \quad \mathcal{R}(S^*) \subseteq \bigvee \{e_i\}_{i=1}^k \subseteq \mathcal{N}(\lambda I - T)$$

so that $S, S^* \in \mathcal{B}_0[\mathcal{H}]$, and hence $T + S$ and $T^* + S^*$ lie in $\mathcal{B}_\infty[\mathcal{H}]$ by Theorem 4.53 (since $\mathcal{B}_0[\mathcal{H}] \subseteq \mathcal{B}_\infty[\mathcal{H}]$). First suppose that $m \leq n$ (and so $k = m$). If x is a vector in $\mathcal{N}(\lambda I - (T + S))$, then $(\lambda I - T)x = Sx$. But $\mathcal{R}(S) \subseteq \mathcal{N}(\overline{\lambda} I - T^*) = \mathcal{R}(\lambda I - T)^\perp$ (Proposition 5.76), and hence $(\lambda I - T)x = Sx = 0$. Therefore, $x \in \mathcal{N}(\lambda I - T) = \operatorname{span}\{e_i\}_{i=1}^m$ so that $x = \sum_{i=1}^m \alpha_i e_i$ (for some family of scalars $\{\alpha_i\}_{i=1}^m$). Thus $0 = Sx = \sum_{j=1}^m \alpha_j S e_j = \sum_{j=1}^m \alpha_j \sum_{i=1}^m \langle e_j\,; e_i \rangle f_i = \sum_{i=1}^m \alpha_i f_i$, which implies that $\alpha_i = 0$ for every $i = 1, \ldots, m$ (reason: $\{f_i\}_{i=1}^m$ is an orthonormal set, thus linearly independent — Proposition 5.34). That is, $x = 0$. Outcome: $\mathcal{N}(\lambda I - (T + S)) = \{0\}$. Hence $\lambda \in \rho(T + S)$ according to Corollary 6.31 (since $T + S \in \mathcal{B}_\infty[\mathcal{H}]$ and $\lambda \neq 0$). Conclusion:

$$m \leq n \quad \text{implies} \quad \mathcal{R}(\lambda I - (T + S)) = \mathcal{H}.$$

Dually, using exactly the same argument,

$$n \leq m \quad \text{implies} \quad \mathcal{R}(\overline{\lambda} I - (T^* + S^*)) = \mathcal{H}.$$

If $m < n$, then $k = m < m + 1 \leq n$ and $f_{m+1} \in \mathcal{R}(\lambda I - (T + S)) = \mathcal{H}$, so that there exists $v \in \mathcal{H}$ for which $(\lambda I - (T + S))v = f_{m+1}$. Hence

$$\begin{aligned} 1 = \langle f_{m+1}\,; f_{m+1} \rangle &= \langle (\lambda I - (T + S))v\,; f_{m+1} \rangle \\ &= \langle (\lambda I - T)v\,; f_{m+1} \rangle - \langle Sv\,; f_{m+1} \rangle = 0, \end{aligned}$$

which is a contradiction. Indeed, $\langle(\lambda I - T)v\,;f_{m+1}\rangle = \langle Sv\,;f_{m+1}\rangle = 0$ for $f_{m+1} \in \mathcal{N}(\overline{\lambda}I - T^*) = \mathcal{R}(\lambda I - T)^\perp$ and $Sv \in \mathcal{R}(S) \subseteq \bigvee\{f_i\}_{i=1}^m$. If $n < m$, then $k = n < n+1 \leq m$, and $e_{n+1} \in \mathcal{R}(\overline{\lambda}I - (T^* + S^*)) = \mathcal{H}$ so that there exists $u \in \mathcal{H}$ for which $(\overline{\lambda}I - (T^* + S^*))u = e_{n+1}$. Hence

$$\begin{aligned} 1 = \langle e_{m+1}\,;e_{m+1}\rangle &= \langle(\overline{\lambda}I - (T^* + S^*))u\,;e_{n+1}\rangle \\ &= \langle(\overline{\lambda}I - T^*)u\,;e_{n+1}\rangle - \langle S^*u\,;e_{n+1}\rangle = 0, \end{aligned}$$

which is a contradiction too (since $e_{n+1} \in \mathcal{N}(\lambda I - T) = \mathcal{R}(\overline{\lambda}I - T^*)^\perp$ and $S^*u \in \mathcal{R}(S^*) \subseteq \bigvee\{e_i\}_{i=1}^n$). Therefore, $m = n$. □

Together, the statements of Propositions 6.29 and 6.35 (or simply, the first identity in Corollary 6.31) are referred to as the *Fredholm Alternative*.

6.7 The Compact Normal Case

Throughout this section $\mathcal{H} \neq \{0\}$ is a complex Hilbert space. Let $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ be a bounded family of complex numbers, let $\{P_\gamma\}_{\gamma\in\Gamma}$ be a resolution of the identity on $\mathcal{H}$, and let $T \in \mathcal{B}[\mathcal{H}]$ be a (bounded) weighted sum of projections (cf. Definition 5.60 and Proposition 5.61):

$$Tx = \sum_{\gamma\in\Gamma} \lambda_\gamma P_\gamma x \quad \text{for every} \quad x \in \mathcal{H}.$$

Proposition 6.36. *Every weighted sum of projections is normal.*

Proof. Note that $\{\overline{\lambda}_\gamma\}_{\gamma\in\Gamma}$ is a bounded family of complex numbers, and consider the weighted sum of projections $T^* \in \mathcal{B}[\mathcal{H}]$ given by

$$T^*x = \sum_{\gamma\in\Gamma} \overline{\lambda}_\gamma P_\gamma x \quad \text{for every} \quad x \in \mathcal{H}.$$

This in fact is the adjoint of $T \in \mathcal{B}[\mathcal{H}]$ since each P_γ is self-adjoint (Proposition 5.81). Indeed, take $x = \sum_{\gamma\in\Gamma} P_\gamma x$ and $y = \sum_{\gamma\in\Gamma} P_\gamma y$ in $\mathcal{H}$ (recall: $\{P_\gamma\}_{\gamma\in\Gamma}$ is a resolution of the identity on $\mathcal{H}$) so that, as $\mathcal{R}(P_\alpha) \perp \mathcal{R}(P_\beta)$ if $\alpha \neq \beta$,

$$\begin{aligned} \langle Tx\,;y\rangle &= \Big\langle \textstyle\sum_{\alpha\in\Gamma}\lambda_\alpha P_\alpha x\,;\sum_{\beta\in\Gamma} P_\beta y\Big\rangle = \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma}\lambda_\alpha\langle P_\alpha x\,;P_\beta y\rangle \\ &= \sum_{\gamma\in\Gamma}\lambda_\gamma\langle P_\gamma x\,;P_\gamma y\rangle = \sum_{\beta\in\Gamma}\sum_{\alpha\in\Gamma}\lambda_\alpha\langle P_\beta x\,;P_\alpha y\rangle \\ &= \Big\langle \textstyle\sum_{\beta\in\Gamma} P_\beta x\,;\sum_{\alpha\in\Gamma}\overline{\lambda}_\alpha P_\alpha y\Big\rangle = \langle x\,;T^*y\rangle. \end{aligned}$$

Moreover, since $P_\gamma^2 = P_\gamma$ for all γ and $P_\alpha P_\beta = P_\beta P_\alpha = O$ if $\alpha \neq \beta$,

$$\begin{aligned} T^*Tx &= \textstyle\sum_{\alpha\in\Gamma}\overline{\lambda}_\alpha P_\alpha\sum_{\beta\in\Gamma}\lambda_\beta P_\beta x = \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma}\overline{\lambda}_\alpha\lambda_\beta P_\alpha P_\beta x \\ &= \textstyle\sum_{\gamma\in\Gamma}|\lambda_\gamma|^2 P_\gamma x = \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma}\lambda_\alpha\overline{\lambda}_\beta P_\alpha P_\beta x \\ &= \textstyle\sum_{\alpha\in\Gamma}\lambda_\alpha P_\alpha\sum_{\beta\in\Gamma}\overline{\lambda}_\beta P_\beta x = TT^*x \end{aligned}$$

for every $x \in \mathcal{H}$. That is, T is normal. □

Particular case: Diagonal operators and, more generally, diagonalizable operators on a separable Hilbert space (as defined in Problem 5.17), are normal operators. In fact, the concept of weighted sum of projections on any Hilbert space can be thought of as a generalization of the concept of diagonalizable operators on a separable Hilbert space. The next proposition shows that such a generalization preserves the spectral properties (compare with Example 6.C).

Proposition 6.37. *If $T \in \mathcal{B}[\mathcal{H}]$ is a weighted sum of projections, then*

$$\sigma_P(T) = \big\{\lambda \in \mathbb{C}\colon\ \lambda = \lambda_\gamma \text{ for some } \gamma \in \Gamma\big\}, \qquad \sigma_R(T) = \varnothing, \qquad \textit{and}$$

$$\sigma_C(T) = \big\{\lambda \in \mathbb{C}\colon\ \lambda \ne \lambda_\gamma \text{ for all } \gamma \in \Gamma \text{ and } \inf_{\gamma\in\Gamma} |\lambda - \lambda_\gamma| = 0\big\}.$$

Proof. Take any $x = \sum_{\gamma\in\Gamma} P_\gamma x$ in $\mathcal{H}$. Recall that $\{P_\gamma\}_{\gamma\in\Gamma}$ is a resolution of the identity on $\mathcal{H}$ so that $\|x\|^2 = \sum_{\gamma\in\Gamma}\|P_\gamma x\|^2$ by Theorem 5.32. Moreover, $\|(\lambda I - T)x\|^2 = \sum_{\gamma\in\Gamma}|\lambda - \lambda_\gamma|^2\,\|P_\gamma x\|^2$ since $(\lambda I - T)x = \sum_{\gamma\in\Gamma}(\lambda - \lambda_\gamma)P_\gamma x$ for any $\lambda \in \mathbb{C}$ (cf. Theorem 5.32 again). If $\mathcal{N}(\lambda I - T) \ne \{0\}$, then there exists an $x \ne 0$ in $\mathcal{H}$ such that $(\lambda I - T)x = 0$, and therefore $\sum_{\gamma\in\Gamma}\|P_\gamma x\|^2 \ne 0$ and $\sum_{\gamma\in\Gamma}|\lambda - \lambda_\gamma|^2\|P_\gamma x\|^2 = 0$, which implies that $\|P_\alpha x\| \ne 0$ for some $\alpha \in \Gamma$ and $|\lambda - \lambda_\alpha|\|P_\alpha x\| = 0$. Thus $\lambda = \lambda_\alpha$. Conversely, take any $\alpha \in \Gamma$ and an arbitrary nonzero vector x in $\mathcal{R}(P_\alpha)$ (recall: $P_\gamma \ne O$ and so $\mathcal{R}(P_\gamma) \ne \{0\}$ for every $\gamma \in \Gamma$). But $\mathcal{R}(P_\alpha) \perp \mathcal{R}(P_\gamma)$ whenever $\alpha \ne \gamma$ so that $\mathcal{R}(P_\alpha) \perp \bigvee_{\alpha\ne\gamma\in\Gamma}\mathcal{R}(P_\gamma)$. Hence $\mathcal{R}(P_\alpha) \subseteq \big(\bigvee_{\alpha\ne\gamma\in\Gamma}\mathcal{R}(P_\gamma)\big)^\perp = \bigwedge_{\alpha\ne\gamma\in\Gamma}\mathcal{R}(P_\gamma)^\perp = \bigwedge_{\alpha\ne\gamma\in\Gamma}\mathcal{N}(P_\gamma)$ (cf. Problem 5.8(a) and Propositions 5.76(a) and 5.81(b)). Thus $x \in \mathcal{N}(P_\gamma)$ for every $\alpha \ne \gamma \in \Gamma$, and so $\|(\lambda_\alpha I - T)x\|^2 = \sum_{\gamma\in\Gamma}|\lambda_\alpha - \lambda_\gamma|^2\|P_\gamma x\|^2 = 0$, which ensures that $\mathcal{N}(\lambda_\alpha I - T) \ne \{0\}$. Outcome: $\mathcal{N}(\lambda I - T) \ne \{0\}$ if and only if $\lambda = \lambda_\alpha$ for some $\alpha \in \Gamma$. That is,

$$\sigma_P(T) = \big\{\lambda \in \mathbb{C}\colon\ \lambda = \lambda_\gamma \text{ for some } \gamma \in \Gamma\big\}.$$

We have just seen that $\mathcal{N}(\lambda I - T) = \{0\}$ if and only if $\lambda \ne \lambda_\gamma$ for all $\gamma \in \Gamma$. In this case (i.e., if $\lambda I - T$ is injective) there exists an inverse $(\lambda I - T)^{-1}$ in $\mathcal{L}[\mathcal{R}(\lambda I - T), \mathcal{H}]$, which is a weighted sum of projections on $\mathcal{R}(\lambda I - T)$:

$$(\lambda I - T)^{-1}x = \sum_{\gamma\in\Gamma}(\lambda - \lambda_\gamma)^{-1}P_\gamma x \quad \text{for every} \quad x \in \mathcal{R}(\lambda I - T).$$

Indeed, if $\lambda \ne \lambda_\gamma$ for all $\gamma \in \Gamma$, then $\sum_{\alpha\in\Gamma}(\lambda - \lambda_\alpha)^{-1}P_\alpha\sum_{\beta\in\Gamma}(\lambda - \lambda_\beta)P_\beta x = \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma}(\lambda - \lambda_\alpha)^{-1}(\lambda - \lambda_\beta)P_\alpha P_\beta x = \sum_{\gamma\in\Gamma}P_\gamma x = x$ for every x in $\mathcal{H}$. Now recall from Proposition 5.61 that $(\lambda I - T)^{-1} \in \mathcal{B}[\mathcal{H}]$ if and only if $\lambda \ne \lambda_\gamma$ for all $\gamma \in \Gamma$ and $\sup_{\gamma\in\Gamma}|\lambda - \lambda_\gamma|^{-1} < \infty$. Equivalently, $(\lambda I - T)^{-1} \in \mathcal{B}[\mathcal{H}]$ if and only if $\inf_{\gamma\in\Gamma}|\lambda - \lambda_\gamma| > 0$. In other words,

$$\rho(T) = \big\{\lambda \in \mathbb{C}\colon\ \inf_{\gamma\in\Gamma}|\lambda - \lambda_\gamma| > 0\big\}.$$

But T is normal by Proposition 6.36, so that $\sigma_R(T) = \varnothing$ (Corollary 6.18), and hence $\sigma_C(T) = (\mathbb{C}\backslash\rho(T))\backslash\sigma_P(T)$. □

Proposition 6.38. *A weighted sum of projections $T \in \mathcal{B}[\mathcal{H}]$ is compact if and only if the following triple condition holds: $\sigma(T)$ is countable, 0 is the only possible accumulation point of $\sigma(T)$, and $\dim \mathcal{R}(P_\gamma) < \infty$ for every γ such that $\lambda_\gamma \neq 0$.*

Proof. Let $T \in \mathcal{B}[\mathcal{H}]$ be a weighted sum of projections.

Claim. $\mathcal{R}(P_\gamma) \subseteq \mathcal{N}(\lambda_\gamma I - T)$ for every γ.

Proof. Take an arbitrary index γ. If $x \in \mathcal{R}(P_\gamma)$, then $x = P_\gamma x$ (Problem 1.4) so that $Tx = TP_\gamma x = \sum_\alpha \lambda_\alpha P_\alpha P_\gamma x = \lambda_\gamma P_\gamma x = \lambda_\gamma x$ (since $P_\alpha \perp P_\gamma$ whenever $\gamma \neq \alpha$), and hence $x \in \mathcal{N}(\lambda_\gamma I - T)$. □

If T is compact, then $\sigma(T)$ is countable and 0 is the only possible accumulation point of $\sigma(T)$ (Corollary 6.34), and $\dim \mathcal{N}(\lambda I - T) < \infty$ whenever $\lambda \neq 0$ (Proposition 6.35) so that $\dim \mathcal{R}(P_\gamma) < \infty$ for every γ such that $\lambda_\gamma \neq 0$ by the above claim. Conversely, if $T = O$, then T is trivially compact. Thus suppose $T \neq O$. Since T is normal (Proposition 6.36), $r(T) > 0$ (reason: the unique normal operator with a null spectral radius is the null operator — see the remark that precedes Corollary 6.28), so that there exists $\lambda \neq 0$ in $\sigma_P(T)$ by Corollary 6.31. If $\sigma(T)$ is countable, then let $\{\lambda_k\}$ be any enumeration of the countable set $\sigma_P(T)\backslash\{0\} = \sigma(T)\backslash\{0\}$. Hence the weighted sum of projections $T \in \mathcal{B}[\mathcal{H}]$ is given by (Proposition 6.37)

$$Tx = \sum_k \lambda_k P_k x \quad \text{for every} \quad x \in \mathcal{H},$$

where $\{P_k\}$ is included in a resolution of the identity on $\mathcal{H}$ (which is itself a resolution of the identity on $\mathcal{H}$ whenever $0 \notin \sigma_P(T)$). If $\{\lambda_k\}$ is a finite set, say $\{\lambda_k\} = \{\lambda_k\}_{k=1}^n$, then $\mathcal{R}(T) = \sum_{k=1}^n \mathcal{R}(P_k)$. If $\dim \mathcal{R}(P_k) < \infty$ for every k, then $\dim \big(\sum_{k=1}^n \mathcal{R}(P_k)\big)^- < \infty$ (according to Problem 5.11), and so T lies in $\mathcal{B}_0[\mathcal{H}] \subseteq \mathcal{B}_\infty[\mathcal{H}]$. Now suppose $\{\lambda_k\}$ is countably infinite. Since $\sigma(T)$ is compact (Corollary 6.12), it follows by Theorem 3.80 and Proposition 3.77 that $\{\lambda_k\}$ has an accumulation point in $\sigma(T)$. If 0 is the only possible accumulation point of $\sigma(T)$, then 0 is the unique accumulation point of $\{\lambda_k\}$. Thus, for each integer $n \geq 1$, consider the partition $\{\lambda_k\} = \{\lambda_{k'}\} \cup \{\lambda_{k''}\}$, where $|\lambda_{k'}| \geq \frac{1}{n}$ and $|\lambda_{k''}| < \frac{1}{n}$. Note that $\{\lambda_{k'}\}$ is a finite subset of $\sigma(T)$ (it has no accumulation point), and hence $\{\lambda_{k''}\}$ is an infinite subset of $\sigma(T)$. Set

$$T_n = \sum_{k'} \lambda_{k'} P_{k'} \in \mathcal{B}[\mathcal{H}] \quad \text{for each} \quad n \geq 1.$$

We have just seen that $\dim \mathcal{R}(T_n) < \infty$. That is, $T_n \in \mathcal{B}_0[\mathcal{H}]$ for every $n \geq 1$. However, since $P_j \perp P_k$ whenever $j \neq k$, we get (cf. Corollary 5.9)

$$\|(T - T_n)x\|^2 = \Big\| \sum_{k''} \lambda_{k''} P_{k''} x \Big\|^2 \leq \sup_{k''} |\lambda_{k''}|^2 \sum_{k''} \|P_{k''} x\|^2 \leq \tfrac{1}{n^2} \|x\|^2$$

for all $x \in \mathcal{H}$, so that $T_n \xrightarrow{u} T$. Hence $T \in \mathcal{B}_\infty[\mathcal{H}]$ by Corollary 4.55. □

Before considering the Spectral Theorem for compact normal operators, we need a few spectral properties of normal operators.

Proposition 6.39. *If $T \in \mathcal{B}[\mathcal{H}]$ is normal, then*

$$\mathcal{N}(\lambda I - T) = \mathcal{N}(\overline{\lambda} I - T^*) \quad \textit{for every} \quad \lambda \in \mathbb{C}.$$

Proof. Take an arbitrary $\lambda \in \mathbb{C}$. If T is normal, then $\lambda I - T$ is normal (cf. proof of Corollary 6.18) so that $\|(\overline{\lambda} I - T^*)x\| = \|(\lambda I - T)x\|$ for every $x \in \mathcal{H}$ by Proposition 6.1(b). □

Proposition 6.40. *Take $\lambda, \mu \in \mathbb{C}$. If $T \in \mathcal{B}[\mathcal{H}]$ is normal, then*

$$\mathcal{N}(\lambda I - T) \perp \mathcal{N}(\mu I - T) \quad \textit{whenever} \quad \lambda \neq \mu.$$

Proof. Suppose $x \in \mathcal{N}(\lambda I - T)$ and $y \in \mathcal{N}(\mu I - T)$ so that $\lambda x = Tx$ and $\mu y = Ty$. Since $\mathcal{N}(\lambda I - T) = \mathcal{N}(\overline{\lambda} I - T^*)$ by the previous proposition, $\overline{\lambda} x = T^* x$. Thus $\langle \mu y \,;\, x \rangle = \langle Ty \,;\, x \rangle = \langle y \,;\, T^* x \rangle = \langle y \,;\, \overline{\lambda} x \rangle = \langle \lambda y \,;\, x \rangle$, and therefore $(\mu - \lambda)\langle y \,;\, x \rangle = 0$, which implies that $\langle y \,;\, x \rangle = 0$ whenever $\mu \neq \lambda$. □

Proposition 6.41. *If $T \in \mathcal{B}[\mathcal{H}]$ is normal, then $\mathcal{N}(\lambda I - T)$ reduces T for every $\lambda \in \mathbb{C}$.*

Proof. Take an arbitrary $\lambda \in \mathbb{C}$ and any $T \in \mathcal{B}[\mathcal{H}]$. Recall that $\mathcal{N}(\lambda I - T)$ is a subspace of $\mathcal{H}$ (Proposition 4.13). Moreover, it is clear that $\mathcal{N}(\lambda I - T)$ is T-invariant (if $Tx = \lambda x$, then $T(Tx) = \lambda(Tx)$). Similarly, $\mathcal{N}(\overline{\lambda} I - T^*)$ is T^*-invariant. Now suppose $T \in \mathcal{B}[\mathcal{H}]$ is a normal operator. Proposition 6.39 says that $\mathcal{N}(\lambda I - T) = \mathcal{N}(\overline{\lambda} I - T^*)$, and so $\mathcal{N}(\lambda I - T)$ also is T^*-invariant. Then $\mathcal{N}(\lambda I - T)$ reduces T (cf. Corollary 5.75). □

Corollary 6.42. *Let $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ be a family of distinct scalars. If $T \in \mathcal{B}[\mathcal{H}]$ is a normal operator, then the topological sum $\big(\sum_{\gamma \in \Gamma} \mathcal{N}(\lambda_\gamma I - T)\big)^-$ reduces T.*

Proof. For each $\gamma \in \Gamma$ write $\mathcal{N}_\gamma = \mathcal{N}(\lambda_\gamma I - T)$, which is a subspace of $\mathcal{H}$ (Proposition 4.13). According to Proposition 6.40, $\{\mathcal{N}_\gamma\}_{\gamma \in \Gamma}$ is a family of pairwise orthogonal subspaces of $\mathcal{H}$. Take an arbitrary $x \in \big(\sum_{\gamma \in \Gamma} \mathcal{N}_\gamma\big)^-$. If Γ is finite, then $\big(\sum_{\gamma \in \Gamma} \mathcal{N}_\gamma\big)^- = \sum_{\gamma \in \Gamma} \mathcal{N}_\gamma$ (Corollary 5.11); otherwise, apply the Orthogonal Structure Theorem (i.e., Theorem 5.16 if Γ is countably infinite or Problem 5.10 if Γ is uncountable). In any case (finite, countably infinite, or uncountable Γ), $x = \sum_{\gamma \in \Gamma} u_\gamma$ with each u_γ in $\mathcal{N}_\gamma$. Moreover, Tu_γ and $T^* u_\gamma$ lie in $\mathcal{N}_\gamma$ for each $\gamma \in \Gamma$ because each $\mathcal{N}_\gamma$ reduces T by Proposition 6.41 (cf. Corollary 5.75). Thus, since T and T^* are linear and continuous, it follows that $Tx = \sum_{\gamma \in \Gamma} Tu_\gamma \in \big(\sum_{\gamma \in \Gamma} \mathcal{N}_\gamma\big)^-$ and $T^* x = \sum_{\gamma \in \Gamma} T^* u_\gamma \in \big(\sum_{\gamma \in \Gamma} \mathcal{N}_\gamma\big)^-$. Therefore, $\big(\sum_{\gamma \in \Gamma} \mathcal{N}_\gamma\big)^-$ reduces T (cf. Corollary 5.75 again). □

Every (bounded) weighted sum of projections is normal (Proposition 6.36), and every compact weighted sum of projections has a countable set of distinct

eigenvalues (Propositions 6.37 and 6.38). The Spectral Theorem for compact normal operators ensures the converse.

Theorem 6.43. (The Spectral Theorem). *If $T \in \mathcal{B}[\mathcal{H}]$ is compact and normal, then there exists a countable resolution of the identity $\{P_k\}$ on $\mathcal{H}$ and a (similarly indexed) bounded set of scalars $\{\lambda_k\}$ such that*

$$T = \sum_k \lambda_k P_k,$$

where $\{\lambda_k\} = \sigma_P(T)$, the set of all (distinct) eigenvalues of T, and each P_k is the orthogonal projection onto the eigenspace $\mathcal{N}(\lambda_k I - T)$. Moreover, if the above countable weighted sum of projections is infinite, then it converges in the (uniform) topology of $\mathcal{B}[\mathcal{H}]$.

Proof. If T is compact and normal, then it has a nonempty point spectrum (Corollary 6.32) and its eigenspaces span $\mathcal{H}$. In other words,

Claim. $\big(\sum_{\lambda\in\sigma_P(T)}\mathcal{N}(\lambda I - T)\big)^- = \mathcal{H}$.

Proof. Set $\mathcal{M} = \big(\sum_{\lambda\in\sigma_P(T)}\mathcal{N}(\lambda I - T)\big)^-$, which is a subspace of $\mathcal{H}$. Suppose $\mathcal{M} \neq \mathcal{H}$ so that $\mathcal{M}^\perp \neq \{0\}$ (Proposition 5.15). Consider the restriction $T|_{\mathcal{M}^\perp}$ of T to $\mathcal{M}^\perp$. If T is normal, then $\mathcal{M}$ reduces T (Corollary 6.42) so that $\mathcal{M}^\perp$ is T-invariant, and hence $T|_{\mathcal{M}^\perp} \in \mathcal{B}[\mathcal{M}^\perp]$ is normal (cf. Problem 6.17). If T is compact, then $T|_{\mathcal{M}^\perp}$ is compact (see Section 4.9). Thus $T|_{\mathcal{M}^\perp}$ is a compact normal operator on the Hilbert space $\mathcal{M}^\perp \neq \{0\}$, and so $\sigma_P(T|_{\mathcal{M}^\perp}) \neq \varnothing$ by Corollary 6.32. That is, there exist $\lambda \in \mathbb{C}$ and $0 \neq x \in \mathcal{M}^\perp$ such that $T|_{\mathcal{M}^\perp}x = \lambda x$ and hence $Tx = \lambda x$. Thus $\lambda \in \sigma_P(T)$ and $x \in \mathcal{N}(\lambda I - T) \subseteq \mathcal{M}$. But this leads to a contradiction, viz., $0 \neq x \in \mathcal{M} \cap \mathcal{M}^\perp = \{0\}$. Outcome: $\mathcal{M} = \mathcal{H}$. □

Since T is compact, the nonempty set $\sigma_P(T)$ is countable (Corollaries 6.32 and 6.34) and bounded (because $T \in \mathcal{B}[\mathcal{H}]$). Then write

$$\sigma_P(T) = \{\lambda_k\}_{k\in N},$$

where $\{\lambda_k\}_{k\in N}$ is a finite or infinite sequence of distinct elements in $\mathbb{C}$ consisting of all eigenvalues of T. Here, either $N = \{1, \ldots, m\}$ for some $m \in \mathbb{N}$ if $\sigma_P(T)$ is finite, or $N = \mathbb{N}$ if $\sigma_P(T)$ is (countably) infinite. Recall that each $\mathcal{N}(\lambda_k I - T)$ is a subspace of $\mathcal{H}$ (Proposition 4.13). Moreover, since T is normal, Proposition 6.40 says that $\mathcal{N}(\lambda_k I - T) \perp \mathcal{N}(\lambda_j I - T)$ whenever $k \neq j$. Thus $\{\mathcal{N}(\lambda_k I - T)\}_{k\in N}$ is a sequence of pairwise orthogonal subspaces of $\mathcal{H}$ such that $\mathcal{H} = \big(\sum_{k\in N}\mathcal{N}(\lambda_k I - T)\big)^-$ by the above claim. Then the sequence $\{P_k\}_{k\in N}$ of the orthogonal projections onto each $\mathcal{N}(\lambda_k I - T)$ is a resolution of the identity on $\mathcal{H}$ (see Theorem 5.59). This implies that $x = \sum_{k\in N} P_k x$ and, since T is linear and continuous, $Tx = \sum_{k\in N} TP_k x$ for every $x \in \mathcal{H}$. But $P_k x \in \mathcal{R}(P_k) = \mathcal{N}(\lambda_k I - T)$, and so $TP_k x = \lambda_k P_k x$, for each $k \in N$ and every $x \in \mathcal{H}$. Hence

$$Tx = \sum_{k\in N} \lambda_k P_k x \quad \text{for every} \quad x \in \mathcal{H}.$$

Conclusion: T is a countable weighted sum of projections. If N is finite, then the theorem is proved. Thus suppose N is infinite (i.e., $N = \mathbb{N}$). In this case, the above identity says that $\sum_{k=1}^{n} \lambda_k P_k \xrightarrow{s} T$ (see the observation that follows the proof of Proposition 5.61). We show next that the above convergence actually is uniform. Indeed, for any $n \in \mathbb{N}$,

$$\Big\|\Big(T - \sum_{k=1}^{n} \lambda_k P_k\Big)x\Big\|^2 = \Big\| \sum_{k=n+1}^{\infty} \lambda_k P_k x\Big\|^2 = \sum_{k=n+1}^{\infty} |\lambda_k|^2 \|P_k x\|^2$$

$$\leq \sup_{k\geq n+1} |\lambda_k|^2 \sum_{k=n+1}^{\infty} \|P_k x\|^2 \leq \sup_{k\geq n} |\lambda_k|^2 \|x\|^2.$$

(Reason: $\mathcal{R}(P_j) \perp \mathcal{R}(P_k)$ whenever $j \neq k$, and $x = \sum_{k=1}^{\infty} P_k x$ so that $\|x\|^2 = \sum_{k=1}^{\infty} \|P_k x\|^2$ — see Corollary 5.9.) Hence

$$0 \leq \Big\|T - \sum_{k=1}^{n} \lambda_k P_k\Big\| = \sup_{\|x\|=1} \Big\|\Big(T - \sum_{k=1}^{n} \lambda_k P_k\Big)x\Big\| \leq \sup_{k\geq n} |\lambda_k|$$

for all $n \in \mathbb{N}$. Since T is compact and since $\{\lambda_n\}$ is an infinite sequence of distinct elements in $\sigma(T)$, it follows by Proposition 6.33 that $\lambda_n \to 0$. Therefore $\lim_n \sup_{k\geq n} |\lambda_k| = \limsup_n |\lambda_n| = 0$, and so $\sum_{k=1}^{n} \lambda_k P_k \xrightarrow{u} T$. □

In other words, if T is a compact and normal operator on a (nonzero) complex Hilbert space $\mathcal{H}$, then the family $\{P_\lambda\}_{\lambda\in\sigma_P(T)}$ of orthogonal projections onto each eigenspace $\mathcal{N}(\lambda I - T)$ is a resolution of the identity on $\mathcal{H}$, and T is a weighted sum of projections. Thus we write

$$T = \sum_{\lambda\in\sigma_P(T)} \lambda P_\lambda,$$

which is to be interpreted pointwise (i.e., $Tx = \sum_{\lambda\in\sigma_P(T)} \lambda P_\lambda x$ for every x in $\mathcal{H}$) as in Definition 5.60. This was naturally identified in Problem 5.16 with the orthogonal direct sum of scalar operators $\bigoplus_{\lambda\in\sigma_P(T)} \lambda I_\lambda$, where $I_\lambda = P_\lambda|_{\mathcal{R}(P_\lambda)}$. Here $\mathcal{R}(P_\lambda) = \mathcal{N}(\lambda I - T)$. Under such a natural identification we also write

$$T = \bigoplus_{\lambda\in\sigma_P(T)} \lambda P_\lambda.$$

These representations are referred to as the *spectral decomposition* of a compact normal operator T. The next result states the Spectral Theorem for compact normal operators in terms of an orthonormal basis for $\mathcal{N}(T)^\perp$ consisting of eigenvectors of T.

Corollary 6.44. *Let $T \in \mathcal{B}[\mathcal{H}]$ be compact and normal.*

(a) *For each* $\lambda \in \sigma_P(T)\backslash\{0\}$ *there is a finite orthonormal basis* $\{e_{k(\lambda)}\}_{k=1}^{n_\lambda}$ *for* $\mathcal{N}(\lambda I - T)$ *consisting entirely of eigenvectors of* T.

(b) *The set* $\{e_k\} = \bigcup_{\lambda\in\sigma_P(T)\backslash\{0\}}\{e_{k(\lambda)}\}_{k=1}^{n_\lambda}$ *is a countable orthonormal basis for* $\mathcal{N}(T)^\perp$ *made up of eigenvectors of* T.

(c) $Tx = \sum_{\lambda\in\sigma_P(T)\backslash\{0\}} \lambda \sum_{k=1}^{n_\lambda} \langle x\,;e_{k(\lambda)}\rangle\, e_{k(\lambda)}$ *for every* $x \in \mathcal{H}$, *so that*

(d) $Tx = \sum_k \mu_k \langle x\,;e_k\rangle e_k$ *for every* $x \in \mathcal{H}$, *where* $\{\mu_k\}$ *is a sequence containing all nonzero eigenvalues of* T *finitely repeated according to the multiplicity of the respective eigenspace.*

Proof. We have already seen that $\sigma_P(T)$ is nonempty and countable (cf. proof of the previous theorem). Recall that $\sigma_P(T) = \{0\}$ if and only if $T = O$ (Corollary 6.32) or, equivalently, if and only if $\mathcal{N}(T)^\perp = \{0\}$ (i.e., $\mathcal{N}(T) = \mathcal{H}$). If $T = O$ (i.e., $T = 0I$), then the above assertions hold trivially ($\sigma_P(T)\backslash\{0\} = \varnothing$, $\{e_k\} = \varnothing$, $\mathcal{N}(T)^\perp = \{0\}$ and $Tx = 0x = 0$ for every $x \in \mathcal{H}$ because the empty sum is null). Thus suppose $T \neq O$ (so that $\mathcal{N}(T)^\perp \neq \{0\}$), and take an arbitrary $\lambda \neq 0$ in $\sigma_P(T)$. According to Proposition 6.35, $\dim\mathcal{N}(\lambda I - T)$ is finite, say, $\dim\mathcal{N}(\lambda I - T) = n_\lambda$ for some *positive* integer n_λ. Then there exists a finite orthonormal basis $\{e_{k(\lambda)}\}_{k=1}^{n_\lambda}$ for the Hilbert space $\mathcal{N}(\lambda I - T) \neq \{0\}$ (cf. Proposition 5.39). This proves (a). Observe that $e_{k(\lambda)}$ is an eigenvector of T for each $k = 1, \ldots, n_\lambda$ (because $0 \neq e_{k(\lambda)} \in \mathcal{N}(\lambda I - T)$).

Claim. $\bigcup_{\lambda\in\sigma_P(T)\backslash\{0\}}\{e_{k(\lambda)}\}_{k=1}^{n_\lambda}$ is an orthonormal basis for $\mathcal{N}(T)^\perp$.

Proof. $\mathcal{H} = \big(\sum_{\lambda\in\sigma_P(T)} \mathcal{N}(\lambda I - T)\big)^-$ (cf. Claim in the proof of Theorem 6.43). Thus, according to Problem 5.8(b,d,e), it follows that

$$\mathcal{N}(T) \;=\; \bigcap_{\lambda\in\sigma_P(T)\backslash\{0\}} \mathcal{N}(\lambda I - T)^\perp \;=\; \Big(\sum_{\lambda\in\sigma_P(T)\backslash\{0\}} \mathcal{N}(\lambda I - T)\Big)^\perp$$

(because $\{\mathcal{N}(\lambda I - T)\}_{\lambda\in\sigma_P(T)}$ is a nonempty family of orthogonal subspaces of $\mathcal{H}$ — Proposition 6.40). Therefore $\mathcal{N}(T)^\perp = \big(\sum_{\lambda\in\sigma_P(T)\backslash\{0\}} \mathcal{N}(\lambda I - T)\big)^-$ (Proposition 5.15), and the claimed result follows by part (a), Proposition 6.40, and Problem 5.11. □

Thus (b) holds since $\{e_k\} = \bigcup_{\lambda\in\sigma_P(T)\backslash\{0\}}\{e_{k(\lambda)}\}_{k=1}^{n_\lambda}$ is countable by Corollary 1.11. Consider the decomposition $\mathcal{H} = \mathcal{N}(T) + \mathcal{N}(T)^\perp$ of Theorem 5.20. Take an arbitrary vector $x \in \mathcal{H}$ so that $x = u + v$ with $u \in \mathcal{N}(T)$ and $v \in \mathcal{N}(T)^\perp$. Let $v = \sum_k \langle v\,;e_k\rangle\, e_k = \sum_{\lambda\in\sigma_P(T)\backslash\{0\}} \sum_{k=1}^{n_\lambda} \langle v\,;e_{k(\lambda)}\rangle\, e_{k(\lambda)}$ be the Fourier series expansion of v (cf. Theorem 5.48) in terms of the orthonormal basis $\{e_k\} = \bigcup_{\lambda\in\sigma_P(T)\backslash\{0\}}\{e_{k(\lambda)}\}_{k=1}^{n_\lambda}$ for the Hilbert space $\mathcal{N}(T)^\perp \neq \{0\}$. Since the operator T is linear and continuous, and since $Te_{k(\lambda)} = \lambda e_{k(\lambda)}$ for each integer $k = 1, \ldots, n_\lambda$ and every $\lambda \in \sigma_P(T)\backslash\{0\}$, it follows that $Tx = Tu + Tv = Tv = \sum_{\lambda\in\sigma_P(T)\backslash\{0\}} \sum_{k=1}^{n_\lambda} \langle v\,;e_{k(\lambda)}\rangle\, Te_{k(\lambda)} = \sum_{\lambda\in\sigma_P(T)\backslash\{0\}} \lambda \sum_{k=1}^{n_\lambda} \langle v\,;e_{k(\lambda)}\rangle\, e_{k(\lambda)}$. However, $\langle x\,;e_{k(\lambda)}\rangle = \langle u\,;e_{k(\lambda)}\rangle + \langle v\,;e_{k(\lambda)}\rangle = \langle v\,;e_{k(\lambda)}\rangle$ because $u \in \mathcal{N}(T)$ and $e_{k(\lambda)} \in \mathcal{N}(T)^\perp$, which proves (c). The preceding assertions lead to (d). □

Remark: *If $T \in \mathcal{B}[\mathcal{H}]$ is compact and normal, and if $\mathcal{H}$ is nonseparable, then $0 \in \sigma_P(T)$ and $\mathcal{N}(T)$ is nonseparable*. Indeed, for $T = O$ the italicized result is trivial ($T = O$ implies $0 \in \sigma_P(T)$ and $\mathcal{N}(T) = \mathcal{H}$). On the other hand, if $T \neq O$, then $\mathcal{N}(T)^\perp \neq \{0\}$ is separable for it has a countable orthonormal basis $\{e_k\}$ (Theorem 5.44 and Corollary 6.44). If $\mathcal{N}(T)$ is separable, then it also has a countable orthonormal basis, say $\{f_k\}$, and hence $\{e_k\} \cup \{f_k\}$ is a countable orthonormal basis for $\mathcal{H} = \mathcal{N}(T) + \mathcal{N}(T)^\perp$ (Problem 5.11) so that $\mathcal{H}$ is separable. Moreover, if $0 \notin \sigma_P(T)$, then $\mathcal{N}(T) = \{0\}$, and therefore $\mathcal{H} = \mathcal{N}(T)^\perp$ is separable.

$\mathcal{N}(T)$ reduces T (Proposition 6.41), and hence $T = T|_{\mathcal{N}(T)^\perp} \oplus O$. By Problem 5.17 and Corollary 6.44(d), *if $T \in \mathcal{B}[\mathcal{H}]$ is compact and normal, then $T|_{\mathcal{N}(T)^\perp} \in \mathcal{B}[\mathcal{N}(T)^\perp]$ is diagonalizable*. Precisely, $T|_{\mathcal{N}(T)^\perp}$ is a diagonal operator with respect to the orthonormal basis $\{e_k\}$ for the separable Hilbert space $\mathcal{N}(T)^\perp$. Generalizing: An operator $T \in \mathcal{B}[\mathcal{H}]$ (not necessarily compact) acting on any Hilbert space $\mathcal{H}$ (not necessarily separable) is *diagonalizable* if there exist a resolution of the identity $\{P_\gamma\}_{\gamma\in\Gamma}$ on $\mathcal{H}$ and a bounded family of scalars $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ such that

$$Tu = \lambda_\gamma u \quad \text{whenever} \quad u \in \mathcal{R}(P_\gamma).$$

Take an arbitrary $x = \sum_{\gamma\in\Gamma} P_\gamma x$ in $\mathcal{H}$. Since T is linear and continuous, $Tx = \sum_{\gamma\in\Gamma} TP_\gamma x = \sum_{\gamma\in\Gamma} \lambda_\gamma P_\gamma x$ so that T is a weighted sum of projections (which is normal by Proposition 6.36). Thus we write (cf. Problem 5.16)

$$T = \sum_{\gamma\in\Gamma} \lambda_\gamma P_\gamma \quad \text{or} \quad T = \bigoplus_{\gamma\in\Gamma} \lambda_\gamma P_\gamma.$$

Conversely, if T is a weighted sum of projections ($Tx = \sum_{\gamma\in\Gamma} \lambda_\gamma P_\gamma x$ for every $x \in \mathcal{H}$), then $Tu = \sum_{\gamma\in\Gamma} \lambda_\gamma P_\gamma u = \sum_{\gamma\in\Gamma} \lambda_\gamma P_\gamma P_\alpha u = \lambda_\alpha u$ for every $u \in \mathcal{R}(P_\alpha)$ (since $P_\gamma P_\alpha = O$ whenever $\gamma \neq \alpha$ and $u = P_\alpha u$ whenever $u \in \mathcal{R}(P_\alpha)$), and hence T is diagonalizable. Outcome: *An operator T on $\mathcal{H}$ is diagonalizable if and only if it is a weighted sum of projections for some bounded sequence of scalars $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ and some resolution of the identity $\{P_\gamma\}_{\gamma\in\Gamma}$ on $\mathcal{H}$*. In this case, $\{P_\gamma\}_{\gamma\in\Gamma}$ is said to *diagonalize* T.

Corollary 6.45. *If $T \in \mathcal{B}[\mathcal{H}]$ is compact, then T is normal if and only if T is diagonalizable. Let $\{P_k\}$ be a resolution of the identity on $\mathcal{H}$ that diagonalizes a compact and normal operator $T \in \mathcal{B}[\mathcal{H}]$ into its spectral decomposition, and take any operator $S \in \mathcal{B}[\mathcal{H}]$. The following assertions are pairwise equivalent.*

(a) *S commutes with T and with T^*.*

(b) *$\mathcal{R}(P_k)$ reduces S for every k.*

(c) *S commutes with every P_k.*

Proof. Take a compact operator T on $\mathcal{H}$. If T is normal, then the Spectral Theorem says that it is diagonalizable. The converse is trivial since a diagonalizable operator is normal. Now suppose T is compact and normal so that

$$T = \sum_k \lambda_k P_k,$$

where $\{P_k\}$ is a resolution of the identity on $\mathcal{H}$ and $\{\lambda_k\} = \sigma_P(T)$ is the set of all (distinct) eigenvalues of T (Theorem 6.43). Recall from the proof of Proposition 6.36 that

$$T^* = \sum_k \overline{\lambda}_k P_k.$$

Take any $\lambda \in \mathbb{C}$. If S commutes with T and with T^*, then $(\lambda I - T)$ commutes with S and with S^*, so that $\mathcal{N}(\lambda I - T)$ is an invariant subspace for both S and S^* (Problem 4.20(c)). Hence $\mathcal{N}(\lambda I - T)$ reduces S (Corollary 5.75), which means that S commutes with the orthogonal projection onto $\mathcal{N}(\lambda I - T)$ (cf. observation that precedes Proposition 5.74). In particular, since $\mathcal{R}(P_k) = \mathcal{N}(\lambda_k I - T)$ for each k (Theorem 6.43), $\mathcal{R}(P_k)$ reduces S for every k, which means that S commutes with every P_k. Then (a)⇒(b)⇔(c). It is readily verified that (c)⇒(a). Indeed, if $SP_k = P_k S$ for every k, then $ST = \sum_k \lambda_k SP_k = \sum_k \lambda_k P_k S = TS$ and $ST^* = \sum_k \overline{\lambda}_k SP_k = \sum_k \overline{\lambda}_k P_k S = T^* S$ (recall that S is linear and continuous). □

6.8 A Glimpse at the General Case

What is the role played by compact operators in the Spectral Theorem? First note that, if T is compact, then its spectrum (and so its point spectrum) is countable. But this is not crucial once we know how to deal with uncountable sums. In particular, we know how to deal with an uncountable weighted sum of projections $Tx = \sum_{\gamma \in \Gamma} \lambda_\gamma P_\gamma x$ (recall that, even in this case, the above sum has only a countable number of nonzero vectors for each x). What really brings a compact operator into play is that a compact normal operator has a nonempty point spectrum and, more than that, it has enough eigenspaces to span $\mathcal{H}$ (see the fundamental claim in the proof of Theorem 6.43). That makes the difference, for a normal (noncompact) operator may have an empty point spectrum (witness: a bilateral shift), or it may have eigenspaces but not enough to span the whole space $\mathcal{H}$ (sample: an orthogonal direct sum of a bilateral shift with an identity).

The general case of the Spectral Theorem is the case that deals with plain normal (not necessarily compact) operators. In fact, the Spectral Theorem survives the lack of compactness if the point spectrum is replaced with the spectrum (which is never empty). But this has a price: a suitable statement of the Spectral Theorem for plain normal operators requires some knowledge of measure theory, and a proper proof requires a sound knowledge of it. We shall not prove the two fundamental theorems of this final section (e.g., see Conway [1] and Radjavi and Rosenthal [1]). Instead, we just state them, and verify some of their basic consequences. Thus we assume here (and only here) that the reader has, at least, some familiarity with measure theory in order

to grasp the definition of *spectral measure* and, therefore, the statement of the Spectral Theorem. Operators will be acting on complex Hilbert spaces $\mathcal{H} \neq \{0\}$ or $\mathcal{K} \neq \{0\}$.

Definition 6.46. Let Ω be a nonempty set in the complex plane $\mathbb{C}$ and let Σ_Ω be the σ-algebra of Borel subsets of Ω. A (complex) *spectral measure* in a (complex) Hilbert space $\mathcal{H}$ is a mapping $P\colon \Sigma_\Omega \to \mathcal{B}[\mathcal{H}]$ such that

(a) $P(\Lambda)$ is an orthogonal projection for every $\Lambda \in \Sigma_\Omega$,

(b) $P(\varnothing) = O$ and $P(\Omega) = I$,

(c) $P(\Lambda_1 \cap \Lambda_2) = P(\Lambda_1)P(\Lambda_2)$ for every $\Lambda_1, \Lambda_2 \in \Sigma_\Omega$,

(d) $P\big(\bigcup_k \Lambda_k\big) = \sum_k P(\Lambda_k)$ whenever $\{\Lambda_k\}$ is a countable collection of pairwise disjoint sets in Σ_Ω (i.e., P is countably additive).

If $\{\Lambda_k\}_{k\in\mathbb{N}}$ is a countably infinite collection of pairwise disjoint sets in Σ_Ω, then the identity in (d) means convergence in the strong topology:

$$\sum_{k=1}^{n} P(\Lambda_k) \xrightarrow{\ s\ } P\Big(\bigcup_{k\in\mathbb{N}} \Lambda_k\Big).$$

In fact, since $\Lambda_j \cap \Lambda_k = \varnothing$ whenever $j \neq k$, it follows by properties (b) and (c) that $P(\Lambda_j)P(\Lambda_k) = P(\Lambda_j \cap \Lambda_k) = P(\varnothing) = O$ for $j \neq k$, so that $\{P(\Lambda_k)\}_{k\in\mathbb{N}}$ is an orthogonal sequence of orthogonal projections in $\mathcal{B}[\mathcal{H}]$. Then, according to Proposition 5.58, $\{\sum_{k=1}^{n} P(\Lambda_k)\}_{n\in\mathbb{N}}$ converges strongly to the orthogonal projection in $\mathcal{B}[\mathcal{H}]$ onto $\big(\sum_{k\in\mathbb{N}} \mathcal{R}(P(\Lambda_k))\big)^- = \bigvee\big(\bigcup_{k\in\mathbb{N}} \mathcal{R}(P(\Lambda_k))\big)$. Therefore, what property (d) says (in the case of a countably infinite collection of pairwise disjoint Borel sets $\{\Lambda_k\}_{k\in\mathbb{N}}$) is that $P\big(\bigcup_{k\in\mathbb{N}} \Lambda_k\big)$ coincides with the orthogonal projection in $\mathcal{B}[\mathcal{H}]$ onto $\bigvee\big(\bigcup_{k\in\mathbb{N}} \mathcal{R}(P(\Lambda_k))\big)$. This generalizes the concept of a resolution of the identity on $\mathcal{H}$. In fact, if $\{\Lambda_k\}_{k\in\mathbb{N}}$ is a partition of Ω, then the orthogonal sequence of orthogonal projections $\{P(\Lambda_k)\}_{k\in\mathbb{N}}$ is such that

$$\sum_{k=1}^{n} P(\Lambda_k) \xrightarrow{\ s\ } P\Big(\bigcup\nolimits_{k\in\mathbb{N}} \Lambda_k\Big) = P(\Omega) = I.$$

Now take $x, y \in \mathcal{H}$ and consider the mapping $p_{x,y}\colon \Sigma_\Omega \to \mathbb{C}$ defined by

$$p_{x,y}(\Lambda) = \langle P(\Lambda)x\,;y\rangle$$

for every $\Lambda \in \Sigma_\Omega$. The mapping $p_{x,y}$ is an ordinary complex-valued (countably additive) measure on Σ_Ω. Let $\varphi\colon \Omega \to \mathbb{C}$ be any bounded Σ_Ω-measurable function. The integral of φ with respect to the measure $p_{x,y}$, viz., $\int \varphi\, dp_{x,y}$, will also be denoted by $\int \varphi(\lambda)\, dp_{x,y}$, or by $\int \varphi\, d\langle P_\lambda x\,;y\rangle$, or by $\int \varphi(\lambda)\, d\langle P_\lambda x\,;y\rangle$. Moreover, there exists a unique $F \in \mathcal{B}[\mathcal{H}]$ such that

$$\langle Fx\,;y\rangle = \int \varphi(\lambda)\, d\langle P_\lambda x\,;y\rangle \quad \text{for every} \quad x, y \in \mathcal{H}.$$

Indeed, let $f\colon \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ be defined by $f(x,y) = \int \phi(\lambda)\, d\langle P_\lambda x\,;y\rangle$ for every x, y in $\mathcal{H}$, which is a sesquilinear form. Since the measure $\langle P(\cdot)x\,;x\rangle$ is positive, $|f(x,x)| \le \int |\phi(\lambda)|\, d\langle P_\lambda x\,;x\rangle \le \|\phi\|_\infty \int d\langle P_\lambda x\,;x\rangle = \|\phi\|_\infty \langle P(\Omega)x\,;x\rangle = \|\phi\|_\infty \langle x\,;x\rangle = \|\phi\|_\infty \|x\|^2$ for every x in $\mathcal{H}$. This implies that f is bounded (i.e., $\sup_{\|x\|=\|y\|=1} |f(x,y)| < \infty$) by the polarization identity (see the remark that follows Proposition 5.4), and so is the linear functional $f(\cdot\,, y)\colon \mathcal{H} \to \mathbb{C}$ for each $y \in \mathcal{H}$. Then, by the Riesz Representation Theorem (Theorem 5.62), for each $y \in \mathcal{H}$ there is a unique $z_y \in \mathcal{H}$ such that $f(x,y) = \langle x\,; z_y\rangle$ for every $x \in \mathcal{H}$. This establishes a mapping $\Phi\colon \mathcal{H} \to \mathcal{H}$ assigning to each $y \in \mathcal{H}$ such a unique $z_y \in \mathcal{H}$ so that $f(x,y) = \langle x\,; \Phi y\rangle$ for every $x, y \in \mathcal{H}$. Φ is unique and lies in $\mathcal{B}[\mathcal{H}]$ (cf. proof of Proposition 5.65(a,b)). Thus $F = \Phi^*$ is the unique operator in $\mathcal{B}[\mathcal{H}]$ for which $\langle Fx\,;y\rangle = f(x,y)$ for every $x, y \in \mathcal{H}$. The notation

$$F = \int \varphi(\lambda)\, dP_\lambda$$

is just a shorthand for the identity $\langle Fx\,;y\rangle = \int \varphi(\lambda)\, d\langle P_\lambda x\,;y\rangle$ for every x, y in $\mathcal{H}$. Observe that $\langle F^*x\,;y\rangle = \langle \Phi x\,;y\rangle = \overline{\langle y\,;\Phi x\rangle} = \overline{f(y,x)} = \int \varphi(\lambda)\, d\langle P_\lambda y\,;x\rangle = \int \overline{\varphi(\lambda)}\, d\langle P_\lambda x\,;y\rangle)$ for every $x, y \in \mathcal{H}$, and hence

$$F^* = \int \overline{\varphi(\lambda)}\, dP_\lambda.$$

If $\psi\colon \Omega \to \mathbb{C}$ is a bounded Σ_Ω-measurable function and $G = \int \psi(\lambda)\, dP_\lambda$, then it can be shown that $FG = \int \varphi(\lambda)\psi(\lambda)\, dP_\lambda$. In particular,

$$F^*F = \int |\varphi(\lambda)|^2\, dP_\lambda = FF^*$$

so that F is normal. The Spectral Theorem states the converse.

Theorem 6.47. (The Spectral Theorem). *If $N \in \mathcal{B}[\mathcal{H}]$ is normal, then there exists a unique spectral measure P on $\Sigma_{\sigma(N)}$ such that*

$$N = \int \lambda\, dP_\lambda.$$

If Λ is a nonempty relatively open subset of $\sigma(N)$, then $P(\Lambda) \neq O$.

The representation $N = \int \lambda\, dP_\lambda$ is usually referred to as the *spectral decomposition* of N. Note that $N^*N = \int |\lambda|^2\, dP_\lambda = NN^*$.

Theorem 6.48. (Fuglede). *Let $N = \int \lambda\, dP_\lambda$ be the spectral decomposition of a normal operator $N \in \mathcal{B}[\mathcal{H}]$. If $S \in \mathcal{B}[\mathcal{H}]$ commutes with N, then S commutes with $P(\Lambda)$ for every $\Lambda \in \Sigma_{\sigma(N)}$.*

In other words, if $SN = NS$, then $SP(\Lambda) = P(\Lambda)S$, and so each subspace $\mathcal{R}(P(\Lambda))$ reduces S, which means that $\{\mathcal{R}(P(\Lambda))\}_{\Lambda\in\Sigma_{\sigma(N)}}$ is a family of reducing subspaces for every operator that commutes with a normal operator

$N = \int \lambda \, dP_\lambda$. If $\sigma(N)$ has a single point, say $\sigma(N) = \{\lambda\}$, then $N = \lambda I$ (by uniqueness of the spectral measure); that is, N is a scalar operator so that every subspace of $\mathcal{H}$ reduces N. Hence, if N is nonscalar, then $\sigma(N)$ has more than one point (and $\dim \mathcal{H} > 1$). If $\lambda, \mu \in \sigma(N)$ and $\lambda \neq \mu$, then let $\mathbb{D}_\lambda$ be the open disk of radius $\frac{1}{2}|\lambda - \mu|$ centered at λ. Set $\Lambda_\lambda = \sigma(N) \cap \mathbb{D}_\lambda$ and $\Lambda'_\lambda = \sigma(N) \backslash \mathbb{D}_\lambda$ in $\Sigma_{\sigma(N)}$ so that $\sigma(N)$ is the disjoint union of Λ_λ and Λ'_λ. Note that $P(\Lambda_\lambda) \neq O$ and $P(\Lambda'_\lambda) \neq O$ (since Λ_λ and $\sigma(N) \backslash \mathbb{D}_\lambda^-$ are nonempty relatively open subsets of $\sigma(N)$, and $\sigma(N) \backslash \mathbb{D}_\lambda^- \subseteq \Lambda'_\lambda$). Then $I = P(\sigma(N)) = P(\Lambda_\lambda \cup \Lambda'_\lambda) = P(\Lambda_\lambda) + P(\Lambda'_\lambda)$, and therefore $P(\Lambda_\lambda) = I - P(\Lambda'_\lambda) \neq I$. Thus

$$\{0\} \neq \mathcal{R}(P(\Lambda_\lambda)) \neq \mathcal{H}.$$

Conclusions: Suppose $\dim \mathcal{H} > 1$. *Every normal operator has a nontrivial reducing subspace*. Actually, *every nonscalar normal operator has a nontrivial hyperinvariant subspace which reduces every operator that commutes with it*. In fact, *an operator is reducible if and only if it commutes with a nonscalar normal operator* or, equivalently, *if and only if it commutes with a nontrivial orthogonal projection* (cf. observation preceding Proposition 5.74).

Corollary 6.49. (Fuglede–Putnam). *If $N_1 \in \mathcal{B}[\mathcal{H}]$ and $N_2 \in \mathcal{B}[\mathcal{K}]$ are normal operators, and if $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ intertwines N_1 to N_2, then X intertwines N_1^* to N_2^** (i.e., *if $XN_1 = N_2X$, then $XN_1^* = N_2^*X$*).

Proof. Let $N = \int \lambda \, dP_\lambda \in \mathcal{B}[\mathcal{H}]$ be normal. Take any $\Lambda \in \Sigma_{\sigma(N)}$ and $S \in \mathcal{B}[\mathcal{H}]$.

Claim. $SN = NS \iff SP(\Lambda) = P(\Lambda)S \iff SN^* = N^*S$.

Proof. If $SN = NS$, then $SP(\Lambda) = P(\Lambda)S$ for every $\Lambda \in \Sigma_{\sigma(N)}$ by Theorem 6.48. Therefore, for every $x, y \in \mathcal{H}$,

$$\begin{aligned} \langle SN^*x \,;y\rangle &= \langle N^*x \,;S^*y\rangle = \int \overline{\lambda} \, d\langle P_\lambda x \,;S^*y\rangle \\ &= \int \overline{\lambda} \, d\langle SP_\lambda x \,;y\rangle = \int \overline{\lambda} \, d\langle P_\lambda Sx \,;y\rangle = \langle N^*Sx \,;y\rangle. \end{aligned}$$

Hence $SN^* = N^*S$ so that $NS^* = S^*N$. Conversely, If $NS^* = S^*N$, then $P(\Lambda)S^* = S^*P(\Lambda)$, and so $SP(\Lambda) = P(\Lambda)S$, for every $\Lambda \in \Sigma_{\sigma(N)}$ (cf. Theorem 6.48 again). Thus $SN = NS$ since, for every $x, y \in \mathcal{H}$,

$$\begin{aligned} \langle SNx \,;y\rangle &= \langle Nx \,;S^*y\rangle = \int \lambda \, d\langle P_\lambda x \,;S^*y\rangle \\ &= \int \lambda \, d\langle SP_\lambda x \,;y\rangle = \int \lambda \, d\langle P_\lambda Sx \,;y\rangle, = \langle NSx \,;y\rangle. \quad \square \end{aligned}$$

Finally, take $N_1 \in \mathcal{B}[\mathcal{H}],\ N_2 \in \mathcal{B}[\mathcal{K}],\ X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$, and consider the operators $N = N_1 \oplus N_2 = \left(\begin{smallmatrix} N_1 & O \\ O & N_2 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} O & O \\ X & O \end{smallmatrix}\right)$ in $\mathcal{B}[\mathcal{H} \oplus \mathcal{K}]$. If N_1 and N_2 are normal, then N is normal. If $XN_1 = N_2X$, then $SN = NS$ and so $SN^* = N^*S$ by the above claim. Hence $XN_1^* = N_2^*X$. $\square$

In particular, the claim in the above proof ensures that $S \in \mathcal{B}[\mathcal{H}]$ *commutes with* N *and with* N^* *if and only if* S *commutes with* $P(\Lambda)$ *or, equivalently,* $\mathcal{R}(P(\Lambda))$ *reduces* S*, for every* $\Lambda \in \Sigma_{\sigma(N)}$ (compare with Corollary 6.45).

Corollary 6.50. *Take* $N_1 \in \mathcal{B}[\mathcal{H}]$*,* $N_2 \in \mathcal{B}[\mathcal{K}]$*, and* $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$*. If* N_1 *and* N_2 *are normal operators and* $XN_1 = N_2X$*, then*

(a) $\mathcal{N}(X)$ *reduces* N_1 *and* $\mathcal{R}(X)^-$ *reduces* N_2*,*

so that $N_1|_{\mathcal{N}(X)^\perp} \in \mathcal{B}[\mathcal{N}(X)^\perp]$ *and* $N_2|_{\mathcal{R}(X)^-} \in \mathcal{B}[\mathcal{R}(X)^-]$*. Moreover,*

(b) $N_1|_{\mathcal{N}(X)^\perp}$ *and* $N_2|_{\mathcal{R}(X)^-}$ *are unitarily equivalent.*

Proof. (a) Since $XN_1 = N_2X$, it follows that $\mathcal{N}(X)$ is N_1-invariant and $\mathcal{R}(X)$ is N_2-invariant (and so $\mathcal{R}(X)^-$ is N_2-invariant — cf. Problem 4.18). Indeed, if $Xx = 0$, then $XN_1x = N_2Xx = 0$; and $N_2Xx = XN_1x \in \mathcal{R}(X)$ for every $x \in \mathcal{H}$. Corollary 6.49 ensures that $XN_1^* = N_2^*X$, and so $\mathcal{N}(X)$ is N_1^*-invariant and $\mathcal{R}(X)^-$ is N_2^*-invariant. Therefore (Corollary 5.75), $\mathcal{N}(X)$ reduces N_1 and $\mathcal{R}(X)^-$ reduces N_2.

(b) Let $X = WQ$ be the polar decomposition of X, where $Q = (X^*X)^{\frac{1}{2}}$ (Theorem 5.89). Observe that $XN_1 = N_2X$ implies $X^*N_2^* = N_1^*X^*$, which in turn implies $X^*N_2 = N_1X^*$ by Corollary 6.49. Then

$$Q^2N_1 = X^*XN_1 = X^*N_2X = N_1X^*X = N_1Q^2$$

so that $QN_1 = N_1Q$ (Theorem 5.85). Therefore, $WN_1Q = WQN_1 = XN_1 = N_2X = N_2WQ$. That is,

$$(WN_1 - N_2W)Q = O.$$

Thus $\mathcal{R}(Q)^- \subseteq \mathcal{N}(WN_1 - N_2W)$, and so

$$\mathcal{N}(Q)^\perp \subseteq \mathcal{N}(WN_1 - N_2W)$$

(since $Q = Q^*$ so that $\mathcal{R}(Q)^- = \mathcal{N}(Q)^\perp$ by Proposition 5.76). Recall that

$$\mathcal{N}(W) = \mathcal{N}(Q) = \mathcal{N}(Q^2) = \mathcal{N}(X^*X) = \mathcal{N}(X)$$

(cf. Propositions 5.76 and 5.86, and Theorem 5.89). If $u \in \mathcal{N}(Q)$ then $N_2Wu = 0$, and $N_1u = N_1|_{\mathcal{N}(X)}u \in \mathcal{N}(X) = \mathcal{N}(W)$ (because $\mathcal{N}(X)$ is N_1-invariant) so that $WN_1u = 0$. Hence

$$\mathcal{N}(Q) \subseteq \mathcal{N}(WN_1 - N_2W).$$

The above displayed inclusions imply that $\mathcal{N}(WN_1 - N_2W) = \mathcal{K}$ (cf. Problem 5.7(b)), which means that $WN_1 = N_2W = O$. Thus

$$WN_1 = N_2W.$$

Now $W = VP$, where $V{:}\mathcal{N}(W)^\perp \to \mathcal{K}$ is an isometry and $P{:}\mathcal{H} \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{N}(W)^\perp$ (Proposition 5.87). Then

$$VPN_1 = N_2VP \quad \text{so that} \quad VPN_1|_{\mathcal{N}(X)^\perp} = N_2VP|_{\mathcal{N}(X)^\perp}.$$

Since $\mathcal{R}(P) = \mathcal{N}(W)^\perp = \mathcal{N}(X)^\perp$ is N_1-invariant (recall: $\mathcal{N}(X)$ reduces N_1), it follows that $N_1(\mathcal{N}(X)^\perp) \subseteq \mathcal{N}(X)^\perp = \mathcal{R}(P)$, and hence

$$VPN_1|_{\mathcal{N}(X)^\perp} = VN_1|_{\mathcal{N}(X)^\perp}.$$

Since $\mathcal{R}(V) = \mathcal{R}(W) = \mathcal{R}(X)^-$ (cf. Theorem 5.89 and the observation that precedes Proposition 5.88), it also follows that

$$N_2VP|_{\mathcal{N}(X)^\perp} = N_2VP|_{\mathcal{R}(P)} = N_2V = N_2|_{\mathcal{R}(X)^-}V.$$

But $V{:}\mathcal{N}(W)^\perp \to \mathcal{R}(V)$ is a unitary transformation (i.e., a surjective isometry) of the Hilbert space $\mathcal{N}(X)^\perp = \mathcal{N}(W)^\perp \subseteq \mathcal{H}$ onto the Hilbert space $\mathcal{R}(X)^- = \mathcal{R}(V) \subseteq \mathcal{K}$. Conclusion:

$$VN_1|_{\mathcal{N}(X)^\perp} = N_2|_{\mathcal{R}(X)^-}V,$$

so that the operators $N_1|_{\mathcal{N}(X)^\perp} \in \mathcal{B}[\mathcal{N}(X)^\perp]$ and $N_2|_{\mathcal{R}(X)^-} \in \mathcal{B}[\mathcal{R}(X)^-]$ are unitarily equivalent. □

An immediate consequence of Corollary 6.50: *If a quasiinvertible linear transformation intertwines two normal operators, then these normal operators are unitarily equivalent*. That is, if $N_1 \in \mathcal{B}[\mathcal{H}]$ and $N_2 \in \mathcal{B}[\mathcal{K}]$ are normal operators, and if $XN_1 = N_2X$, where $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ is such that $\mathcal{N}(X) = \{0\}$ (equivalently, $\mathcal{N}(X)^\perp = \mathcal{H}$) and $\mathcal{R}(X)^- = \mathcal{K}$, then $UN_1 = N_2U$ for a unitary $U \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$. This happens, in particular, when X is invertible (i.e., if X is in $\mathcal{G}[\mathcal{H}, \mathcal{K}]$). Outcome: *Two similar normal operators are unitarily equivalent*.

Applying Theorems 6.47 and 6.48 we saw that normal operators (on a complex Hilbert space of dimension greater than 1) have a nontrivial invariant subspace. This also is the case for compact operators (on a complex Banach space of dimension greater than 1). The ultimate result along this line was presented by Lomonosov in 1973: *An operator has a nontrivial invariant subspace if it commutes with a nonscalar operator that commutes with a nonzero compact operator*. In fact, *every nonscalar operator that commutes with a nonscalar compact operator* (itself, in particular) *has a nontrivial hyperinvariant subspace*. Recall that, on an infinite-dimensional normed space, the only scalar compact operator is the null operator. On a finite-dimensional normed space every operator is compact, and hence every operator on a complex finite-dimensional normed space of dimension greater than 1 has a nontrivial invariant subspace and, if it is nonscalar, a nontrivial hyperinvariant subspace as well. This prompts the most celebrated open question in operator theory, namely, the *invariant subspace problem*: *Does every operator* (on

an infinite-dimensional complex separable Hilbert space) *have a nontrivial invariant subspace*? All the qualifications are crucial here. Observe that the operator $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on $\mathbb{R}^2$ has no nontrivial invariant subspace (when acting on the Euclidean *real* space but, of course, it has a nontrivial invariant subspace when acting on the unitary *complex* space $\mathbb{C}^2$). Thus the preceding question actually refers to complex spaces and, henceforward, we assume that all spaces are complex. The problem has a negative answer if we replace Hilbert space with Banach space. This (the invariant subspace problem in a Banach space) remained as an open question for a long period up to the mid-1980s, when it was solved by Read (1984) and Enflo (1987), who constructed a Banach-space operator without a nontrivial invariant subspace. As we have just seen, the problem has an affirmative answer in a finite-dimensional space (of dimension greater than 1). It has an affirmative answer in a nonseparable Hilbert space too. Indeed, let T be any operator on a nonseparable Hilbert space $\mathcal{H}$, and let x be any nonzero vector in $\mathcal{H}$. Consider the orbit of x under T, $\{T^n x\}_{n\geq 0}$, so that $\bigvee\{T^n x\}_{n\geq 0} \neq \{0\}$ is an invariant subspace for T (cf. Problem 4.23). Since $\{T^n x\}_{n\geq 0}$ is a countable set, $\bigvee\{T^n x\}_{n\geq 0} \neq \mathcal{H}$ by Proposition 4.9(b). Hence $\bigvee\{T^n x\}_{n\geq 0}$ is a nontrivial invariant subspace for T. Completeness and boundedness are also crucial here. In fact, it can be shown that (1) there is an operator on an infinite-dimensional complex separable (incomplete) inner product space which has no nontrivial invariant subspace, and that (2) there is a (not necessarily bounded) linear transformation of a complex separable Hilbert space into itself without nontrivial invariant subspaces. However, for bounded linear operators on an infinite-dimensional complex separable Hilbert space, the invariant subspace problem remains a recalcitrant open question.

Suggested Reading

Akhiezer and Glazman [1], [2]
Arveson [2]
Bachman and Narici [1]
Beals [1]
Beauzamy [1]
Berberian [1], [3]
Berezansky, Sheftel, and Us [1], [2]
Clancey [1]
Colojoară and Foiaş [1]
Conway [1], [2], [3]
Douglas [1]
Dowson [1]
Dunford and Schwartz [3]
Fillmore [1], [2]
Furuta [1]
Gustafson and Rao [1]
Halmos [1], [4]
Helmberg [1]
Herrero [1]
Kubrusly [1], [2]
Martin and Putinar [1]
Naylor and Sell [1]
Pearcy [1], [2]
Putnam [1]
Radjavi and Rosenthal [1], [2]
Riesz and Sz.-Nagy [1]
Sunder [1]
Sz.-Nagy and Foiaş [1]
Taylor and Lay [1]
Weidmann [1]
Xia [1]
Yoshino [1]

Problems

Problem 6.1. Let $\mathcal{H}$ be a Hilbert space. Show that the set of all normal operators from $\mathcal{B}[\mathcal{H}]$ is closed in $\mathcal{B}[\mathcal{H}]$.

Hint: $(T^* - S^*)(T - S) + (T^* - S^*)S + S^*(T - S) = T^*T - S^*S$, and hence $\|T^*T - S^*S\| \le \|T - S\|^2 + 2\|S\|\|T - S\|$ for every $T, S \in \mathcal{B}[\mathcal{H}]$. Verify the above inequality. Now let $\{N_n\}_{n=1}^{\infty}$ be a sequence of normal operators in $\mathcal{B}[\mathcal{H}]$ that converges in $\mathcal{B}[\mathcal{H}]$ to $N \in \mathcal{B}[\mathcal{H}]$. Check that

$$\begin{aligned} \|N^*N - NN^*\| &= \|N^*N - N_n^*N_n + N_nN_n^* - NN^*\| \\ &\le \|N_n^*N_n - N^*N\| + \|N_nN_n^* - NN^*\| \\ &\le 2\big(\|N_n - N\|^2 + 2\|N\|\|N_n - N\|\big). \end{aligned}$$

Conclude: *The (uniform) limit of a uniformly convergent sequence of normal operators is normal*. Finally, apply the Closed Set Theorem.

Problem 6.2. Let S and T be normal operators acting on the same Hilbert space. Prove the following assertions.

(a) αT is normal for every scalar α.

(b) If $S^*T = TS^*$, then $S + T$, TS and ST are normal operators.

(c) $T^{*n}T^n = T^nT^{*n} = (T^*T)^n = (TT^*)^n$ for every integer $n \ge 0$.

Hint: Problem 5.24 and Proposition 6.1.

Problem 6.3. Let T be a contraction on a Hilbert space $\mathcal{H}$. Show that

(a) $T^{*n}T^n \xrightarrow{s} A$,

(b) $O \le A \le I$ (i.e., $A \in \mathcal{B}^+[\mathcal{H}]$ and $\|A\| \le 1$; a nonnegative contraction),

(c) $T^{*n}AT^n = A$ for every integer $n \ge 0$.

Hint: Take $T \in \mathcal{B}[\mathcal{H}]$ with $\|T\| \le 1$. Use Proposition 5.84, Problem 5.49 and Proposition 5.68, Problems 4.45(a), 5.55, and 5.24(a).

According to Problem 5.54 *a contraction T is strongly stable if and only if $A = O$*. Since $A \ge O$, it follows by Proposition 5.81 that A is an orthogonal projection if and only if it is idempotent (i.e., if and only if $A = A^2$). In general, A may not be a projection.

(d) Consider the operator $T = \text{shift}(\alpha, 1, 1, 1, \ldots)$ in $\mathcal{B}[\ell_+^2]$, a weighted shift on $\mathcal{H} = \ell_+^2$ with $|\alpha| \in (0, 1)$, and show that T is a contraction for which $A = \text{diag}(|\alpha|^2, 1, 1, 1, \ldots)$ in $\mathcal{B}[\ell_+^2]$ is not a projection.

(e) Show that $A = A^2$ if and only if $AT = TA$.

Hint: Use part (c) to show that $\langle ATx\,;TAx\rangle = \|Ax\|^2$. Since $\|T\| \le 1$, check that $\|ATx - TAx\|^2 \le \|ATx\|^2 - \|Ax\|^2$. Recalling that $\|T\| \le 1$, $\|A\| \le 1$, and using part (c), show that $\|Ax\|^2 \le \|ATx\|^2 \le \|A^{\frac{1}{2}}Tx\|^2 = \langle T^*ATx\,;x\rangle = \langle Ax\,;x\rangle = \|A^{\frac{1}{2}}x\|^2$. Conclude: $A = A^2$ implies $AT = TA$. For the converse, use parts (a) and (c).

(f) Show that $A = A^2$ if T is a normal contraction.

Hint: Problems 6.2(c) and 5.24.

Remark: It can be shown that $A = A^2$ if T is a cohyponormal contraction.

Problem 6.4. Consider the Hilbert space $L^2(\mathbb{T})$ of Example 5.L(c), where $\mathbb{T}$ denotes the unit circle about the origin of the complex plane. Recall that, in this context, the terms "bounded function", "equality", "inequality", "belongs", and "for all" are interpreted in the sense of equivalence classes. Let $\varphi\colon \mathbb{T} \to \mathbb{C}$ be a bounded function. Show that

(a) φf lies in $L^2(\mathbb{T})$ for every $f \in L^2(\mathbb{T})$.

Thus consider the mapping $M_\varphi\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined by

$$M_\varphi f \;=\; \varphi f \quad \text{ for every } \quad f \in L^2(\mathbb{T}).$$

That is, $(M_\varphi f)(z) = \varphi(z)f(z)$ for all $z \in \mathbb{T}$. This mapping is called the *multiplication operator* on $L^2(\mathbb{T})$. It is easy to show that M_φ is linear and bounded (i.e., $M_\varphi \in \mathcal{B}[L^2(\mathbb{T})]$). Prove the following propositions.

(b) $\|M_\varphi\| = \|\varphi\|_\infty$.

Hint: Show that $\|M_\varphi f\| \le \|\varphi\|_\infty \|f\|$ for every $f \in L^2(\mathbb{T})$. Take any $\varepsilon > 0$ and set $\mathbb{T}_\varepsilon = \{z \in \mathbb{T}\colon \|\varphi\|_\infty - \varepsilon < |\varphi(z)|\}$. Let f_ε be the characteristic function of $\mathbb{T}_\varepsilon$. Show that $f_\varepsilon \in L^2(\mathbb{T})$ and $\|M_\varphi f_\varepsilon\| \ge (\|\varphi\|_\infty - \varepsilon)\|f - \varepsilon\|$.

(c) $M_\varphi^* g = \overline{\varphi}\, g$ for every $g \in L^2(\mathbb{T})$.

(d) M_φ is a normal operator.

(e) M_φ is unitary if and only if $\varphi(z) \in \mathbb{T}$ for all $z \in \mathbb{T}$.

(f) M_φ is self-adjoint if and only if $\varphi(z) \in \mathbb{R}$ for all $z \in \mathbb{T}$.

(g) M_φ is nonnegative if and only if $\varphi(z) \ge 0$ for all $z \in \mathbb{T}$.

(h) M_φ is positive if and only if $\varphi(z) > 0$ for all $z \in \mathbb{T}$.

(i) M_φ is strictly positive if and only if $\varphi(z) \ge \alpha > 0$ for all $z \in \mathbb{T}$.

Problem 6.5. If T is a quasinormal operator, then

(a) $(T^*T)^n T = T(T^*T)^n$ for every $n \ge 0$,

(b) $|T^n| = |T|^n$ for every $n \ge 0$, and

(c) $T^n \xrightarrow{s} O \iff |T|^n \xrightarrow{s} O \iff |T|^n \xrightarrow{w} O$.

Hint: Prove (a) by induction. (b) holds trivially for $n = 0, 1$, for every operator T. If T is a quasinormal operator (so that (a) holds) and if $|T^n| = |T|^n$ for some $n \geq 1$, then verify that $|T^{n+1}|^2 = T^*T^{*n}T^nT = T^*|T^n|^2T = T^*|T|^{2n}T = T^*(T^*T)^nT = T^*T(T^*T)^n = (T^*T)^{n+1} = |T|^{2(n+1)} = (|T|^{n+1})^2$. Now conclude the induction that proves (b) by recalling that the square root is unique. Use Problem 5.61(d) and part (b) to prove (c).

Problem 6.6. Every quasinormal operator is hyponormal. Give a direct proof.

Hint: Let T be an operator on a Hilbert space $\mathcal{H}$. Take any $x = u + v \in \mathcal{H} = \mathcal{N}(T^*) + \mathcal{N}(T^*)^\perp = \mathcal{N}(T^*) + \mathcal{R}(T)^-$, with $u \in \mathcal{N}(T^*)$ and $v \in \mathcal{R}(T)^-$, so that $v = \lim_n v_n$ where $\{v_n\}$ is an $\mathcal{R}(T)$-valued sequence (cf. Propositions 4.13, 5.20, 5.76 and 3.27). Set $D = T^*T - TT^*$. Verify that $\langle Du\,;u\rangle = \|Tu\|^2$. If T is quasinormal (i.e., $DT = O$), then $\langle Du\,;v\rangle = \lim_n\langle u\,;Dv_n\rangle = 0$, $\langle Dv\,;u\rangle = \lim_n\langle Dv_n\,;u\rangle = 0$, and $\langle Dv\,;v\rangle = \lim_n \langle Dv_n\,;v\rangle = 0$. Thus $\langle Dx\,;x\rangle \geq 0$.

Problem 6.7. If $T \in \mathcal{G}[\mathcal{H}]$ is hyponormal, then T^{-1} is hyponormal.

Hint: $O \leq D = T^*T - TT^*$. Then (Problem 5.51(a)) $O \leq T^{-1}DT^{-1*}$. Show that $I \leq T^{-1}T^*TT^{*-1}$ and so $T^*T^{-1}T^{*-1}T \leq I$ (Problems 1.10 and 5.53(b)). Verify: $O \leq T^{-1*}(I - T^*T^{-1}T^{*-1}T)T^{-1}$. Conclude: T^{-1} is hyponormal.

Problem 6.8. If $T \in \mathcal{G}[\mathcal{H}]$ is hyponormal and both T and T^{-1} are contractions, then T is normal.

Hint: $\|Tx\| = \|T(T^*)^{-1}T^*x\| \leq \|T^*x\|$, and so T is cohyponormal.

Remark: The above statement is just a particular case of the following general result. *If T is invertible and both T and T^{-1} are contractions, then T is unitary* (and $\|T\| = \|T^{-1}\| = 1$) — see Proposition 5.73(a,d, and c).

Problem 6.9. Take any operator $T \in \mathcal{B}[\mathcal{H}]$ on a Hilbert space $\mathcal{H}$ and take an arbitrary vector $x \in \mathcal{H}$. Show that

(a) $T^*Tx = \|T\|^2x$ if and only if $\|Tx\| = \|T\|\,\|x\|$.

Hint: If $\|Tx\| = \|T\|\,\|x\|$, then $\langle T^*Tx\,;\|T\|^2x\rangle = \|T\|^4\|x\|^2$. Therefore, $\|T^*Tx - \|T\|^2x\|^2 = \|T^*Tx\|^2 - \|T\|^4\|x\|^2 \leq (\|T^*T\|^2 - \|T\|^4)\|x\|^2 = 0$.

Now consider the set

$$\mathcal{M} = \big\{x \in \mathcal{H}\colon\ \|Tx\| = \|T\|\,\|x\|\big\}$$

and prove the following assertions.

(b) $\mathcal{M}$ is a subspace of $\mathcal{H}$.

Hint: $\mathcal{M} = \mathcal{N}(\|T\|^2I - T^*T)$.

(c) If T is hyponormal, then $\mathcal{M}$ is T-invariant.

Hint: $\|T(Tx)\| \le \|T\|\|Tx\| = \|\|T\|^2 x\| = \|T^*Tx\| \le \|T(Tx)\|$ if $x \in \mathcal{M}$.

(d) If T is normal, then $\mathcal{M}$ reduces T.

Hint: $\mathcal{M}$ is invariant for both T and T^* whenever T is normal.

Note: $\mathcal{M}$ may be trivial (examples: $T = I$ and $T = \operatorname{diag}(\{\frac{k}{k+1}\}_{k=1}^{\infty})$).

Problem 6.10. Let $\mathcal{H} \ne \{0\}$ and $\mathcal{K} \ne \{0\}$ be complex Hilbert spaces. Take $T \in \mathcal{B}[\mathcal{H}]$ and $W \in \mathcal{G}[\mathcal{H}, \mathcal{K}]$ arbitrary. Recall that $\mathcal{H}$ and $\mathcal{K}$ are unitarily equivalent, according to Problem 5.70. Show that

$$\sigma_P(T) = \sigma_P(WTW^{-1}) \quad \text{and} \quad \rho(T) = \rho(WTW^{-1}).$$

Thus conclude that (see Proposition 6.17)

$$\sigma_R(T) = \sigma_R(WTW^{-1}) \quad \text{and} \quad \sigma(T) = \sigma(WTW^{-1}).$$

Finally, verify that

$$\sigma_C(T) = \sigma_C(WTW^{-1}).$$

Outcome: *Similarity preserves each part of the spectrum*, and so *similarity preserves the spectral radius*: $r(T) = r(WTW^{-1})$. That is, if $T \in \mathcal{B}[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{K}]$ are similar (i.e., if $T \approx S$), then $\sigma_P(T) = \sigma_P(S)$, $\sigma_R(T) = \sigma_R(S)$, $\sigma_C(T) = \sigma_C(S)$, and so $r(T) = r(S)$. Use Problem 4.41 to show that *unitary equivalence also preserves the norm* (i.e., if $T \cong S$, then $\|T\| = \|S\|$).

Note: Similarity preserves nontrivial invariant subspaces (Problem 4.29).

Problem 6.11. Let $A \in \mathcal{B}[\mathcal{H}]$ be a self-adjoint operator on a complex Hilbert space $\mathcal{H} \ne \{0\}$. Use Corollary 6.18(d) to check that $\pm i \in \rho(A)$ so that $A + iI$ and $A - iI$ both lie in $\mathcal{G}[\mathcal{H}]$. Consider the operator

$$U = (A - iI)(A + iI)^{-1} = (A + iI)^{-1}(A - iI)$$

in $\mathcal{G}[\mathcal{H}]$, where $U^{-1} = (A + iI)(A - iI)^{-1} = (A - iI)^{-1}(A + iI)$ (cf. Corollary 4.23). Note that A commutes with $(A + iI)^{-1}$ and with $(A - iI)^{-1}$ because every operator commutes with its resolvent function. Show that

(a) U is unitary,

(b) $U = I - 2i(A + iI)^{-1} = 2A(A + iI)^{-1} - I$,

(c) $1 \in \rho(U)$ and $A = i(I + U)(I - U)^{-1} = i(I - U)^{-1}(I + U)$.

Hint: (a) $\|(A \pm iI)x\|^2 = \|Ax\|^2 + \|x\|^2$ (since $A = A^*$ and so $2\operatorname{Re}\langle Ax\,; ix\rangle = 0$) for every $x \in \mathcal{H}$. Take any $y \in \mathcal{H}$ so that $y = (A + iI)x$ for some $x \in \mathcal{H}$ (recall: $\mathcal{R}(A + iI) = \mathcal{H}$). Then $Uy = (A - iI)x$ and $\|Uy\|^2 = \|(A - iI)x\|^2 = \|(A + iI)x\|^2 = \|y\|^2$ so that U is an isometry. (b) $A - iI = -2iI + (A + iI) = 2A - (A + iI)$. (c) $(I - U)^{-1} = \frac{1}{2i}(A + iI)$ and $I + U = I + (A - iI)(A + iI)^{-1}$.

Conversely, let $U \in \mathcal{G}[\mathcal{H}]$ be a unitary operator with $1 \in \rho(U)$ (so that $I - U$ lies in $\mathcal{G}[\mathcal{H}]$) and consider the operator

$$A \,=\, i(I+U)(I-U)^{-1} \,=\, i(I-U)^{-1}(I+U)$$

in $\mathcal{B}[\mathcal{H}]$. Recall again: U commutes with $(I-U)^{-1}$. Show that

(d) $A \,=\, iI + 2iU(I-U)^{-1} \,=\, -iI + 2i(I-U)^{-1}$,

(e) A is self-adjoint,

(f) $\pm i \in \rho(A)$ and $U \,=\, (A-iI)(A+iI)^{-1} \,=\, (A+iI)^{-1}(A-iI)$.

Hint: (d) $i(I+U) = i(I-U) + 2iU = -i(I-U) + 2iI$. (e) $(I-U)^{-1*}U^* = (I-U^*)^{-1}U^{-1} \,=\, (U(I-U^*))^{-1} \,=\, -(I-U)^{-1}$, and therefore $A^*= -iI - 2i(I-U)^{-1*}U^* \,=\, -iI + 2i(I-U)^{-1} \,=\, A$. (f) Using (d) we get $A - iI \,=\, 2iU(I-U)^{-1}$ and $A + iI = 2i(I-U)^{-1}$, so that $(A+iI)^{-1} = \frac{1}{2i}(I-U)$.

Summing up: Set $U = (A-iI)(A+iI)^{-1}$ for an arbitrary self-adjoint operator A. U is unitary with $1 \in \rho(U)$ and $i(I+U)(I-U)^{-1} = A$. Conversely, set $A = i(I+U)(I-U)^{-1}$ for any unitary operator U with $1 \in \rho(U)$. A is self-adjoint and $(A-iI)(A+iI)^{-1} = U$.

Outcome: There is a one-to-one correspondence between the class of all self-adjoint operators and the class of all unitary operators for which 1 belongs to the resolvent set, namely, a mapping $A \mapsto (A-iI)(A+iI)^{-1}$ with inverse $U \mapsto i(I+U)(I-U)^{-1}$. If A is a self-adjoint operator, then the unitary operator $U = (A-iI)(A+iI)^{-1}$ is called the *Cayley transform* of A. What is behind such a one-to-one correspondence is the *Möbius transformation* $z \mapsto \frac{z-i}{z+i}$, which maps the open upper half plane onto the open unit disk, and the extended real line onto the unit circle.

Remark: Take any A in $\mathcal{L}[\mathcal{D}(A), \mathcal{H}]$ where $\mathcal{D}(A)$, the domain of A, is a linear manifold of $\mathcal{H}$. Suppose $A - I$ is injective (i.e., $1 \notin \sigma_P(A)$) so that it has an inverse (not necessarily bounded) on its range; that is, $A - I$ has an inverse $(A-I)^{-1}$ on $\mathcal{R}(A-I)$. Then set

$$S \,=\, (A+I)(A-I)^{-1}$$

in $\mathcal{L}[\mathcal{R}(A-I), \mathcal{H}]$. Observe that, on $\mathcal{R}(A-I)$,

$$S - I \,=\, 2(A-I)^{-1} \quad \text{and} \quad S + I \,=\, 2A(A-I)^{-1}.$$

Indeed, $S = (A - I + 2I)(A-I)^{-1} = (A-I)(A-I)^{-1} + 2(A-I)^{-1}$ and $S = (2A - A + I)(A-I)^{-1} = 2A(A-I)^{-1} - (A-I)(A-I)^{-1}$. Thus $S - I$ is injective (i.e., $1 \notin \sigma_P(S)$ because $(A-I)^{-1}$ in $\mathcal{L}[\mathcal{R}(A-I), \mathcal{H}]$ is injective) so that it has an inverse $(S-I)^{-1} = \frac{1}{2}(A-I)$ on its range $\mathcal{R}(S-I)$, and hence

$$(S+I)(S-I)^{-1} \,=\, A.$$

The domain of $(S-I)^{-1}$, which is the range of $S - I$, coincides with the domain of A so that $\mathcal{R}(S-I) = \mathcal{D}(A)$. Thus, similarly, on $\mathcal{R}(S-I) = \mathcal{D}(A)$,

$$A - I = 2(S - I)^{-1} \quad \text{and} \quad A + I = 2S(S - I)^{-1}.$$

In fact, $A = (S - I + 2I)(S - I)^{-1} = (S - I)(S - I)^{-1} + 2(S - I)^{-1}$ and $A = (2S - S + I)(S - I)^{-1} = 2S(S - I)^{-1} - (S - I)(S - I)^{-1}$. It is also usual to refer to S as the *Cayley transform* of A. Even if $\mathcal{D}(A) = \mathcal{H}$, we may not apply Corollary 4.24 to infer boundedness for the inverse of $S - I$ because the domain of $S - I$ is $\mathcal{R}(A - I)$, which may not be a Banach space. It is worth noticing that if $A \in \mathcal{B}[\mathcal{H}]$ is self-adjoint with $1 \in \rho(A)$, then $S \in \mathcal{B}[\mathcal{H}]$ may not be unitary. Example: $A = \alpha I$ with $1 \neq \alpha \in \mathbb{R}$ is self-adjoint and $A - I$ has a bounded inverse, but $S = \frac{\alpha+1}{\alpha-1} I$ is not unitary if $\left|\frac{\alpha+1}{\alpha-1}\right| \neq 1$ (i.e., if $\alpha \neq 0$).

Problem 6.12. Let $T \in \mathcal{B}[\mathcal{H}]$ be any operator on a complex Hilbert space $\mathcal{H}$ of dimension greater than 1. Prove the following assertions.

(a) $\mathcal{N}(\lambda I - T)$ is a subspace of $\mathcal{H}$, which is T-invariant for every $\lambda \in \mathbb{C}$. Moreover, $T|_{\mathcal{N}(\lambda I - T)} = \lambda I : \mathcal{N}(\lambda I - T) \to \mathcal{N}(\lambda I - T)$. That is, if restricted to the invariant subspace $\mathcal{N}(\lambda I - T)$, then T acts as a scalar operator on $\mathcal{N}(\lambda I - T)$, and hence $T|_{\mathcal{N}(\lambda I - T)}$ is normal.

Remark: Obviously, if $\mathcal{N}(\lambda I - T) = \{0\}$ (in other words, if $\lambda \notin \sigma_P(T)$), then $T|_{\mathcal{N}(\lambda I - T)} = \lambda I : \{0\} \to \{0\}$ coincides with the null operator: on the null space every operator is null or, equivalently, the only operator on the null space is the null operator.

(b) $\mathcal{N}(\lambda I - T) \subseteq \mathcal{N}(\overline{\lambda} I - T^*)$ if and only if $\mathcal{N}(\lambda I - T)$ reduces T.

Hint: Take x in $\mathcal{N}(\lambda I - T)$ so that $Tx = \lambda x$ and $Tx \in \mathcal{N}(\lambda I - T)$ because $\mathcal{N}(\lambda I - T)$ is T-invariant. If $\mathcal{N}(\lambda I - T) \subseteq \mathcal{N}(\overline{\lambda} I - T^*)$, then $T^* x = \overline{\lambda} x$. Thus $T(T^* x) = T(\overline{\lambda} x) = \overline{\lambda} T x = \overline{\lambda} \lambda x = \lambda \overline{\lambda} x = \lambda (T^* x)$, and therefore $T^* x \in \mathcal{N}(\lambda I - T)$. Now apply Corollary 5.75. Conversely, if $\mathcal{N}(\lambda I - T)$ reduces T, then $\mathcal{N}(\lambda I - T)$ reduces $\lambda I - T$. Consider the decomposition $\mathcal{H} = \mathcal{N}(\lambda I - T) \oplus \mathcal{N}(\lambda I - T)^{\perp}$. Thus $\lambda I - T = O \oplus S$ for some operator S on $\mathcal{N}(\lambda I - T)^{\perp}$ (since $(\lambda I - T)|_{\mathcal{N}(\lambda I - T)} = 0$). Hence $\overline{\lambda} I - T^* = O \oplus S^*$. Then $\mathcal{N}(\overline{\lambda} I - T^*) = \mathcal{N}(\lambda I - T) \oplus \mathcal{N}(S^*)$. So $\mathcal{N}(\lambda I - T) \subseteq \mathcal{N}(\overline{\lambda} I - T^*)$.

(c) An operator T is reducible if and only if $\lambda I - T$ is reducible for every λ.

Hint: Recall that *an operator is reducible if and only if it commutes with a nontrivial orthogonal projection*; that is, if and only if it commutes with an orthogonal projection onto a nontrivial subspace (cf. observation that precedes Proposition 5.74).

(d) Every eigenspace of a nonscalar operator T is a nontrivial hyperinvariant subspace for T (i.e., if $\lambda \in \sigma_P(T)$, then $\{0\} \neq \mathcal{N}(\lambda I - T) \neq \mathcal{H}$, and $\mathcal{N}(\lambda I - T)$ is S-invariant for every S that commutes with T).

(e) If T is nonscalar and $\sigma_P(T) \cup \sigma_R(T) \neq \varnothing$ (i.e., $\sigma_P(T) \cup \sigma_P(T^*) \neq \varnothing$), then T has a nontrivial hyperinvariant subspace.

(f) If T has no nontrivial invariant subspace, then $\sigma(T) = \sigma_C(T)$.

Hint: Problems 4.20(c) and 4.26, and Propositions 5.74 and 6.17.

Problem 6.13. We have already seen in Section 6.3 that $\sigma(T^{-1}) = \sigma(T)^{-1} = \{\lambda^{-1} \in \mathbb{C}\colon\ \lambda \in \sigma(T)\}$ for every $T \in \mathcal{G}[\mathcal{X}]$, where $\mathcal{X} \neq \{0\}$ is a complex Banach space. Exhibit a diagonal operator T in $\mathcal{G}[\mathbb{C}^2]$ for which $r(T^{-1}) \neq r(T)^{-1}$.

Problem 6.14. Let T be an arbitrary operator on a complex Banach space $\mathcal{X} \neq \{0\}$, take any $\lambda \in \rho(T)$ (so that $(\lambda I - T) \in \mathcal{G}[\mathcal{X}]$), and set

$$d \,=\, d(\lambda, \sigma(T)),$$

the distance of λ to $\sigma(T)$. Since $\sigma(T)$ is nonempty (bounded) and closed, if follows that d is a positive real number (cf. Problem 3.43(b)). Show that the spectral radius of the inverse of $\lambda I - T$ coincides with the inverse of the distance of $\lambda \in \rho(T)$ to the spectrum of T. That is,

(a) $$r((\lambda I - T)^{-1}) \,=\, d^{-1}.$$

Hint: Since $d = \inf_{\mu \in \sigma(T)} |\lambda - \mu|$, it follows that $d^{-1} = \sup_{\mu \in \sigma(T)} |\lambda - \mu|^{-1}$. Why? Recall from the Spectral Mapping Theorem (Theorem 6.19) that $\sigma(\lambda I - T) = \{\lambda - \mu \in \mathbb{C}\colon\ \mu \in \sigma(T)\}$. Since $\sigma((\lambda I - T)^{-1}) = \sigma(\lambda I - T)^{-1} = \{\mu^{-1} \in \mathbb{C}\colon\ \mu \in \sigma(\lambda I - T)\}$, then $\sigma((\lambda I - T)^{-1}) = \{(\lambda - \mu)^{-1} \in \mathbb{C}\colon\ \mu \in \sigma(T)\}$.

Now let $\mathcal{X}$ be a Hilbert space and prove the following implication.

(b) $$\text{If } T \text{ is hyponormal, then } \|(\lambda I - T)^{-1}\| = d^{-1}.$$

Hint: If T is hyponormal, then $\lambda I - T$ is hyponormal (cf. proof of Corollary 6.18) and so is $(\lambda I - T)^{-1}$ (Problem 6.7). Hence $(\lambda I - T)^{-1}$ is normaloid by Proposition 6.10. Apply (a).

Problem 6.15. Let $\mathcal{M}$ be a subspace of a Hilbert space $\mathcal{H}$ and take $T \in \mathcal{B}[\mathcal{H}]$. If $\mathcal{M}$ is T-invariant, then $(T|_{\mathcal{M}})^* = PT^*|_{\mathcal{M}}$ in $\mathcal{B}[\mathcal{M}]$, where $P\colon \mathcal{H} \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{M}$.

Hint: Use Proposition 5.81 to verify that, for every $u, v \in \mathcal{M}$, $\langle (T|_{\mathcal{M}})^* u \,;\, v\rangle = \langle u \,;\, T|_{\mathcal{M}} v\rangle = \langle u \,;\, Tv\rangle = \langle u \,;\, TPv\rangle = \langle PT^* u \,;\, v\rangle = \langle PT^*|_{\mathcal{M}}\, u \,;\, v\rangle$.

In other words, if $\mathcal{M}$ is T-invariant, then $T(\mathcal{M}) \subseteq \mathcal{M}$ (so that $T|_{\mathcal{M}}$ lies in $\mathcal{B}[\mathcal{M}]$), but $T^*(\mathcal{M})$ may not be included in $\mathcal{M}$; it has to be projected there: $PT^*(\mathcal{M}) \subseteq \mathcal{M}$ (so that $PT^*|_{\mathcal{M}}$ lies in $\mathcal{B}[\mathcal{M}]$ and coincides with $(T|_{\mathcal{M}})^*$). If $\mathcal{M}$ reduces T (i.e., if $\mathcal{M}$ also is T^*-invariant), then $T^*(\mathcal{M})$ does not need to be projected on $\mathcal{M}$; it is already there (i.e., if $\mathcal{M}$ reduces T, then $T^*(\mathcal{M}) \subseteq \mathcal{M}$ and $(T|_{\mathcal{M}})^* = T^*|_{\mathcal{M}}$ — see Corollary 5.75).

Problem 6.16. Let $\mathcal{M}$ be an invariant subspace for $T \in \mathcal{B}[\mathcal{H}]$.

(a) If T is hyponormal, then $T|_{\mathcal{M}}$ is hyponormal.

(b) If T is hyponormal and $T|_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces T.

Hint: $T|_{\mathcal{M}} \in \mathcal{B}[\mathcal{M}]$. Use Problem 6.15 (and Propositions 5.81 and 6.6) to show that $\|(T|_{\mathcal{M}})^* u\| \le \|T^* u\| \le \|T|_{\mathcal{M}} u\|$ for every $u \in \mathcal{M}$. If $T|_{\mathcal{M}}$ is normal, say $T|_{\mathcal{M}} = N$, then $T = \left(\begin{smallmatrix} N & X \\ O & Y \end{smallmatrix}\right)$ in $\mathcal{B}[\mathcal{M} \oplus \mathcal{M}^\perp]$ (cf. Example 2.O and Proposition 5.51). Since N is normal and T is hyponormal, verify that

$$O \le D = T^*T - TT^* = \begin{pmatrix} -XX^* & N^*X - XY^* \\ X^*N - YX^* & X^*X + Y^*Y - YY^* \end{pmatrix}.$$

Take u in $\mathcal{M}$, set $x = (u, 0)$ in $\mathcal{M} \oplus \mathcal{M}^\perp$, and show that $\langle Dx\,;x\rangle = -\|X^*u\|^2$. Conclude: $T = \left(\begin{smallmatrix} N & O \\ O & Y \end{smallmatrix}\right) = N \oplus Y$, and hence $\mathcal{M}$ reduces T.

Problem 6.17. This is a rather important result. Let $\mathcal{M}$ be an invariant subspace for a normal operator $T \in \mathcal{B}[\mathcal{H}]$. Show that

$$T|_{\mathcal{M}} \text{ is normal if and only if } \mathcal{M} \text{ reduces } T.$$

Hint: If T is normal, then it is hyponormal. Apply Problem 6.16 to verify that $\mathcal{M}$ reduces T whenever $T|_{\mathcal{M}}$ is normal. Conversely, if $\mathcal{M}$ reduces T, then $T = N_1 \oplus N_2$ on $\mathcal{M} \oplus \mathcal{M}^\perp$, with $N_1 = T|_{\mathcal{M}} \in \mathcal{B}[\mathcal{M}]$ and $N_2 = T|_{\mathcal{M}^\perp} \in \mathcal{B}[\mathcal{M}^\perp]$. Now verify that both N_1 and N_2 are normal operators whenever T is normal.

Problem 6.18. Let T be a compact operator on a complex Hilbert space $\mathcal{H}$ and let $\mathbb{D}$ be the open unit disk about the origin of the complex plane.

(a) Show that $\sigma_P(T) \subseteq \mathbb{D}$ implies $T^n \xrightarrow{u} O$.

Hint: Corollary 6.31 and Proposition 6.22.

(b) Show that $T^n \xrightarrow{w} O$ implies $\sigma_P(T) \subseteq \mathbb{D}$.

Hint: If $\lambda \in \sigma_P(T)$, then verify that there exists a unit vector x in $\mathcal{H}$ such that $T^n x = \lambda^n x$ for every positive integer n. Thus $|\lambda|^n \to 0$, and hence $|\lambda| < 1$, whenever $T^n \xrightarrow{w} O$ (cf. Proposition 5.67).

Conclude: *The concepts of weak, strong, and uniform stabilities coincide for a compact operator on a complex Hilbert space.*

Problem 6.19. If $T \in \mathcal{B}[\mathcal{H}]$ is hyponormal, then

$$\mathcal{N}(\lambda I - T) \subseteq \mathcal{N}(\overline{\lambda} I - T^*) \quad \text{for every} \quad \lambda \in \mathbb{C}.$$

Hint: Adapt the proof of Proposition 6.39.

Problem 6.20. Take $\lambda, \mu \in \mathbb{C}$. If $T \in \mathcal{B}[\mathcal{H}]$ is hyponormal, then

$$\mathcal{N}(\lambda I - T) \perp \mathcal{N}(\mu I - T) \quad \text{whenever} \quad \lambda \ne \mu.$$

Hint: Adapt the proof of Proposition 6.40 by using Problem 6.19.

Problem 6.21. If $T \in \mathcal{B}[\mathcal{H}]$ is hyponormal, then $\mathcal{N}(\lambda I - T)$ reduces T for every $\lambda \in \mathbb{C}$.

Hint: Adapt the proof of Proposition 6.41. First observe that, if $\lambda x = Tx$, then $T^*x = \overline{\lambda}x$ (by Problem 6.19). Next verify that $\lambda T^*x = \lambda\overline{\lambda}x = \overline{\lambda}\lambda x = \overline{\lambda}Tx = T\overline{\lambda}x = TT^*x$. Then conclude: $\mathcal{N}(\lambda I - T)$ is T^*-invariant. Note: $T|_{\mathcal{N}(\lambda I - T)}$ is a scalar operator on $\mathcal{N}(\lambda I - T)$, and so a normal operator (Problem 6.12(a)).

An operator is called *completely nonnormal* if it has no normal direct summand (i.e., if it has no nonzero reducing subspace on which it acts as a normal operator; equivalently, if the restriction of it every nonzero reducing subspace is not normal). A *pure hyponormal* is a completely nonnormal hyponormal operator. Use the above result to prove the following assertion.

A pure hyponormal operator has an empty point spectrum.

Problem 6.22. Let $T \in \mathcal{B}[\mathcal{H}]$ be a hyponormal operator. Show that

$$\mathcal{M} = \big(\textstyle\sum_{\lambda \in \sigma_P(T)} \mathcal{N}(\lambda I - T)\big)^- \text{ reduces } T \text{ and } T|_{\mathcal{M}} \text{ is normal.}$$

Hint: If $\sigma_{P(T)} = \varnothing$, then the result is trivial (for the empty sum is null). Thus suppose $\sigma_P(T) \neq \varnothing$. First note that $\{\mathcal{N}(\lambda I - T)\}_{\lambda \in \sigma_P(T)}$ is an orthogonal family of nonzero subspaces of the Hilbert space $\mathcal{M}$ (Problem 6.20). Now choose one of the following methods.

(1) Adapt the proof of Corollary 6.42, with the help of Problems 6.20 and 6.21, to verify that $\mathcal{M}$ reduces T. Use Theorem 5.59 and Problem 5.10 to check that the family $\{P_\lambda\}_{\lambda \in \sigma_P(T)}$ consisting of the nonzero orthogonal projections $P_\lambda \in \mathcal{B}[\mathcal{M}]$ onto each $\mathcal{N}(\lambda I - T)$ is a resolution of the identity on $\mathcal{M}$. Take any $u \in \mathcal{M}$. Verify that $u = \sum_\lambda P_\lambda u$, and so $T|_{\mathcal{M}}u = Tu = \sum_\lambda TP_\lambda u = \sum_\lambda \lambda P_\lambda u$ (reason: $P_\lambda u \in \mathcal{N}(\lambda I - T)$, where the sums run over $\sigma_P(T)$). Conclude that $T|_{\mathcal{M}} \in \mathcal{B}[\mathcal{M}]$ is a weighted sum of projections. Apply Proposition 6.36.

(2) Use Example 5.J and Problem 5.10 to identify the topological sum $\mathcal{M}$ with the orthogonal direct sum $\bigoplus_{\lambda \in \sigma_P(T)} \mathcal{N}(\lambda I - T)$. Since each $\mathcal{N}(\lambda I - T)$ reduces T (Problem 6.21), it follows that $\mathcal{M}$ reduces T, and also that each $\mathcal{N}(\lambda I - T)$ reduces $T|_{\mathcal{M}} \in \mathcal{B}[\mathcal{M}]$. Therefore, $T|_{\mathcal{M}} = \bigoplus_{\lambda \in \sigma_P(T)} T|_{\mathcal{N}(\lambda I - T)}$. But each $T|_{\mathcal{N}(\lambda I - T)}$ is normal (in fact, a scalar operator — Problem 6.12), which implies that $T|_{\mathcal{M}}$ is normal (actually, a weighted sum of projections).

Problem 6.23. Every compact hyponormal operator is normal.

Hint: Let $T \in \mathcal{B}[\mathcal{H}]$ be a compact hyponormal operator on a Hilbert space $\mathcal{H}$. Consider the subspace $\mathcal{M}$ of Problem 6.22. If $\lambda \in \sigma_P(T|_{\mathcal{M}^\perp})$, then there is a *nonzero* vector $v \in \mathcal{M}^\perp$ such that $\lambda v = T|_{\mathcal{M}^\perp}v = Tv$, and hence $v \in \mathcal{N}(\lambda I - T) \subseteq \mathcal{M}$, which is a contradiction. Thus $\sigma_P(T|_{\mathcal{M}^\perp}) = \varnothing$. Recall that $T|_{\mathcal{M}^\perp}$ is compact (Section 4.9) and hyponormal (Problem 6.16). Then $\mathcal{M}^\perp = \{0\}$ (use Corollary 6.32). Apply Problem 6.22 to show that T is normal.

Remark: According to the above result, on a finite-dimensional Hilbert space, quasinormality, subnormality, and hyponormality all collapse to normality (and so isometries become unitaries — see Problem 4.38(d)).

Problem 6.24. Let $T \in \mathcal{B}[\mathcal{H}]$ be a weighted sum of projections on a complex Hilbert space $\mathcal{H} \neq \{0\}$. That is,

$$Tx = \sum_{\gamma \in \Gamma} \lambda_\gamma P_\gamma x \quad \text{for every} \quad x \in \mathcal{H},$$

where $\{P_\gamma\}_{\gamma \in \Gamma}$ is a resolution of the identity on $\mathcal{H}$ with $P_\gamma \neq O$ for all $\gamma \in \Gamma$, and $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ is a (similarly indexed) bounded family of scalars. Recall from Proposition 6.36 that T is normal. Now prove the following equivalences.

(a) T is unitary $\iff$ $\lambda_\gamma \in \mathbb{T}$ for all γ $\iff$ $\sigma(T) \subseteq \mathbb{T}$.

(b) T is self-adjoint $\iff$ $\lambda_\gamma \in \mathbb{R}$ for all γ $\iff$ $\sigma(T) \subseteq \mathbb{R}$.

(c) T is nonnegative $\iff$ $\lambda_\gamma \subseteq [0, \infty)$ for all γ $\iff$ $\sigma(T) \subseteq [0, \infty)$.

(d) T is positive $\iff$ $\lambda_\gamma \subseteq (0, \infty)$ for all γ.

(e) T is strictly positive $\iff$ $\lambda_\gamma \subseteq [\alpha, \infty)$ for all γ $\iff$ $\sigma(T) \subseteq [\alpha, \infty)$.

(f) T is a projection $\iff$ $\lambda_\gamma \subseteq \{0, 1\}$ for all γ $\iff$ $\sigma(T) = \sigma_P(T) \subseteq \{0, 1\}$.

Note: In part (a), $\mathbb{T}$ denotes the unit circle about the origin of the complex plane. In part (e), α is some positive real number. In part (f), projection means orthogonal projection (Proposition 6.2).

Problem 6.25. Let T be an operator on a complex (nonzero) Hilbert space $\mathcal{H}$. Show that

(a) T is diagonalizable if and only if $\mathcal{H}$ has an orthogonal basis made up of eigenvectors of T.

Hint: If $\{e_\gamma\}$ is an orthonormal basis for $\mathcal{H}$, where each e_γ is an eigenvector of T, then (by the Fourier Series Theorem) show that the resolution of the identity on $\mathcal{H}$ of Proposition 5.57 diagonalizes T. Conversely, if T is diagonalizable, every nonzero vector in $\mathcal{R}(P_\gamma)$ is an eigenvector of T. Let B_γ be an orthonormal basis for the Hilbert space $\mathcal{R}(P_\gamma)$. Since $\big(\sum_\gamma \mathcal{R}(P_\gamma)\big)^- = \mathcal{H}$ (Theorem 5.59 and Problem 5.10), use Problem 5.11 to show that $\bigcup_\gamma B_\gamma$ is an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of T.

If there exists an orthonormal basis $\{e_\gamma\}_{\gamma \in \Gamma}$ for $\mathcal{H}$ and a bounded family of scalars $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ such that $Tx = \sum_{\gamma \in \Gamma} \lambda_\gamma \langle x \,; e_\gamma \rangle e_\gamma$ for every $x \in \mathcal{H}$, then we say that T is (or acts as) a *diagonal operator with respect to the basis* $\{e_\gamma\}_{\gamma \in \Gamma}$ (cf. Problem 5.17). Use part (a) to show that

(b) T is diagonalizable if and only if it is a diagonal operator with respect to some orthonormal basis for $\mathcal{H}$.

Now let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal basis for $\mathcal{H}$ and consider the Hilbert space ℓ^2_Γ of Example 5.K. Let $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ be a bounded family of scalars and consider the mapping $D\colon \ell^2_\Gamma \to \ell^2_\Gamma$ defined by

$$Dz = \{\lambda_\gamma\zeta_\gamma\}_{\gamma\in\Gamma} \quad \text{for every} \quad z = \{\zeta_\gamma\}_{\gamma\in\Gamma} \in \ell^2_\Gamma.$$

In fact, $Dz \in \ell^2_\Gamma$ for all $z \in \ell^2_\Gamma$, $D \in \mathcal{B}[\ell^2_\Gamma]$, and $\|D\| = \sup_{\gamma\in\Gamma}|\lambda_\gamma|$ (*hint*: Example 4.H). This is called a *diagonal operator* on ℓ^2_Γ. Show that

(c) T is diagonalizable if and only if it is unitarily equivalent to a diagonal operator.

Hint: Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal basis for $\mathcal{H}$ and consider the natural mapping (cf. Theorem 5.48) $U\colon \mathcal{H} \to \ell^2_\Gamma$ given by

$$Ux = \{\langle x\,;e_\gamma\rangle\}_{\gamma\in\Gamma} \quad \text{for every} \quad x = \textstyle\sum_{\gamma\in\Gamma}\langle x\,;e_\gamma\rangle e_\gamma.$$

Verify that U is unitary (i.e., a linear surjective isometry — see the proof of Theorem 5.49), and use part (b) to show that the diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\;T\;} & \mathcal{H} \\ {\scriptstyle U}\big\downarrow & & \big\uparrow{\scriptstyle U^*} \\ \ell^2_\Gamma & \xrightarrow{\;D\;} & \ell^2_\Gamma \end{array}$$

commutes if and only if T is diagonalizable, where D is a diagonal operator on the Hilbert space ℓ^2_Γ.

Problem 6.26. If T is a normal operator on a complex Hilbert space $\mathcal{H}$, then

(a) T is unitary if and only if $\sigma(T) \subseteq \mathbb{T}$,

(b) T is self-adjoint if and only if $\sigma(T) \subseteq \mathbb{R}$,

(c) T is nonnegative if and only if $\sigma(T) \subseteq [0,\infty)$,

(d) T is strictly positive if and only if $\sigma(T) \subseteq [\alpha,\infty)$ for some $\alpha > 0$,

(e) T is an orthogonal projection if and only if $\sigma(T) \subseteq \{0,1\}$.

Hint: Recall that $\mathbb{T}$ stands for the unit circle about the origin of the complex plane. Half of this problem was solved in Corollary 6.18. To verify the other half, use the spectral decomposition (Theorem 6.47), $T = \int_{\sigma(T)} \lambda\, dP_\lambda$, which is an abbreviation of $\langle Tx\,;x\rangle = \int_{\sigma(T)} \lambda\, d\langle P_\lambda x\,;x\rangle$ for every $x \in \mathcal{H}$; and recall that $T^* = \int_{\sigma(T)} \overline{\lambda}\, dP_\lambda$ and $T^*T = \int_{\sigma(T)} |\lambda|^2\, dP_\lambda = TT^*$.

Problem 6.27. Let $T \in \mathcal{B}[\mathcal{H}]$ be a hyponormal operator. Prove the following implications.

(a) If $\sigma(T) \subseteq \mathbb{T}$, then T is unitary.

Hint: If $\sigma(T)$ is included in the unit circle $\mathbb{T}$, then $0 \in \rho(T)$, and so T is in $\mathcal{G}[\mathcal{H}]$. Since T is hyponormal, verify that $\|T\| = r(T) = 1$. Use Problem 6.7 to check that T^{-1} is hyponormal. Show that $\|T^{-1}\| = 1$ (recall: $\sigma(T^{-1}) = \sigma(T)^{-1} = \mathbb{T}$) and conclude from Problem 6.8 that T is unitary.

(b) If $\sigma(T) \subseteq \mathbb{R}$, then T is self-adjoint.

Hint: Take any $0 \neq \alpha \in \mathbb{R}$. Since $\alpha i \in \rho(T)$ and T is hyponormal, use Problem 6.14 to show that $\|(\alpha i - T)^{-1}\| = d^{-1} \leq |\alpha|^{-1}$. Now verify that

$$\begin{aligned} \alpha^2 \|(\alpha i I - T)^{-1}(\alpha i I - T)x\|^2 &\leq \alpha^2 \|(\alpha i - T)^{-1}\|^2 \|(\alpha i I - T)x\|^2 \\ &\leq \alpha^2 \|x\|^2 + \|Tx\|^2 - 2\,\mathrm{Re}\,\langle \alpha i x\,; Tx\rangle, \end{aligned}$$

thus $-2\alpha\,\mathrm{Im}\,\langle Tx\,; x\rangle = -2\,\mathrm{Im}\,\langle \alpha x\,; Tx\rangle = 2\,\mathrm{Re}\,\langle \alpha i x\,; Tx\rangle \leq \|Tx\|^2$, and so $\mathrm{Im}\,\langle Tx\,; x\rangle = 0$ (with $\alpha = -\frac{1}{2}$), for every $x \in \mathcal{H}$. Use Proposition 5.79.

(c) If $\sigma(T) \subseteq [0, \infty)$, then T is nonnegative.

(d) If $\sigma(T) \subseteq [\alpha, \infty)$ for some $\alpha > 0$, then T is strictly positive.

(e) If $\sigma(T) \subseteq \{0, 1\}$, then T is an orthogonal projection.

Hint: Use part (b) and Problem 6.26 to prove (c), (d), and (e).

Problem 6.28. Prove the following assertion.

(a) An isolated point of the spectrum of a normal operator is an eigenvalue.

Hint: Let $N = \int \lambda\, dP_\lambda$ be a normal operator on a Hilbert space $\mathcal{H}$ and let λ_0 be an isolated point of $\sigma(N)$. Apply Theorems 6.47 and 6.48 to show that:

(1) $P(\{\lambda_0\}) \neq O$,

(2) $\mathcal{R}(P(\{\lambda_0\})) \neq \{0\}$ reduces N,

(3) $N|_{\mathcal{R}(P(\{\lambda_0\}))} = NP(\{\lambda_0\}) = \int \lambda \chi_{\{\lambda_0\}}\, dP_\lambda = \lambda_0 P(\{\lambda_0\})$, where $\chi_{\{\lambda_0\}}$ is the characteristic function of $\{\lambda_0\}$, and

(4) $(\lambda_0 I - N)u = \lambda_0 P(\{\lambda_0\})u - N|_{\mathcal{R}(P(\{\lambda_0\}))}u = 0$ if $u \in \mathcal{R}(P(\{\lambda_0\}))$.

An important result in operator theory is the *Riesz Decomposition Theorem*, which reads as follows.

> *If T is an operator on a complex Hilbert space, and if $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are disjoint nonempty closed sets in $\mathbb{C}$, then T has a complementary (not necessarily orthogonal) pair of nontrivial invariant subspaces $\{\mathcal{M}_1, \mathcal{M}_2\}$ such that $\sigma(T|_{\mathcal{M}_1}) = \sigma_1$ and $\sigma(T|_{\mathcal{M}_2}) = \sigma_2$.*

Use the Riesz Decomposition Theorem to prove the next assertion.

(b) An isolated point of the spectrum of a hyponormal operator is an eigenvalue.

Hint: Let λ_1 be an isolated point of the spectrum $\sigma(T)$ of a hyponormal operator $T \in \mathcal{B}[\mathcal{H}]$. Show that $\sigma(T) = \{\lambda_1\} \cup \sigma_2$ for some nonempty closed set σ_2

that does not contain λ_1. Apply the Riesz Decomposition Theorem to ensure that T has a nontrivial invariant subspace $\mathcal{M}$ such that $\sigma(T|_{\mathcal{M}}) = \{\lambda_1\}$. Set $H = T|_{\mathcal{M}}$ on $\mathcal{M} \neq \{0\}$. Show that $\lambda_1 I - H$ is a hyponormal (thus normaloid) operator and $\sigma(\lambda_1 I - H) = \{0\}$. Conclude that $T|_{\mathcal{M}} = H = \lambda_1 I$ in $\mathcal{B}[\mathcal{M}]$.

Remark: An operator is *isoloid* if isolated points of the spectrum are eigenvalues. Thus item (b) simply says that *hyponormal operators are isoloid*.

Apply Problem 6.21 to prove the following assertion.

(c) A pure hyponormal operator has no isolated point in its spectrum.

Problem 6.29. Let S and T be normal operators acting on the same Hilbert space. Prove the following assertion.

If $ST = TS$, then $S + T$, TS, and ST are normal operators.

Hint: Corollary 6.49 and Problem 6.2. (Compare with Problem 6.2(b).)

Problem 6.30. The operators in this problem act on a complex Hilbert space of dimension greater than 1. Recall from Problem 4.22:

(a) Every nilpotent operator has a nontrivial invariant subspace.

However, it is still an open question whether every quasinilpotent operator has a nontrivial invariant subspace. In other words, the invariant subspace problem remains unanswered for quasinilpotent operators on a Hilbert space (but not for quasinilpotent operators on a nonreflexive Banach space, where the answer is in the negative: the existence of quasinilpotent operators on ℓ^1 without nontrivial invariant subspaces was proved by Read in 1997). Now shift from nilpotent to normal operators, and recall from the Spectral Theorem:

(b) Every normal operator has a nontrivial invariant subspace.

Next prove the following propositions.

(c) Every quasinormal operator has a nontrivial invariant subspace.

Hint: $(T^*T - TT^*)T = O$. Use Problem 4.21.

(d) Every isometry has a nontrivial invariant subspace.

Every subnormal operator has a nontrivial invariant subspace. This is a deep result proved by S. Brown in 1978. However, it is still unknown whether every hyponormal operator has a nontrivial invariant subspace.

Problem 6.31. Consider the direct sum $\bigoplus_k T_k \in \mathcal{B}[\bigoplus_k \mathcal{H}_k]$ with each T_k in $\mathcal{B}[\mathcal{H}_k]$, where the Hilbert space $\bigoplus_k \mathcal{H}_k$ is the (orthogonal) direct sum of a countable collection $\{\mathcal{H}_k\}$ of Hilbert spaces (as in Examples 5.F and 5.G and Problems 4.16 and 5.28). Also see Problems 5.71 and 5.73. Show that

(a) $\bigoplus_k T_k$ is normal, quasinormal, or hyponormal if and only if each T_k is.

(b) If every T_k is normaloid, then so is $\bigoplus_k T_k$. However the converse fails.

Hint: Set $T = \bigoplus_k T_k$ and use Problems 4.16(b) and 5.71(b) to verify that $\|T^n\| = \sup_k \|T_k^n\| = \sup_k \|T_k\|^n = (\sup_k \|T_k\|)^n = \|T\|^n$ if every T_k is normaloid. Show that $T = S \oplus Q$ is normaloid if S is a normaloid nonstrict contraction ($\|S^n\| = \|S\|^n = 1$) and Q is a nilpotent nonzero contraction ($\|Q^2\| = 0 \neq \|Q\| \leq 1$). Example: $\|T^n\| = \|T\|^n = 1$ for $T = 1 \oplus \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

Problem 6.32. Take any operator T acting on a normed space. Lemma 6.8 says that the sequence $\{\|T^n\|^{\frac{1}{n}}\}$ converges and its limit was denoted by $r(T)$. That is, $r(T) = \lim_n \|T^n\|^{\frac{1}{n}}$. Now consider the following assertions.

(a) $r(T) = \|T\|$.

(b) $\|T^n\| = \|T\|^n$ for every integer $n \geq 1$.

(c) $\|T^n\| = \|T\|^n$ for every integer $n \geq m$ for some integer $m \geq 1$.

(d) $\|T^{1+jk}\| = \|T\|^{1+jk}$ for every integer $j \geq 0$ for some integer $k \geq 1$.

A normaloid operator was defined as one for which (a) holds true. Proposition 6.9 says that (a) and (b) are equivalent. Show that the above assertions are all equivalent so that T is normaloid if and only if any of them holds true.

Hint: (b) trivially implies (c) and (d), and (c) implies (a) as (b) implies (a). (d) implies (a) as follows. If (d) holds, then there is a subsequence $\{T^{n_j}\}$ of $\{T^n\}$, viz., $T^{n_j} = T^{1+jk}$, such that $\lim_j \|T^{n_j}\|^{1/n_j} = \lim_j (\|T\|^{n_j})^{1/n_j} = \|T\|$. Since the convergent sequence $\{\|T^n\|^{1/n}\}$ has a subsequence that converges to $\|T\|$, then $\{\|T^n\|^{1/n}\}$ converges itself to the same limit. Thus (a) holds true.

Remark: An independent proof of (c)⇒(b) without using (a): If $\|T^n\| = \|T\|^n$ for every $n \geq m$ for some $m \geq 1$, then, for every $1 \leq k \leq m$,

$$\|T\|^k = \frac{\|T\|^m}{\|T\|^{m-k}} = \frac{\|T^m\|}{\|T\|^{m-k}} \leq \frac{\|T\|^{m-k}\|T^k\|}{\|T\|^{m-k}} = \|T^k\| \leq \|T\|^k$$

so that $\|T^k\| = \|T\|^k$, and hence $\|T^n\| = \|T\|^n$ holds for every $n \geq 1$.

Problem 6.33. An operator $T \in \mathcal{B}[\mathcal{X}]$ on a normed space $\mathcal{X}$ is called *k-paranormal* if there exists an integer $k \geq 1$ such that

$$\|Tx\|^{k+1} \leq \|T^{k+1}x\| \|x\|^k \quad \text{for every} \quad x \in \mathcal{X}.$$

A *paranormal* operator is simply a 1-paranormal operator: T is *paranormal* if

$$\|Tx\|^2 \leq \|T^2x\| \|x\| \quad \text{for every} \quad x \in \mathcal{X}.$$

Prove the following assertions.

(a) Every hyponormal operator is paranormal (if $\mathcal{X}$ is a Hilbert space).

Hint: If $T^*T - TT^* \geq O$, then $T^*(T^*T - TT^*)T \geq O$, which means that $|T|^4 \leq |T^2|^2$. Thus $|T|^2 \leq |T^2|$ (cf. remark in Problem 5.59). So $\|Tx\|^2 = \langle |T|^2 x \,;x\rangle \leq \langle |T^2| x \,;x\rangle \leq \| |T^2| x\| \|x\| = \|T^2x\| \|x\|$ by Problem 5.61(c).

(b) Every k-paranormal operator is normaloid.

Hint: Suppose T is a nonzero k-paranormal. Take any integer $j \geq 1$. Thus

$$\begin{aligned} \|T^j x\|^{k+1} &= \|T(T^{j-1}x)\|^{k+1} \leq \|T^{k+1}(T^{j-1}x)\| \, \|T^{j-1}x\|^k \\ &= \|T^{k+j}x\| \, \|T^{j-1}x\|^k \leq \|T^{k+j}\| \, \|T^{j-1}\|^k \|x\|^{k+1} \end{aligned}$$

for every $x \in \mathcal{H}$, which implies $\|T^j\|^{k+1} \leq \|T^{k+j}\| \, \|T^{j-1}\|^k$. Therefore, if $\|T^j\| = \|T\|^j$ for some $j \geq 1$ (which holds tautologically for $j = 1$), then $\|T\|^{(k+1)j} = (\|T\|^j)^{k+1} = \|T^j\|^{k+1} \leq \|T^{k+j}\| \, \|T^{j-1}\|^k \leq \|T^{k+j}\| \, \|T\|^{(j-1)k}$ so that $\|T\|^{k+j} = \|T\|^{(k+1)j} / \|T\|^{(j-1)k} \leq \|T^{k+j}\| \leq \|T\|^{k+j}$, and hence $\|T^{k+j}\| = \|T\|^{k+j}$. Thus it is proved by induction that

$$\|T^{1+jk}\| = \|T\|^{1+jk}$$

for every $j \geq 1$. This yields a subsequence $\{T^{n_j}\}$ of $\{T^n\}$, say $T^{n_j} = T^{1+jk}$, such that $\lim_j \|T^{n_j}\|^{1/n_j} = \lim_j (\|T\|^{n_j})^{1/n_j} = \|T\|$. Since $\{\|T^n\|^{1/n}\}$ is a convergent sequence that converges to $r(T)$ (Lemma 6.8), and since it has a subsequence that converges to $\|T\|$, then the sequence $\{\|T^n\|^{1/n}\}$ must converge itself to the same limit (Proposition 3.5). Hence $r(T) = \|T\|$.

Remark: These classes of operators are related by proper inclusion as follows.

$$\text{Hyponormal} \subset \text{Paranormal} \subset k\text{-Paranormal} \subset \text{Normaloid}.$$

Problem 6.34. Consider the setup of the previous problem. Recall that a part of an operator is the restriction of it to an invariant subspace. Prove:

(a) Every part of a k-paranormal operator is again k-paranormal.

Hint: $\|(T|_{\mathcal{M}})u\|^{k+1} = \|Tu\|^{k+1} \leq \|T^{k+1}u\| \, \|u\|^k = \|(T|_{\mathcal{M}})^{k+1}u\| \, \|u\|^k$.

(b) The inverse of an invertible paranormal operator is again paranormal.

Hint: If $T \in \mathcal{B}[\mathcal{X}]$ is an invertible (with a bounded inverse: $T^{-1} \in \mathcal{B}[\mathcal{X}]$) k-paranormal operator on a normed space $\mathcal{X}$ for some $k \geq 1$, then

$$\|x\|^{k+1} = \|TT^{-1}x\|^{k+1} \leq \|T^{k+1}(T^{-1}x)\| \|T^{-1}x\|^k = \|T^k x\| \|T^{-1}x\|^k$$

for every x in $\mathcal{X}$. Since T^k is invertible, take an arbitrary y in $\mathcal{X} = \mathcal{R}(T^k)$. Thus $y = T^k x$, and hence $x = (T^k)^{-1}y = T^{-k}y$, which implies that $T^{-1}x = T^{-(k+1)}y$, for some x in $\mathcal{X}$. Therefore, by the above inequality,

$$\|T^{-k}y\|^{k+1} \leq \|y\| \, \|T^{-(k+1)}y\|^k$$

for every y in $\mathcal{X}$. Thus by setting $k = 1$ it follows that if T is 1-paranormal, then $\|T^{-1}y\|^2 \leq \|T^{-2}y\| \, \|y\|$ for every y in $\mathcal{X}$, and so T^{-1} is 1-paranormal.

Remark: Is the inverse of an invertible k-paranormal operator (for $k > 1$) still k-paranormal? So far, this seems to be an open question.

(c) If $T \in \mathcal{B}[\mathcal{X}]$ is paranormal then, for every $x \in \mathcal{X}$ and every integer $n \geq 1$,

$$\|T^n x\| \, \|Tx\| \leq \|T^{n+1}x\| \, \|x\|.$$

Hint: If T is paranormal, $\|T^{n+1}x\|^2\|Tx\| \le \|T^{n+2}x\|\,\|T^nx\|\,\|Tx\|$. If the claimed result holds, $\|T^nx\|\,\|Tx\| \le \|T^{n+1}x\|\,\|x\|$. Conclude the induction.

Problem 6.35. Take any $C \in \mathcal{B}[\mathcal{H}]$ on a Hilbert space $\mathcal{H}$. Consider the subset

$$\mathcal{M}_C = \big\{(Cx, Cx) \in \mathcal{H} \oplus \mathcal{H}\colon\ x \in \mathcal{H}\big\}$$

of $\mathcal{H} \oplus \mathcal{H}$. Prove the following assertions.

(a) If C commutes with $T \in \mathcal{B}[\mathcal{H}]$, then $\mathcal{M}_C$ is $(T \oplus T)$-invariant.

(b) If C has a closed range, then $\mathcal{M}_C$ is a subspace of $\mathcal{H} \oplus \mathcal{H}$.

Take any operator T in $\mathcal{B}[\mathcal{H}]$ and take an arbitrary operator C in the commutant $\{T\}'$ of T. Prove the following propositions.

(c) $\big\|(T \oplus T)|_{\mathcal{M}_C}^n\big\| = \sup_{Cx\neq 0} \frac{\|T^nCx\|}{\|Cx\|} = \sup_{Cx\neq 0} \frac{\|CT^nx\|}{\|Cx\|}$.

(d) If T is an isometry, then $\big\|(T \oplus T)|_{\mathcal{M}_C}^n\big\| = 1$.

(e) If C is an isometry, then $\big\|(T \oplus T)|_{\mathcal{M}_C}^n\big\| = \|T^n\|$.

(f) If T is normaloid and C is an isometry, then $(T \oplus T)|_{\mathcal{M}_C}$ is normaloid.

(g) If T is unitary, then $(T \oplus T)|_{\mathcal{M}_T}$ and $(T \oplus T)|_{\mathcal{M}_{T^{-1}}}$ are normaloid.

(h) Assertion (f) may fail if C is not an isometry. For instance, let $Q \in \mathcal{B}[\mathbb{C}^3]$ be a nilpotent contraction of index 3 ($\|Q^3\| = 0 < \|Q^2\| \le \|Q\| \le 1$). Take $T = 1 \oplus Q$ in $\mathcal{B}[\mathbb{C}^4]$. Now take $C = 0 \oplus Q$ in $\mathcal{B}[\mathbb{C}^4]$, which clearly commutes with T (in fact, $C = T - T^3$) and has a closed range. (Why?) Show that T is a normaloid operator for which $(T \oplus T)|_{\mathcal{M}_C}$ is not normaloid.

Hints: (a) $(T \oplus T)(Cx, Cx) = (TCx, TCx) = (CTx, CTx)$ if $CT = TC$.

(b) It is clear (isn't it?) that $\mathcal{M}_C$ is a linear manifold of $\mathcal{H} \oplus \mathcal{H}$. If $\mathcal{R}(C)$ is closed, then $\mathcal{R}(C) \oplus \mathcal{R}(C)$ is closed in $\mathcal{H} \oplus \mathcal{H}$ since the *orthogonal* direct sum of closed subspaces is closed (Theorem 5.10 and Proposition 5.24). Take any $\mathcal{M}_C$-valued sequence $\{z_n\}$, say $z_n = (Cu_n, Cu_n)$ for each n, that converges to $z = (Cx, Cy)$ in $\mathcal{R}(C) \oplus \mathcal{R}(C)$. Thus $\|z_n - z\|^2 = \|(Cu_n, Cu_n) - (Cx, Cy)\|^2 = \|C(u_n - x), C(u_n - y)\|^2 = \|C(u_n - x)\|^2 + \|C(u_n - y)\|^2 \to 0$, so that

$$\|C(u_n - x)\| \to 0 \quad \text{and} \quad \|C(u_n - y)\| \to 0.$$

Hence $Cu_n \to Cx$ and $Cu_n \to Cy$, and so $Cx = Cy$. Then $z = (Cx, Cx)$ lies in $\mathcal{M}_C$, and so $\mathcal{M}_C$ is closed in $\mathcal{R}(C) \oplus \mathcal{R}(C)$ (Theorem 3.30). Thus $\mathcal{M}_C$ is closed in $\mathcal{H} \oplus \mathcal{H}$ since $\mathcal{R}(C) \oplus \mathcal{R}(C)$ is closed in $\mathcal{H} \oplus \mathcal{H}$ (Problem 3.38(c)).

(c) By (a) $\big\|(T \oplus T)|_{\mathcal{M}_C}^n\big\| = \sup_{Cx\neq 0} \frac{\|(T\oplus T)^n(Cx,Cx)\|}{\|(Cx,Cx)\|} = \sup_{Cx\neq 0} \frac{\|T^nCx\|}{\|Cx\|}$.

(d) By (c) and Proposition 4.37(c) $\big\|(T \oplus T)|_{\mathcal{M}_C}^n\big\| = \sup_{Cx\neq 0} \frac{\|Cx\|}{\|Cx\|} = 1$.

(e) By (c) and Proposition 4.37(b) $\big\|(T \oplus T)|_{\mathcal{M}_C}^n\big\| = \sup_{x\neq 0} \frac{\|T^nx\|}{\|x\|} = \|T^n\|$.

(f) By (e) $\big\|(T\oplus T)|_{\mathcal{M}_C}^{\,n}\big\| = \|T^n\| = \|T\|^n = \big\|(T\oplus T)|_{\mathcal{M}_C}\big\|^n$.

(g) Particular case of (f).

(h) Since $T^n = 1\oplus Q^n$ and since $\|Q^3\| = 0 < \|Q^2\| \le \|Q\| \le 1$, it follows that $\|T^n\| = 1 = \|T\|^n$, and T is normaloid. Since $TC = 0\oplus Q^2 \ne O$ and $T^2C = 0\oplus Q^3 = O$, it follows by (c) that $\big\|(T\oplus T)|_{\mathcal{M}_C^{\,2}}\big\| = 0$, while (still by (c))

$$\big\|(T\oplus T)|_{\mathcal{M}_C}\big\| = \sup_{(0\oplus Q)x\ne 0} \frac{\|(0\oplus Q^2)x\|}{\|(0\oplus Q)x\|} = \sup_{Qy\ne 0} \frac{\|Q^2y\|}{\|Qy\|} \ge \frac{\|Q^2y_0\|}{\|Qy_0\|} > 0$$

for every $y_0 \in \mathcal{H}\backslash\mathcal{N}(Q^2)$. Thus $\big\|(T\oplus T)|_{\mathcal{M}_C^{\,2}}\big\| \ne \big\|(T\oplus T)|_{\mathcal{M}_C}\big\|^2$.

Problem 6.36. If Λ_1 and Λ_2 are sets of complex numbers, then define their product $\Lambda_1\cdot\Lambda_2$ as the set in $\mathbb{C}$ consisting of all products $\lambda_1\lambda_2$ with $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$. It is plain that $\Lambda_1\cdot\Lambda_2 = \Lambda_2\cdot\Lambda_1$. Let $|\Lambda|^2$ denote the set of nonnegative numbers consisting of the squared absolute values $|\lambda|^2$ of all $\lambda \in \Lambda$ (i.e., the set of all nonnegative numbers of the form $\overline{\lambda}\lambda$). Recall that $\sigma(T^*) = \sigma(T)^*$ for every Hilbert space operator T (Proposition 6.17). Exhibit T such that

(a) $T \cong T^* \ne T$ (i.e., T is unitarily equivalent to its adjoint without being self-adjoint), and

(b) $|\sigma(T)|^2 \ne \sigma(T^*)\cdot\sigma(T) \not\subseteq \mathbb{R}$ (i.e., $\sigma(T^*)\cdot\sigma(T)$ is not only different from $|\sigma(T)|^2$, but it is not even a subset of the real line).

Hints: Take $T = \begin{pmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}$ and $U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ in $\mathcal{B}[\mathbb{C}^3]$.

(a) U is a symmetry such that $UT = T^*U$.

(b) $\sigma(T^*)\cdot\sigma(T) = \{i, 1, -i\}\cdot\{-i, 1, i\} = \{1, i, -1, -i\}$ and $|\sigma(T)|^2 = \{1\}$.

Problem 6.37. Take $T \in \mathcal{B}[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{K}]$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and the (orthogonal) direct sum $T\oplus S \in \mathcal{B}[\mathcal{H}\oplus\mathcal{K}]$. Prove the following assertions.

(a) $\sigma(T\oplus S) = \sigma(T)\cup\sigma(S)$.

Hint: $\rho(T\oplus S) = \rho(T)\cap\rho(S)$.

(b) $\sigma_P(T\oplus S) = \sigma_P(T)\cup\sigma_P(S)$.

Remark: Assertion (a) can be extended to a *finite* direct sum, $\sigma\big(\bigoplus_{i=1}^n T_i\big) = \bigcup_{i=1}^n \sigma(T_i)$, but not to countably infinite direct sums. For instance, let $\{q_k\}_{k=1}^\infty$ be any enumeration of all rational numbers in the interval $[0,1]$. Consider the direct sum $\bigoplus_{k=1}^\infty T_k = \operatorname{diag}\big(\{q_k\}_{k=1}^\infty\big)$ in $\mathcal{B}\big[\bigoplus_{k=1}^\infty \mathbb{C}\big] = \mathcal{B}[\ell_+^2]$ of the bounded family $\{T_k\}_{k=1}^\infty$, each $T_k = q_k$ on $\mathbb{C}$, so that $\bigcup_{k=1}^\infty \sigma(T_k) = \mathbb{Q}\cap[0,1]$, which is not closed in $\mathbb{C}$. But the spectrum is closed. Actually, for every bounded family $\{T_k\}_{k=1}^\infty$ of operators in $\mathcal{B}[\mathcal{H}_k]$ we have $\big(\bigcup_{k=1}^\infty \sigma(T_k)\big)^- \subseteq \sigma\big(\bigoplus_{k+1}^\infty T_k\big)$ but, in general, equality may not hold. However, assertion (b) can be extended to countable direct sums: $\sigma_P\big(\bigoplus_{k=1}^\infty T_k\big) = \bigcup_{k=1}^\infty \sigma_P(T_k)$.

(c) $\sigma_{P_4}(T \oplus S) \not\subseteq \sigma_{P_4}(T) \cup \sigma_{P_4}(S)$.

Hint: $\sigma_{P_1}(T) \cap \sigma_{R_1}(S) \subseteq \sigma_{P_4}(T \oplus S)$ — see Problem 5.7(e).

(d) $\sigma_C(T) \cup \sigma_C(S) \not\subseteq \sigma_C(T \oplus S)$.

Hint: Set $T = 0$ on $\mathcal{H} = \mathbb{C}$ and $S = \operatorname{diag}\big(\{\frac{1}{k}\}_{k+1}^{\infty}\big)$ on $\mathcal{K} = \ell_+^2$ so that $T \oplus S = \operatorname{diag}\big(0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\big)$ on $\mathcal{H} \oplus \mathcal{K}$. Verify that $0 \in \sigma_C(S) \cap \sigma_P(T \oplus S)$.

Problem 6.38. Recall that a quasinilpotent $T \in \mathcal{B}[\mathcal{X}]$ on a Banach space $\mathcal{X}$ is an operator such that $\sigma(T) = \{0\}$ (equivalently, such that $r(T) = 0$). Prove the following assertion. If $0 \notin \sigma(T)$ (i.e., if $0 \in \rho(T)$), then every restriction of T to a nonzero invariant subspace is not quasinilpotent; that is,

$$0 \notin \sigma(T) \quad \Longrightarrow \quad \sigma(T|_{\mathcal{M}}) \neq \{0\} \text{ for every } T\text{-invariant subspace } \mathcal{M} \neq \{0\}.$$

Hint: If $0 \in \rho(T)$, then T is invertible, thus bounded below, and so is $T|_{\mathcal{M}}$. Hence $0 \in \rho(T|_{\mathcal{M}}) \cup \sigma_{R_1}(T|_{\mathcal{M}})$ (cf. diagram of Section 6.2). If $0 \in \sigma_{R_1}(T|_{\mathcal{M}})$, then $\sigma(T|_{\mathcal{M}}) \neq \{0\}$ because $\sigma_{R_1}(T|_{\mathcal{M}})$ is open.

Problem 6.39. Let $T \in \mathcal{B}[\mathcal{X}]$ be an operator on a complex Banach space.

(a) Use the spectral radius condition for uniform stability of Proposition 6.22 to prove the following result.

$$\inf_n \|T^n\| < 1 \quad \text{implies} \quad \|T^n\| \to 0.$$

Hint: If $\inf_n \|T^n\| < 1$, then there is an integer $n_0 \geq 0$ such that $\|T^{n_0}\| < 1$. Thus $r(T)^{n_0} = r(T^{n_0}) \leq \|T^{n_0}\| < 1$, and hence $r(T) < 1$ or, equivalently, $\|T^n\| \to 0$ (cf. Corollary 6.20, its remarks, and Proposition 6.22).

(b) This is the strong stability version of the above result. For every $x \in \mathcal{X}$,

$$\sup_n \|T^n\| < \infty \text{ and } \inf_n \|T^n x\| = 0 \quad \text{implies} \quad \|T^n x\| \to 0.$$

Hint: $\|T^{n+1}x\| \leq \|T\|\|T^n x\|$. If $\inf_n \|T^n x\| = 0$, then $\liminf_n \|T^n x\| = 0$, and therefore there exists a subsequence $\{\|T^{n_k}x\|\}$ of $\{\|T^n x\|\}$ such that $\lim_k \|T^{n_k}x\| \to 0$. If $\sup_n \|T^n\| < \infty$, then take any $\varepsilon > 0$ and let $k_\varepsilon = k_\varepsilon(x)$ be any integer such that $\|T^{n_{k_\varepsilon}}x\| \leq \varepsilon / \sup_n \|T^n\|$. Thus, $\|T^n x\| \leq \|T^{n-n_{k_\varepsilon}}\|\|T^{n_{k_\varepsilon}}x\| \leq \sup_n \|T^n\|\|T^{n_{k_\varepsilon}}x\| < \varepsilon$, and hence $\|T^n x\| \to 0$.

Problem 6.40. Let $T \in \mathcal{B}[\mathcal{X}]$ and $S \in \mathcal{B}[\mathcal{Y}]$ be operators on complex Banach spaces. Apply the preceding problem to prove the following propositions.

(a) $$\|T^n\|\|S^n\| \to 0 \quad \text{implies} \quad \|T^n\| \to 0 \quad \text{or} \quad \|S^n\| \to 0.$$

Hint: According to Problem 6.39(a),

$$\inf_n \|T^n\| = 0 \quad \Longrightarrow \quad \|T^n\| \to 0.$$

Now, under the assumption that $\|T^n\|\|S^n\| \to 0$,

$$\inf_n \|T^n\| > 0 \quad \Longrightarrow \quad \|S^n\| \to 0.$$

In fact, since $\|T^{n+1}\| \le \|T\|\|T^n\|$, it follows that $\inf_n \|T^n\| = 0$ if and only if $\liminf_n \|T^n\| = 0$. Thus $\inf_n \|T^n\| > 0$ if and only if $\liminf_n \|T^n\| > 0$. In this case, since $\|T^n\|\|S^n\| \to 0$, it follows that $\|S^n\| \to 0$.

(b) $\sup_n \|T^n\|\|S^n\| < \infty$ implies $\sup_n \|T^n\| < \infty$ or $\sup_n \|S^n\| < \infty$.

Hint: Suppose $\sup_n \|S^n\| = \infty$ so that $\inf_n \|S^n\| \ge 1$ by Problem 6.39(a). In this case, $\sup_n \|T^n\| < \infty$ if and only if $\sup_n(\|T^n\|\|S^n\|) < \infty$. But if $\sup_n(\|T^n\|\|S^n\|) < \infty$, then $\sup_n \|T^n\| \inf_n \|S^n\| < \infty$. Thus, by Problem 3.10(d), $\sup_n \|T^n\| < \infty$ because $\inf_n \|S^n\| \ge 1$.

(c) The above implications do not hold for general (not power) sequences. Exhibit two sequences of operators on $\mathcal{X}$, $\{T_n\}$ and $\{S_n\}$, such that

$$\|T_n\|\|S_n\| \to 0 \quad \text{but} \quad \limsup_n \|T_n\| = \limsup_n \|S_n\| = \infty.$$

Hint: $T_n = nI$ if n is odd and $T_n = \frac{1}{n^2} I$ if n is even, and $S_n = \frac{1}{n^2} I$ if n is odd and $S_n = nI$ if n is even (or, more drastically, $T_n = O$ if n is even and $S_n = O$ if n is odd).

Problem 6.41. Consider the setup of the previous problem. Prove the following strong stability version of the uniform stability result of Problem 6.40(a).

If $\sup_n \|T^n\| < \infty$ or $\sup_n \|S^n\| < \infty$, and if $\|T^n x\|\|S^n y\| \to 0$ for every x in $\mathcal{X}$ and every y in $\mathcal{Y}$, then $\|T^n x\| \to 0$ for every x in $\mathcal{X}$ or $\|S^n y\| \to 0$ for every y in $\mathcal{Y}$.

Hint: If $\sup_n \|T^n\| < \infty$ then, by Problem 6.39(b),

$$\inf_n \|T^n x\| = 0 \text{ for every } x \in \mathcal{X} \quad \Longrightarrow \quad \|T^n x\| \to 0 \text{ for every } x \in \mathcal{X}.$$

Now, under the assumption that $\|T^n x\|\|S^n y\| \to 0$ for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,

$$\inf_n \|T^n x\| > 0 \text{ for some } x \in \mathcal{X} \quad \Longrightarrow \quad \|S^n y\| \to 0 \text{ for every } y \in \mathcal{Y}.$$

Indeed, if $\inf_n \|T^n x\| > 0$ for some $x \in \mathcal{X}$, then $\liminf_n \|T^n x\| > 0$. Take an arbitrary y in $\mathcal{Y}$. Since $\|T^n x\|\|S^n y\| \to 0$, it follows that $\|S^n y\| \to 0$.

Problem 6.42. Let T and S be operators on a complex Hilbert space $\mathcal{H}$.

(a) Show that ST and TS are proper contractions whenever S is a contraction and T is a proper contraction.

(b) Thus show that the point spectrum $\sigma_P(T^*T)$ lies in the open unit disk if T is a proper contraction.

Conclude: *The concepts of proper and strict contraction coincide for compact operators on a complex Hilbert space.*

Hint: Suppose T is compact. Verify that T^*T is compact (Proposition 4.54). Suppose, in addition, that T is a proper contraction. Use (b) to infer that the spectrum $\sigma(T^*T)$, which is always closed, also lies in the open unit disk (since $\sigma(K)\backslash\{0\} = \sigma_P(K)\backslash\{0\}$ whenever K is compact — Corollary 6.31). Thus verify that $\|T\|^2 = r(T^*T) < 1$.

Problem 6.43. If Λ_1 and Λ_2 are arbitrary subsets of $\mathbb{C}$ and $p\colon \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is any polynomial in two variables (with complex coefficients), then set

$$p(\Lambda_1, \Lambda_2) \,=\, \big\{p(\lambda_1, \lambda_2) \in \mathbb{C}\colon\ \lambda_1 \in \Lambda_1,\ \lambda_2 \in \Lambda_2\big\};$$

in particular, with $\Lambda^* = \{\overline{\lambda} \in \mathbb{C}\colon\ \lambda \in \Lambda\}$,

$$p(\Lambda, \Lambda^*) \,=\, \big\{p(\lambda, \overline{\lambda}) \in \mathbb{C}\colon\ \lambda \in \Lambda\big\}.$$

Let $\mathcal{A}$ be a maximal commutative subalgebra of a complex unital Banach algebra $\mathcal{B}$ (i.e., a commutative subalgebra of $\mathcal{B}$ that is not included in any other commutative subalgebra of $\mathcal{B}$). Consider the algebra $\mathbb{C}$ (of all complex numbers). Let $\hat{\mathcal{A}}$ denote the collection of all algebra homomorphisms of $\mathcal{A}$ onto $\mathbb{C}$ (see Problem 2.31). An important result in spectral theory reads as follows. Let $\mathcal{X}$ be a complex Banach space.

If $\mathcal{A}$ is a maximal commutative subalgebra of $\mathcal{B}[\mathcal{X}]$, then, for every $T \in \mathcal{A}$,

$$\sigma(T) \,=\, \big\{\phi(T) \in \mathbb{C}\colon\ \phi \in \hat{\mathcal{A}}\big\}.$$

(See Problem 2.31.) Use this result to prove the following extension of the Spectral Mapping Theorem for polynomials, which is referred to as the *Spectral Mapping Theorem for Normal Operators*. Let $\mathcal{H}$ be a complex Hilbert space.

If $T \in \mathcal{B}[\mathcal{H}]$ is normal and $p(\cdot, \cdot)$ is a polynomial in two variables, then

$$\sigma(p(T, T^*)) \,=\, p(\sigma(T), \sigma(T^*)) \,=\, \big\{p(\lambda, \overline{\lambda}) \in \mathbb{C}\colon\ \lambda \in \sigma(T)\big\}.$$

Hint: Since T is normal, use Zorn's Lemma to show that there exists a maximal commutative subalgebra $\mathcal{A}_T$ of $\mathcal{B}[\mathcal{H}]$ containing T and T^*. Verify that

$$\phi(p(T, T^*)) \,=\, p(\phi(T), \phi(T^*))$$

for every homomorphism $\phi\colon \mathcal{A}_T \to \mathbb{C}$. Thus

$$\sigma(p(T, T^*)) \,=\, \big\{p(\phi(T), \phi(T^*)) \in \mathbb{C}\colon\ \phi \in \hat{\mathcal{A}}_T\big\}.$$

Take any surjective homomorphism $\phi\colon \mathcal{A}_T \to \mathbb{C}$ (i.e., any $\phi \in \hat{\mathcal{A}}_T$). Consider the Cartesian decomposition $T = A + iB$, where $A, B \in \mathcal{B}[\mathcal{H}]$ are self-adjoint operators, and so $T^* = A - iB$ (Problem 5.46). Thus $\phi(T) = \phi(A) + i\phi(B)$

and $\phi(T^*) = \phi(A) - i\phi(B)$. Verify that $\{\phi(A) \in \mathbb{C}\colon\ \phi \in \hat{\mathcal{A}}_T\} = \sigma(A) \subset \mathbb{R}$ because A is self-adjoint (cf. Corollary 6.18). Hence $\phi(A) \in \mathbb{R}$ and $\phi(B) \in \mathbb{R}$. Thus conclude that

$$\phi(T^*) \,=\, \overline{\phi(T)}.$$

Hence, since $\sigma(T^*) = \sigma(T)^*$ for every $T \in \mathcal{B}[\mathcal{H}]$ by Proposition 6.17,

$$\begin{aligned}\sigma(p(T,T^*)) \,&=\, \big\{p(\phi(T),\overline{\phi(T)}) \in \mathbb{C}\colon\ \phi \in \hat{\mathcal{A}}_T\big\}\\ &=\, \big\{p(\lambda,\overline{\lambda}) \in \mathbb{C}\colon\ \lambda \in \{\phi(T) \in \mathbb{C}\colon\ \phi \in \hat{\mathcal{A}}_T\}\big\}\\ &=\, \big\{p(\lambda,\overline{\lambda}) \in \mathbb{C}\colon\ \lambda \in \sigma(T)\big\} \,=\, p(\sigma(T),\sigma(T)^*) \,=\, p(\sigma(T),\sigma(T^*)).\end{aligned}$$

Problem 6.44. Let T be a normal operator on a complex Hilbert space, and consider its spectral decomposition $T = \int \lambda\, dP_\lambda$ (Theorem 6.47). Show that

$$p(T,T^*) \,=\, \int p(\lambda,\overline{\lambda})\, dP_\lambda$$

for every polynomial $p\colon \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ in two variables with complex coefficients.

Hint: Let P be the spectral measure on $\Sigma_{\sigma(T)}$ for the spectral decomposition of T. Recall from Section 6.8 that if $\varphi, \psi\colon \sigma(T) \to \mathbb{C}$ are bounded $\Sigma_{\sigma(T)}$-measurable functions, and if $F = \int \psi(\lambda)\, dP_\lambda$ and $G = \int \psi(\lambda)\, dP_\lambda$, then $FG = \int \varphi(\lambda)\,\psi(\lambda)\, dP_\lambda$. This is enough to ensure that $T^i T^{*j} = \int \lambda^i \overline{\lambda}^j\, dP_\lambda$, and so

$$\sum_{i,j=0}^{m} \alpha_{i.j} T^i T^{*j} \,=\, \sum_{i,j=0}^{m} \alpha_{i.j} \int_{\sigma(T)} \lambda^i \overline{\lambda}^j\, dP_\lambda \,=\, \int_{\sigma(T)} \Big(\sum_{i,j=0}^{m} \alpha_{i.j} \lambda^i \overline{\lambda}^j \Big)\, dP_\lambda.$$

Remark: Since a polynomial in one variable can be thought of as a particular case of a polynomial in two variables, it follows that

$$p(T) \,=\, \int p(\lambda)\, dP_\lambda$$

for every polynomial $p\colon \mathbb{C} \to \mathbb{C}$ with complex coefficients.

Problem 6.45. Let T be a nonnegative operator on a complex Hilbert space. Consider its spectral decomposition $T = \int \lambda\, dP_\lambda$. Use Problem 6.44 to show that

$$T^{\frac{1}{2}} \,=\, \int \lambda^{\frac{1}{2}}\, dP_\lambda.$$

Hint: Consider the nonnegative square root $T^{\frac{1}{2}}$ of T (Theorem 5.85). Recall that $\sigma(T^{\frac{1}{2}}) = \sigma(T)^{\frac{1}{2}}$ (Section 6.3). Let P be the spectral measure on $\Sigma_{\sigma(T)}$ for the spectral decomposition of T, and let P' be the spectral measure on $\Sigma_{\sigma(T^{1/2})}$ for the spectral decomposition of $T^{\frac{1}{2}}$. Thus

$$T^{\frac{1}{2}} \,=\, \int_{\sigma(T^{\frac{1}{2}})} \lambda\, dP'_\lambda, \quad \text{so that} \quad T \,=\, (T^{\frac{1}{2}})^2 \,=\, \int_{\sigma(T)^{\frac{1}{2}}} \lambda^2\, dP'_\lambda$$

by Problem 6.44, and hence

$$\int_{\sigma(T)^{\frac{1}{2}}} \lambda^2 \, dP'_\lambda = \int_{\sigma(T)} \lambda \, dP_\lambda, \quad \text{which implies} \quad \int_{\sigma(T)} \lambda \, dP'_\lambda = \int_{\sigma(T)} \lambda \, dP_\lambda.$$

Since the spectral decomposition is unique (Theorem 6.47), it follows that $P' = P$, and therefore

$$T^{\frac{1}{2}} = \int_{\sigma(T)^{\frac{1}{2}}} \lambda \, dP'_\lambda = \int_{\sigma(T)} \lambda^{\frac{1}{2}} \, dP_\lambda.$$

Remark: Since polynomials $p(T, T^*)$ and $p(T)$ are compact and normal whenever T is (see Proposition 4.54 and Problem 4.29), and since the nonnegative square root $T^{\frac{1}{2}}$ is compact whenever a nonnegative T is compact (cf. Problem 5.62), it follows by the preceding two problems and Theorem 6.43 that

$$p(T, T^*) = \sum_{\sigma_P(T)} p(\lambda, \overline{\lambda}) \, dP_\lambda, \quad p(T) = \sum_{\sigma_P(T)} p(\lambda) \, dP_\lambda, \quad T^{\frac{1}{2}} = \sum_{\sigma_P(T)} \lambda^{\frac{1}{2}} \, dP_\lambda$$

whenever T is compact and normal (and nonnegative for the case of $T^{\frac{1}{2}}$).

Problem 6.46. Consider the setup of Problem 6.4. Let $M_\varphi \in \mathcal{B}[L^2(\mathbb{T})]$ be the multiplication operator where $\varphi\colon \mathbb{T} \to \mathbb{T} \subseteq \mathbb{C}$ is the identity function ($\varphi(\lambda) = \lambda$ for all $\lambda \in \mathbb{T}$). Use Problem 5.33 and Example 6.E to show that

(a) $\sigma(M_\varphi) = \sigma_C(M_\varphi) = \mathbb{T}$.

Now consider Definition 6.46. For each $\Lambda \in \Sigma_{\mathbb{T}}$ let $\chi_\Lambda\colon \mathbb{T} \to \{0, 1\}$ be its characteristic function, and take the multiplication operator $M_{\chi_\Lambda} \in \mathcal{B}[L^2(\mathbb{T})]$. Show that the mapping $P\colon \Sigma_{\mathbb{T}} \to \mathcal{B}[\mathcal{H}]$ defined by

(b) $P(\Lambda) = M_{\chi_\Lambda}$ for every $\Lambda \in \Sigma_{\mathbb{T}}$ is a spectral measure in $L^2(\mathbb{T})$.

This in fact is the unique (and natural) spectral measure in $L^2(\mathbb{T})$ that leads to the spectral decomposition of the unitary operator M_φ, namely,

$$M_\varphi = \int \lambda \, dP_\lambda,$$

in the sense that the complex-valued measure $p_{x,y}\colon \Sigma_{\mathbb{T}} \to \mathbb{C}$ defined by

$$p_{x,y}(\Lambda) = \langle P(\Lambda)x \,;\, y\rangle = \langle M_{\chi_\Lambda} x \,;\, y\rangle = \int \chi_\Lambda \, x(\lambda) \, \overline{y(\lambda)} \, d\lambda = \int_\Lambda x(\lambda) \, \overline{y(\lambda)} \, d\lambda$$

for every $\Lambda \in \Sigma_{\mathbb{T}}$ is such that, for each $x, y \in L^2(\mathbb{T})$,

$$\langle M_\varphi \, x \,;\, y\rangle = \int \varphi(\lambda) \, dp_{x,y} = \int \lambda \, x(\lambda) \, \overline{y(\lambda)} \, d\lambda.$$

References

N.I. Akhiezer and I.M. Glazman

[1] *Theory of Linear Operators in Hilbert Space – Volume I*, Pitman, London, 1981; reprinted: Dover, New York, 1993.

[2] *Theory of Linear Operators in Hilbert Space – Volume II*, Pitman, London, 1981; reprinted: Dover, New York, 1993.

W. Arveson

[1] *An Invitation to C*-Algebras*, Springer, New York, 1976.

[2] *A Short Course on Spectral Theory*, Springer, New York, 2002.

G. Bachman and L. Narici

[1] *Functional Analysis*, Academic Press, New York, 1966; reprinted: Dover, Mineola, 2000.

A.V. Balakrishnan

[1] *Applied Functional Analysis*, 2nd edn. Springer, New York, 1981.

S. Banach

[1] *Theory of Linear Operations*, North-Holland, Amsterdam, 1987.

R. Beals

[1] *Topics in Operator Theory*, The University of Chicago Press, Chicago, 1971.

B. Beauzamy

[1] *Introduction to Operator Theory and Invariant Subspaces*, North-Holland, Amsterdam, 1988.

S.K. Berberian

[1] *Notes on Spectral Theory*, Van Nostrand, New York, 1966.

[2] *Lectures in Functional Analysis and Operator Theory*, Springer, New York, 1974.

[3] *Introduction to Hilbert Space*, 2nd edn. Chelsea, New York, 1976.

C.S. Kubrusly, *The Elements of Operator Theory*, DOI 10.1007/978-0-8176-4998-2,

© Springer Science+Business Media, LLC 2011

Y.M. Berezansky, Z.G. Sheftel, and G.F. Us
[1] *Functional Analysis – Volume I*, Birkhäuser, Basel, 1996.
[2] *Functional Analysis – Volume II*, Birkhäuser, Basel, 1996.

K.G. Binmore
[1] *The Foundations of Analysis – A Straightforward Introduction – Book I: Logic, Sets and Numbers*, Cambridge University Press, Cambridge, 1980.

G. Birkhoff and S. MacLane
[1] *A Survey of Modern Algebra*, Macmillan, New York, 1941; 3rd edn. 1965.

A. Brown and C. Pearcy
[1] Spectra of tensor products of operators, *Proc. Amer. Math. Soc.* **17** (1966), 162–166.
[2] *Introduction to Operator Theory I – Elements of Functional Analysis*, Springer, New York, 1977.
[3] *An Introduction to Analysis*, Springer, New York, 1995.

S.W. Brown
[1] Some invariant subspaces for subnormal operators, *Integral Equations Operator Theory* **1** (1978), 310–333.

G. Cantor
[1] Ein Beitrag zur Mannigfaltigkeitslehre, *J. für Math.* **84** (1878), 242–258.

K. Clancey
[1] *Seminormal Operators*, Springer, Berlin, 1979.

P.J. Cohen
[1] The independence of the continuum hypothesis, *Proc. Nat. Acad. Sci.* **50** (1963), 1143–1148.
[2] *Set Theory and the Continuum Hypothesis*, W.A. Benjamin, New York, 1966.

I. Colojoară and C. Foiaş
[1] *Theory of Generalized Spectral Operators*, Gordon & Breach, New York, 1968.

J.B. Conway
[1] *A Course in Functional Analysis*, 2nd edn. Springer, New York, 1990.
[2] *The Theory of Subnormal Operators*, Mathematical Surveys and Monographs Vol. 36, Amer. Math. Soc., Providence, 1991.
[3] *A Course in Operator Theory*, Graduate Studies in Mathematics Vol. 21, Amer. Math. Soc., Providence, 2000.

J.N. Crossley, C.J. Ash, C.J Brickhill, J.C. Stillwell, and N.H. Williams
[1] *What is Mathematical Logic*, Oxford University Press, Oxford, 1972.

K.R. Davidson
[1] *C^*-Algebras by Example*, Fields Institute Monographs Vol. 6, Amer. Math. Soc., Providence, 1996.

M.M. Day
[1] *Normed Linear Spaces*, 3rd edn. Springer, Berlin, 1973.

J. Dieudonné
[1] *Foundations of Modern Analysis*, Academic Press, New York, 1969.

R.G. Douglas
[1] *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972; 2nd edn. Springer, New York, 1998.

H.R. Dowson
[1] *Spectral Theory of Linear Operators*, Academic Press, London, 1978.

J. Dugundji
[1] *Topology*, Allyn & Bacon, Boston, 1966; reprinted: 1978.

N. Dunford and J.T. Schwartz
[1] *Linear Operators – Part I: General Theory*, Interscience, New York, 1958; reprinted: Wiley, New York, 1988.
[2] *Linear Operators – Part II: Spectral Theory – Self Adjoint Operators in Hilbert Space*, Interscience, New York, 1963; reprinted: Wiley, New York, 1988.
[3] *Linear Operators – Part III: Spectral Operators*, Interscience, New York, 1971; reprinted: Wiley, New York, 1988.

A. Dvoretzky and C.A. Rogers
[1] Absolute and unconditional convergence in normed linear spaces, *Proc. Nat. Acad. Sci.* **36** (1950), 192–197.

P. Enflo
[1] A counterexample to the approximation problem in Banach spaces, *Acta Math.* **130** (1973), 309–317.
[2] On the invariant subspace problem for Banach spaces, *Acta Math.* **158** (1987), 213–313.

S. Feferman
[1] Some applications of the notion of forcing and generic sets, *Fund. Math.* **56** (1965), 325–345.

P.A. Fillmore
[1] *Notes on Operator Theory*, Van Nostrand, New York, 1970.
[2] *A User's Guide to Operator Algebras*, Wiley, New York, 1996.

A.A. Fraenkel, Y. Bar-Hillel, and A. Levy
[1] *Foundations of Set Theory*, 2nd edn. North-Holland, Amsterdam, 1973.

T. Furuta
[1] *Invitation to Linear Operators: From Matrices to Bounded Linear Operators on a Hilbert Space*, Taylor & Francis, London, 2001.

B.R. Gelbaum and J.M.H. Olmsted
[1] *Counterexamples in Analysis*, Holden-Day, San Francisco, 1964; reprinted: Dover, Mineola, 2003.

K. Gödel
[1] Consistence-proof for the generalized continuum-hypothesis, *Proc. Nat. Acad. Sci.* **25** (1939), 220–224.

C. Goffman and G. Pedrick
[1] *A First Course in Functional Analysis*, Prentice-Hall, Englewood Cliffs, 1965; 2nd edn. Chelsea, New York, 1983.

I.C. Gohberg and M.G. Kreĭn
[1] *Introduction to Nonselfadjoint Operators*, Translations of Mathematical Monographs Vol. 18, Amer. Math. Soc., Providence, 1969.

S. Goldberg
[1] *Unbounded Linear Operators*, McGraw-Hill, New York, 1966; reprinted: Dover, Mineola, 1985, 2006.

K.E. Gustafson and D.K.M. Rao
[1] *Numerical Range*, Springer, New York, 1997.

P.R. Halmos
[1] *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, 2nd edn. Chelsea, New York, 1957; reprinted: AMS Chelsea, Providence, 1998.
[2] *Finite-Dimensional Vector Spaces*, Van Nostrand, New York, 1958; reprinted: Springer, New York, 1974.
[3] *Naive Set Theory*, Van Nostrand, New York, 1960; reprinted: Springer, New York, 1974.
[4] *A Hilbert Space Problem Book*, Van Nostrand, New York, 1967; 2nd edn. Springer, New York, 1982.
[5] *Linear Algebra Problem Book*, Dolciani Mathematical Expositions, No. 16. Math. Assoc. America, Washington, D.C., 1995.

G. Helmberg
[1] *Introduction to Spectral Theory in Hilbert Space*, North-Holland, Amsterdam, 1969.

D. Herrero
[1] *Approximation of Hilbert Space Operators – Volume. 1*, 2nd edn. Longman, Harlow, 1989.

I.N. Herstein
[1] *Topics in Algebra*, Blaisdell, New York, 1964; 2nd edn. Xerox, Lexington, 1975.
[2] *Abstract Algebra*, Macmillan, New York, 1986; 3rd edn. Prentice-Hall, Upper Saddle River, 1996.

E. Hille and R.S. Phillips
[1] *Functional Analysis and Semi-Groups*, Colloquium Publications Vol. 31, Amer. Math. Soc., Providence, 1957; reprinted: 1974.

K. Hoffman and R. Kunze
[1] *Linear Algebra*, 2nd edn. Prentice-Hall, Englewood Cliffs, 1971.

V.I. Istrăţescu
[1] *Introduction to Linear Operator Theory*, Marcel Dekker, New York, 1981.

L.V. Kantorovich and G.P. Akilov
[1] *Functional Analysis*, 2nd edn. Pergamon Press, Oxford, 1982.

I. Kaplansky
[1] *Linear Algebra and Geometry: A Second Course*, Allyn & Bacon, Boston, 1969; revised edition: Dover, Mineola, 2003.
[2] *Set Theory and Metric Spaces*, Allyn & Bacon, Boston, 1972; reprinted: Chelsea, New York, 1977.

T. Kato
[1] *Perturbation Theory for Linear Operators*, 2nd edn. Springer, Berlin, 1980; reprinted: 1995.

J.L. Kelley
[1] *General Topology*, Van Nostrand, New York, 1955; reprinted: Springer, New York, 1975.

A.N. Kolmogorov and S.V. Fomin
[1] *Introductory Real Analysis*, Prentice-Hall, Englewood Cliffs, 1970; reprinted: Dover, New York, 1975.

E. Kreyszig
[1] *Introductory Functional Analysis with Applications*, Wiley, New York, 1978, reprinted: 1989.

C.S. Kubrusly
[1] *An Introduction to Models and Decompositions in Operator Theory*, Birkhäuser, Boston, 1997.
[2] *Hilbert Space Operators: A Problem Solving Approach*, Birkhäuser, Boston, 2003.

P.D. Lax
[1] *Functional Analysis*, Wiley, New York, 2002.
[2] *Linear Algebra and Its Applications*, 2nd edn. Wiley, Hoboken, 2007.

V.I. Lomonosov
[1] Invariant subspaces for the family of operators which commute with a completely continuous operator, *Functional Anal. Appl.* **7** (1973), 213–214.

S. MacLane and G. Birkhoff
[1] *Algebra*, Macmillan, New York, 1967; 3rd edn. Chelsea, New York, 1988.

M. Martin and M. Putinar
[1] *Lectures on Hyponormal Operators*, Birkhäuser, Basel, 1989.

I.J. Maddox
[1] *Elements of Functional Analysis*, 2nd edn. Cambridge University Press, Cambridge, 1988.

J.N. McDonald and N.A Weiss
[1] *Real Analysis*, Academic Press, San Diego, 1999.

G.H. Moore
[1] *Zermelo's Axiom of Choice*, Springer, New York, 1982.

J.R. Munkres
[1] *Topology: A First Course*, Prentice-Hall, Englewood Cliffs, 1975.

G. Murphy
[1] *C*-Algebras and Operator Theory*, Academic Press, San Diego, 1990.

A.W. Naylor and G.R. Sell
[1] *Linear Operator Theory in Engineering and Science*, Holt, Rinehart & Winston, New York, 1971; reprinted: Springer, New York, 1982.

C.M. Pearcy
[1] *Some Recent Developments in Operator Theory*, CBMS Regional Conference Series in Mathematics No. 36, Amer. Math. Soc., Providence, 1978.
[2] *Topics in Operator Theory*, Mathematical Surveys No. 13, Amer. Math. Soc., Providence, 2nd pr. 1979.

A. Pietsch
[1] *History of Banach Spaces and Linear Operators*, Birkhäuser, Boston, 2007.

C.R. Putnam
[1] *Commutation Properties of Hilbert Space Operators and Related Topics*, Springer, Berlin, 1967.

H. Radjavi and P. Rosenthal
[1] *Invariant Subspaces*, Springer, New York, 1973; 2nd edn. Dover, Mineola, 2003.
[2] *Simultaneous Triangularization*, Springer, New York, 2000.

C.J. Read
[1] A solution to the invariant subspace problem, *Bull. London Math. Soc.* **16** (1984), 337–401.
[2] Quasinilpotent operators and the invariant subspace problem, *J. London Math, Soc.* **56** (1997), 595–606.

M. Reed and B. Simon
[1] *Methods of Modern Mathematical Physics I: Functional Analysis*, 2nd edn. Academic Press, New York, 1980.

F. Riesz and B. Sz.-Nagy
[1] *Functional Analysis*, Frederick Ungar, New York, 1955; reprinted: Dover, New York, 1990.

A.P. Robertson and W.I. Robertson
[1] *Topological Vector Spaces*, 2nd edn. Cambridge University Press, Cambridge, 1973; reprinted: 1980.

S. Roman
[1] *Advanced Linear Algebra*, 3rd edn. Springer, New York, 2008.

H.L. Royden
[1] *Real Analysis*, 3rd edn. Macmillan, New York, 1988.

W. Rudin
[1] *Functional Analysis*, 2nd edn. McGraw-Hill, New York, 1991.

B.P. Rynne and M.A. Youngson
[1] *Linear Functional Analysis*, 2nd edn. Springer, London, 2008.

R. Schatten
[1] *Norm Ideals of Completely Continuous Operators*, Springer, Berlin, 1970.

L. Schwartz
[1] *Analyse – Topologie Générale et Analyse Fonctionnelle*, 2ème édn. Hermann, Paris, 1970.

W. Sierpinski
[1] L'hypothese généralisée du continu et l'axiome du choix, *Fund. Math.* **34** (1947), 1–5.

G.F. Simmons
[1] *Introduction to Topology and Modern Analysis*, McGraw-Hill, New York, 1963; reprinted: Krieger, Melbourne, 1983.

D.R. Smart
[1] *Fixed Point Theorems*, Cambridge University Press, Cambridge, 1974.

M.H. Stone
[1] *Linear Transformations in Hilbert Space*, Colloquium Publications Vol. 15, Amer. Math. Soc., Providence, 1932; reprinted: 1990.

V.S. Sunder
[1] *Functional Analysis – Spectral Theory*, Birkhäuser, Basel, 1998.

P. Suppes
[1] *Axiomatic Set Theory*, Van Nostrand, Princeton, 1960; reprinted: Dover, New York, 1972.

W.A. Sutherland
[1] *Introduction to Metric and Topological Spaces*, Oxford University Press, Oxford, 1975.

B. Sz.-Nagy and C. Foiaş
[1] *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, 1970.

A.E. Taylor and D.C. Lay
[1] *Introduction to Functional Analysis*, 2nd edn. Wiley, New York, 1980; enlarged edn. of A.E. Taylor, 1958; reprinted: Krieger, Melbourne, 1986.

R.L. Vaught
[1] *Set Theory – An Introduction*, 2nd edn. Birkhäuser, Boston, 1995.

J. Weidmann
[1] *Linear Operators in Hilbert Spaces*, Springer, New York, 1980.

R.L. Wilder
[1] *Introduction to the Foundation of Mathematics*, 2nd edn. Wiley, New York, 1965; reprinted: Krieger, Malabar, 1983.

D. Xia
[1] *Spectral Theory of Hyponormal Operators*, Birkhäuser, Basel, 1983.

B.H. Yandell
[1] *The Honors Class: Hilbert's Problems and Their Solvers*, A K Peters, Natick, 2002.

T. Yoshino
[1] *Introduction to Operator Theory*, Longman, Harlow, 1993.

K. Yosida
[1] *Functional Analysis*, 6th edn. Springer, Berlin, 1981; reprinted: 1995.

Index

Abelian group, 38
absolute homogeneity, 200
absolutely convergent series, 203
absolutely convex set, 271
absolutely homogeneous functional, 200
absolutely homogeneous metric, 200
absolutely summable family, 341, 343
absolutely summable sequence, 203
absorbing set, 271
accumulation point, 117–119
additive Abelian group, 38
additive mapping, 55
additively invariant metric, 199
additivity, 310
adherent point, 117, 118
adjoint, 376, 384–388, 456
algebra, 82
algebra homomorphism, 84
algebra isomorphism, 84
algebra with identity, 82
algebraic complement, 68–72, 289
algebraic conjugate, 56
algebraic dual, 56
algebraic linear transformation, 80
algebraic operator, 282
algebraically disjoint, 67
Almost Orthogonality Lemma, 238
annihilator, 302
antisymmetric relation, 8
approximate eigenvalue, 454
approximate point spectrum, 454
approximation spectrum, 454
Arzelà–Ascoli Theorem, 164, 198
ascent, 85
associative binary operation, 37
Axiom of Choice, 15
Axiom of Empty Set, 23
Axiom of Extension, 23
Axiom of Foundation, 23
Axiom of Infinity, 23
Axiom of Pairing, 23
Axiom of Power Set, 23
Axiom of Regularity, 23
Axiom of Replacement, 23
Axiom of Restriction, 23
Axiom of Separation, 23
Axiom of Specification, 23
Axiom of Substitution, 23
Axiom of Union, 23

backward bilateral shift, 293, 421
backward unilateral shift, 249, 292, 420, 476
Baire Category Theorem, 146–148
Baire metric, 188
Baire space, 148
balanced set, 271
Banach algebra, 224
Banach Fixed Point Theorem, 133
Banach limit, 303
Banach space, 202, 211, 216, 221, 235, 272, 294

Banach–Steinhaus Theorem, 244, 294
Banach–Tarski Lemma, 11
barrel, 272
barreled space, 272, 294
base, 125, 272
basis, 48, 227, 351
Bessel inequality, 352
best linear approximation, 329
bidual, 267
bijective function, 5
bilateral ideal, 83
bilateral shift, 292, 421, 424, 471
bilateral weighted shift, 472, 473
bilinear form, 309
bilinear functional, 309
binary operation, 37
block diagonal operator, 280
Bolzano–Weierstrass Property, 157
Boolean sum, 4
boundary, 182
boundary point, 182
bounded above, 9, 10, 225
bounded away from zero, 227, 273
bounded below, 9, 10, 225, 230
bounded family, 209
bounded function, 10, 89, 273
bounded increments, 169
bounded inverse, 229, 230
Bounded Inverse Theorem, 230
bounded linear operator, 222
bounded linear transformation, 217
bounded sequence, 14, 90, 129, 204
bounded set, 9, 89, 154, 271, 273
bounded variation, 186
boundedly complete lattice, 10
Browder Fixed Point Theorem, 308

C*-algebra, 393
canonical basis for $\mathbb{F}^n$, 54
canonical bilateral shift, 422, 472
canonical orthonormal basis for $\mathbb{F}^n$, 359
canonical orthonormal basis for ℓ^2_+, 360
canonical unilateral shift, 420, 472
Cantor set, 191
Cantor–Bernstein Theorem, 11, 17
cardinal number, 16, 22
cardinality, 14, 16, 18, 23, 51, 53
Cartesian decomposition, 428
Cartesian product, 4, 13, 66, 178, 190, 194, 208
Cauchy criterion, 128, 275, 341
Cauchy sequence, 128, 129, 135, 155, 185–188, 272
Cayley transform, 503, 504
chain, 12
characteristic function, 16, 17, 29
Chebyshev–Hermite functions, 361
Chebyshev–Hermite polynomials, 361
Chebyshev–Laguerre functions, 361
Chebyshev–Laguerre polynomials, 361
clopen set, 185
closed ball, 102
closed convex hull, 271
Closed Graph Theorem, 231
closed linear transformation, 287–289
closed map, 115
closed set, 114–116, 118, 120, 129, 130, 150, 151
Closed Set Theorem, 120
closed subspace, 181
closure, 115–117, 181, 185
cluster point, 117, 119
codimension, 70, 81
codomain, 5
coefficients, 277, 357
cohyponormal operator, 447
coisometry, 388, 420, 421
collinear vectors, 406
comeager set, 145
commensurable topologies, 108
commutant, 284
commutative algebra, 83
commutative binary operation, 38
commutative diagram, 6
commutative group, 38
commutative ring, 39
commuting operators, 282, 491, 494, 495, 511
compact extension, 258
compact linear transformation, 252–258, 300, 301, 306, 426, 427
compact mapping, 308
compact operator, 256, 433, 435, 438, 478–482, 486, 488–491, 497, 506, 507, 518, 520
compact restriction, 258

compact set, 149–151, 160, 161, 195, 196, 198, 238, 239
compact space, 150–152, 156, 158–164, 194, 196
compatible topology, 270
complementary linear manifolds, 338
complementary projection, 71, 290, 364
complementary subspaces, 289, 290, 336, 339, 365
complete lattice, 10, 45, 178, 214, 282
complete set, 272
complete space, 129–133, 146–149, 157–161, 186–190, 193
completely continuous, 252
completely nonnormal operator, 507
completely nonunitary contraction, 440
completion, 139, 142, 143, 242–244, 337
complex Banach space, 202
complex field, 40
complex Hilbert space, 315
complex inner product space, 311
complex linear space, 41
complex normed space, 201
complex-valued function, 5
complex-valued sequence, 13
composition of functions, 6
compression spectrum, 454
condensation point, 181
conditionally compact, 151
cone, 80
conjugate homogeneous, 310, 375
conjugate space, 266
connected set, 185
connected space, 185
connectedness, 185
consistent system of axioms, 3
constant function, 5
continuity, 98, 108
continuity of inner product, 407, 408
continuity of inversion, 297
continuity of metric, 179
continuity of norm, 202
continuity of scalar multiplication, 271
continuity of vector addition, 270, 271
continuous composition, 105, 177
continuous extension, 136
Continuous Extension Theorem, 264
continuous function, 98–105, 115, 124, 151, 152, 161, 183, 218
continuous inverse, 225, 230
Continuous Inverse Theorem, 230
continuous linear extension, 239–241
continuous linear transformation, 217
continuous projection, 223, 289, 290, 299, 300
continuous restriction, 177
continuous spectrum, 453
Continuum Hypothesis, 23
contraction, 99, 196, 220, 297, 396, 439–442, 499
Contraction Mapping Theorem, 133
contrapositive proof, 2
convergence, 95, 108
convergence-preserving map, 101
convergent nets, 98, 105
convergent sequence, 95–98, 105, 108, 128, 129
convergent series, 203, 274, 275, 345
convex functional, 200
convex hull, 75, 271
convex linear combination, 75
convex set, 75, 271
convex space, 272
coordinates, 49
coset, 43
countable set, 18
countably infinite set, 18
covering, 8, 149
cyclic subspace, 283
cyclic vector, 283

De Morgan laws, 4, 26
decomposition, 67, 71–74, 339, 365, 402, 428, 489, 494, 510
decreasing function, 10
decreasing increments, 186
decreasing sequence, 14, 31
dense in itself, 128
dense linear manifold, 213, 239–243, 259, 294, 329, 330, 415
dense set, 123, 124, 147, 148, 181, 326
dense subspace, 124, 136, 138, 139
densely embedded, 139
densely intertwined, 283
denumerable set, 18
derived set, 117, 120
descent, 85
diagonal mapping, 174, 184, 220, 289

diagonal operator, 220, 226, 248, 251, 256, 298, 397, 414, 439, 472, 508
diagonal procedure, 22, 156, 163
diagonalizable operator, 414, 469, 491, 508, 509
diameter, 89
difference equation, 79, 134
dimension, 53, 76, 78, 81, 356
direct proof, 2
direct sum, 66–68, 74, 208–210, 222–224, 279, 280, 289, 319–321, 323, 334, 335, 338–340, 365, 389, 413, 419, 438–441, 489, 511, 515
direct sum decomposition, 68, 71–74, 339, 365
direct summand, 74, 279
directed downward, 10
directed set, 10
directed upward, 10
disconnected set, 185
disconnected space, 185
disconnection, 185
discontinuous function, 99
discrete dynamical system, 79
discrete metric, 107
discrete set, 127, 185
discrete space, 107
discrete topology, 107
disjoint linear manifolds, 67
disjoint sets, 4
disjointification, 30
distance, 87, 91
distance function, 87
distributive laws, 39
division ring, 39
domain, 5
Dominated Extension Theorems, 264
doubleton, 4
dual space, 266

ε-net, 153
eigenspace, 453, 488, 489, 504
eigenvalue, 453, 454, 488–490, 510
eigenvector, 453, 454, 490, 508
embedding, 6
empty function, 29
empty sum, 346
equicontinuous, 162, 198, 294
equiconvergent sequences, 172, 187
equivalence, 232
equivalence class, 7
equivalence relation, 7
equivalent metrics, 108, 110, 178
equivalent norms, 233, 234
equivalent sets, 14
equivalent spaces, 232
Euclidean metric, 89
Euclidean norm, 204, 316
Euclidean space, 89, 204, 316
eventually constant, 109
eventually in, 105
expansion, 49, 277, 357
exponentially decreasing increments, 186
extended nonnegative integers, 85
extension by continuity, 260
extension of a function, 6
extension ordering, 29
extension over completion, 142, 143, 243, 244, 258, 338
exterior, 123
exterior point, 123

F-space, 272, 273
F_σ, 149
field, 40
final space, 401
finite sequence, 13
finite set, 15
finite-dimensional space, 53, 60, 65, 76, 79, 81, 234–239, 246, 253, 269, 291, 302, 343, 351, 379, 382, 425, 468, 508
finite-dimensional transformation, 79, 252, 255
finite-rank transformation, 79, 252, 253, 255, 291, 301, 427, 480
first category set, 145–149
fixed point, 6, 11, 26, 70, 133, 308
Fixed Point Theorems, 133, 308
Fourier coefficients, 357
Fourier series expansion, 357
Fourier Series Theorem, 356
Fréchet space, 272, 273
Fredholm Alternative, 484
Fubini Theorem, 387
Fuglede Theorem, 494
Fuglede–Putnam Theorem, 495

full direct sum, 208
function, 4

G_δ, 149
Gelfand–Beurling formula, 461
Gelfand–Naimark Theorem, 393
Generalized Continuum Hypothesis, 23
Gram–Schmidt process, 354
graph, 4
greatest lower bound, 9
group, 38, 231

Haar wavelet, 361
Hahn Interpolation Theorem, 180
Hahn–Banach Theorem, 259–264
Hamel basis, 48–51, 351, 355
Hausdorff Maximal Principle, 24
Hausdorff space, 180
Heine–Borel Theorem, 160
Hermitian operator, 393
Hermitian symmetric functional, 310
Hermitian symmetry, 310
Hilbert basis. 351
Hilbert cube, 196
Hilbert space, 315, 375
Hilbert–Schmidt operator, 434, 435
Hölder conjugates, 165
Hölder inequalities, 165, 166, 168
homeomorphic spaces, 110, 152
homeomorphism, 110–113, 115, 152
homogeneity, 310
homogeneous mapping, 55
homomorphism, 84
hyperinvariant linear manifold, 284
hyperinvariant subspace, 284, 285
hyperplane, 81
hyponormal operator, 447–450, 457, 501, 502, 505–507, 509–513
hyponormal restriction, 505

ideal, 83
idempotent, 7, 26, 70, 300
identity element, 38, 39, 82
identity map, 6
identity operator, 224
image of a point, 5
image of a set, 5
inclusion map, 6
incommensurable topologies, 108
increasing function, 10, 11
increasing sequence, 14, 31
independent system of axioms, 3
index set, 12
indexed family, 12
indexing, 13
indiscrete topology, 107
induced equivalence relation, 8
induced topology, 106, 201, 312, 313
induced uniform norm, 220
inductive set, 2
infimum, 9, 10, 14
infinite diagonal matrix, 220
infinite sequence, 13
infinite set, 15
infinite series, 203
infinite-dimensional space, 53, 353, 356
initial segment, 13
initial space, 401
injection, 6
injective function, 5, 26, 27
injective linear transformation, 56
injective mapping, 5, 11, 16
inner product, 310
inner product axioms, 310
inner product space, 311
inner product space $\ell^2_+(\mathcal{X})$, 321
integer-valued function, 5
integer-valued sequence, 13
interior, 121
interior point, 123
intertwined operators, 283, 495, 497
intertwining transformation, 283
invariant linear manifold, 73, 74, 281
invariant set, 6
invariant subspace, 281–284, 389, 390, 497, 498, 502, 505, 506, 511
invariant subspace problem, 497, 511
inverse element, 38, 83
inverse image, 5
Inverse Mapping Theorem, 230
inverse of a function, 7, 27, 225
inversely induced topology, 181
invertible contraction, 297
invertible element of $\mathcal{B}[\mathcal{X},\mathcal{Y}]$, 230
invertible function, 7, 27
invertible linear transformation, 58
invertible operator in $\mathcal{B}[\mathcal{X}]$, 231
involution, 27, 393, 427, 439, 457

irreflexive spaces, 269
isolated point, 127, 128, 144, 510, 511
isoloid operator, 511
isometric isomorphism, 241, 243, 244, 268, 269, 291, 293, 302, 335, 337
isometric spaces, 111
isometrically equivalent operators, 394
isometrically equivalent spaces, 111, 139, 142, 375
isometrically isomorphic spaces, 241, 243, 267, 268, 294, 303, 335, 375
isometry, 111, 196, 241, 292, 293, 298, 300, 336, 388, 439, 444, 467, 511
isomorphic algebras, 84
isomorphic equivalence, 63, 64, 66
isomorphic linear spaces, 59, 62–64
isomorphism, 58–60, 63–65

Jensen inequality, 167

k-paranormal operator, 512, 513
k-paranormal restriction, 513
kernel, 55
Kronecker delta, 53
Kronecker function, 53

lattice, 10, 29, 45, 214, 281, 282
Laurent expansion, 461
Law of the Excluded Middle, 2
least upper bound, 9
least-squares, 426
left ideal, 83
left inverse, 27
limit, 31, 95, 98, 171
limit inferior, 31, 171
limit superior, 31, 171
linear algebra, 82
linear basis, 48
linear combination, 45, 46
linear composition, 78
linear dimension, 53, 353, 356
linear equivalence relation, 42
linear extension, 56, 259–262
linear functional, 55
linear manifold, 43, 210, 329, 338
linear restriction, 56, 78
linear space, 40
linear space $\mathcal{L}[\mathcal{X}, \mathcal{Y}]$, 56, 78
linear span, 45
linear topology, 270
linear transformation, 55, 56, 62, 64
linear variety, 81
linearly independent set, 46, 349
linearly ordered set, 12
Liouville Theorem, 451
Lipschitz condition, 99
Lipschitz constant, 99
Lipschitzian mapping, 99, 196, 218
locally compact space, 195
locally convex space, 272
Lomonosov Theorem, 497
lower bound, 9, 10
lower limit, 171
lower semicontinuity, 176

map, 5
mapping, 5
mathematical induction, 2, 13
matrix, 61, 62, 65
Matrix Inversion Lemma, 84
maximal commutative subalgebra, 518
maximal element, 9
maximal linear variety, 81
maximal orthonormal set, 350, 351, 353
maximum, 9
meager set, 145
metric, 87
metric axioms, 87
metric function, 87
metric generated by a norm, 201, 202
metric generated by a quasinorm, 272
metric space, 87
metrizable, 107
minimal element, 9
minimum, 9
Minkowski inequalities, 166, 167
Möbius transformation, 503
modus ponens, 2
monotone function, 10
monotone sequence, 14, 31
multiplication operator, 500, 520
multiplicity, 420, 421, 453
mutually orthogonal projections, 368

natural embedding, 268, 269
natural isomorphism, 65, 67, 339
natural mapping, 8, 44, 66, 290, 338
natural projection, 224, 290

neighborhood, 103, 180
neighborhood base, 272
net, 14
neutral element, 38
Newton's Second Law, 197
nilpotent linear transformation, 80
nilpotent operator, 282, 461, 473, 511
nondegenerate interval, 21
nondenumerable set, 18
nonmeager set, 145
nonnegative contraction, 431, 499
nonnegative functional, 200
nonnegative homogeneity, 200
nonnegative operator, 395–398, 429–433, 439, 457, 500, 508–510
nonnegative quadratic form, 310
nonnegativeness, 87, 200, 310
nonreflexive spaces, 270
nonstrict contraction, 220, 221
nontrivial hyperinvariant subspace, 285, 286, 495, 497, 504
nontrivial invariant subspace, 281–284, 286, 389, 497, 498, 504, 510, 511
nontrivial linear manifold, 43
nontrivial projection, 70, 457
nontrivial reducing subspace, 489, 495
nontrivial ring, 39
nontrivial subset, 3
nontrivial subspace, 212, 281
norm, 200, 219
norm axioms, 200
norm induced by an inner product, 312, 313, 315, 405
norm topology, 201, 313
normal extension, 446
normal operator, 443–446, 450, 457, 484, 487–491, 494–497, 499–501, 506–511, 518, 519
normal restriction, 506
normaloid operator, 449, 450, 463, 467, 468, 481, 512–514
normed algebra, 224
normed linear space, 201
normed space, 201
normed space $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$, 219, 220
normed spaces $\ell_+^p(\mathcal{X})$ and $\ell_+^\infty(\mathcal{X})$, 210
normed vector space, 201
nowhere continuous, 100
nowhere dense, 144, 145, 148
nuclear operator, 434
null function, 42
null operator, 248, 381, 468
null space, 55, 218
null transformation, 56, 219
nullity, 78
numerical radius, 466–468
numerical range, 465

one-to-one correspondence, 5, 11, 14
one-to-one mapping, 5
open ball, 102, 103
open map, 110
Open Mapping Theorem, 227
open neighborhood, 103
open set, 102–106
open subspace, 181
operator, 222
operator algebra $\mathcal{B}[\mathcal{X}]$, 224, 231, 393
operator convergence, 246–250, 295, 305, 306, 378–382, 398, 415–418
operator matrix, 281
operator norm property, 222
orbit, 282
ordered n-tuples, 13
ordered pair, 4
ordering, 8
order-preserving correspondence, 25
ordinal number, 25
origin of a linear space, 41, 44
orthogonal complement, 326, 336, 339
orthogonal dimension, 353, 356, 362
orthogonal direct sum, 323, 335, 339, 340, 365, 389, 411, 413, 419–421, 438–441, 491, 511, 515, 516
orthogonal family, 348, 352, 353, 368
Orthogonal Normalization Lemma, 413
orthogonal projection, 364–373, 395, 439, 444, 457, 508, 509
orthogonal projection onto $\mathcal{M}$, 366, 367, 370, 390, 488, 505
orthogonal sequence, 322, 323, 368
orthogonal set, 321, 322, 349, 353, 356
Orthogonal Structure Theorem, 332, 371, 410
orthogonal subspaces, 324, 325, 332, 334–336, 339, 340, 365, 487, 506
orthogonal wavelet, 361
orthogonality, 321

orthonormal basis, 351–354, 357, 359–362
orthonormal family, 352, 353, 367
orthonormal sequence, 322, 323
orthonormal set, 349–351, 354

p-integrable functions, 94
p-summable family, 209, 341, 345
p-summable sequence, 90
pair, 4
parallelogram law, 313
paranormal operator, 512, 513
Parseval identity, 357
part of an operator, 446
partial isometry, 401–403
partial ordering, 8
partially ordered set, 9
partition, 8
perfect set, 128, 149
point of accumulation, 117–120
point of adherence, 117, 118
point of continuity, 99
point of discontinuity, 99
point spectrum, 453
pointwise bounded, 162, 244
pointwise convergence, 97, 245
pointwise totally bounded, 162, 198
polar decomposition, 402–405, 444, 445
polarization identities, 313, 406, 494
Polish space, 136
polynomial, 62, 79, 80, 282, 459, 518–520
positive functional, 200
positive operator, 395, 397, 429, 430, 432, 439, 500, 508
positive quadratic form, 310
positiveness, 87, 200, 310
power bounded operator, 248, 306, 381
power inequality, 466
power of a function, 7
power sequence, 248, 282, 299, 381, 417, 448
power set, 4
precompactness, 156
pre-Hilbert space, 311
pre-image, 5
Principle of Contradiction, 2
Principle of Mathematical Induction, 2, 13
Principle of Recursive Definition, 13
Principle of Superposition, 77
Principle of Transfinite Induction, 36
product metric, 169
product of cardinal numbers, 36
product space, 169, 178, 184, 190, 194
product topology, 194
projection, 70–73, 82, 223, 299
projection on $\mathcal{M}$, 71–73
projection operator, 223
Projection Theorem, 336, 339, 365
proof by contradiction, 2
proof by induction, 2
proper contraction, 220, 518
proper linear manifold, 43
proper subset, 3
proper subspace, 212, 238, 264
proportional vectors, 406
pseudometric, 93
pseudometric space, 93
pseudonorm, 200, 205
pure hyponormal operator, 507
Pythagorean Theorem, 322, 348

quadratic form, 310
quasiaffine transform, 285
quasiaffinity, 285
quasiinvertible transformation, 285
quasinilpotent operator, 461, 468, 511
quasinorm, 272
quasinormal operator, 444, 446, 450, 501, 508, 511
quasinormed space, 272
quasisimilar operators, 285
quasisimilarity, 285
quotient algebra, 83, 84
quotient norm, 216
quotient space, 7, 42–44, 69, 83, 93, 140, 205–207, 215, 216, 242, 318

range, 5
rank, 78
rare set, 144
real Banach space, 202
real field, 40
real Hilbert space, 315
real inner product space, 311
real linear space, 41
real normed space, 201

real-valued function, 5
real-valued sequence, 13
reducible operator, 389, 495, 504
reducing subspace, 389, 390, 487, 491, 494, 504, 506, 507
reflexive relation, 7
reflexive spaces, 268, 269, 307, 375
relation, 4
relative complement, 3
relative metric, 88
relative topology, 181
relatively closed, 181
relatively compact, 151, 160
relatively open, 181
residual set, 145, 147, 149
residual spectrum, 453
resolution of the identity, 368–371, 413, 484, 488, 489, 491, 493, 508
resolvent function, 451
resolvent identity, 451
resolvent set, 450, 451
restriction of a function, 5
Riemann–Lebesgue Lemma, 424
Riesz Decomposition Theorem, 510
Riesz Lemma, 238
Riesz Representation Theorem, 374
right ideal, 83
right inverse, 27
ring, 39
ring with identity, 39
Russell paradox, 24

scalar, 40
scalar multiplication, 40
scalar operator, 221, 281, 469, 504
scalar product, 310
Schauder basis, 277, 286, 301
Schauder Fixed Point Theorem, 308
Schwarz inequality, 311
second category set, 145–147
second dual, 267
self-adjoint operator, 393–398, 427–429, 431, 439, 457, 500, 502, 503, 508–510, 515
self-indexing, 13
semicontinuity, 176
semi-inner product, 317
semi-inner product space, 317
semigroup property, 28
seminorm, 200, 205
seminormal operator, 448, 450
separable space, 124–127, 154, 184, 213, 257, 267, 277, 291, 294, 353, 354, 359–363, 369, 383, 491, 498
sequence, 13
sequence of partial sums, 170, 203
sequentially compact set, 157
sequentially compact space, 157–159
sesquilinear form, 309, 310
sesquilinear functional, 309, 310
set, 3
shift, 249, 292, 293, 419–424, 442, 450, 470–478
similar linear transformations, 64, 80
similar operators, 286, 293, 418, 497
similarity, 64, 65, 80, 286, 293, 294, 409, 497, 502
simply ordered set, 12
singleton, 4
span, 45, 46, 213
spanned linear manifold, 46
spanned subspace, 213
spanning set, 213
spectral decomposition, 489, 494
Spectral Mapping Theorem, 459, 518
spectral measure, 493
spectral radius, 461–464, 466–468, 502, 505, 512
Spectral Theorem, 488, 494
spectraloid operator, 467, 468
spectrum, 450–457, 468–478, 480–482, 485, 502, 509–511, 515, 516, 518
spectrum diagram, 453
spectrum partition, 453
square root, 398, 431–433, 460, 519
square root algorithm, 175
square-summable family, 341, 347, 348
square-summable net, 320
square-summable sequence, 319, 322, 323, 334, 335
stability, 248–250, 296, 306, 381, 415, 420–422, 431, 432, 442, 463, 464, 500, 506, 516, 517
strict contraction, 99, 133, 220, 297, 396, 518
strictly decreasing function, 10
strictly decreasing sequence, 14
strictly increasing function, 10

strictly increasing sequence, 14
strictly positive operator, 395–397, 429, 430, 432, 439, 457, 500, 508–510
strong convergence, 246–248, 295, 298, 300, 369, 370, 373, 378, 379, 398, 416–418, 433, 499
strong limit, 246
strong stability, 248, 306, 381, 431, 506, 516, 517
stronger topology, 108
strongly bounded, 244, 407
strongly closed, 250, 298, 381, 382
strongly stable coisometry, 420, 421
strongly stable contraction, 442, 499
strongly stable operator, 248–250
subadditive functional, 200
subadditivity, 200
subalgebra, 83
subcovering, 150
sublattice, 10
sublinear functional, 200
subnormal operator, 446, 448, 450, 511
subsequence, 14, 98, 129, 155, 157
subset, 3
subspace of a metric space, 88, 181
subspace of a normed space, 210–214, 216, 218, 236, 324–329, 332, 334–336, 339, 340, 365, 366, 401, 410–412, 440, 478
subspace of a topological space, 181
Successive Approximation Method, 133
sum of cardinal numbers, 36
sum of linear manifolds, 44, 45, 67, 68, 81, 214, 215
summable family, 340–348
summable sequence, 203, 274, 275, 279
sup-metric, 92, 93
sup-norm, 211, 212
supremum, 9, 10, 14
surjective function, 5, 26, 27
surjective isometry, 111, 139–143, 292, 337, 388, 404
symmetric difference, 4, 26
symmetric functional, 310
symmetric relation, 7
symmetry, 87, 427

Tietze Extension Theorem, 180
Tikhonov Theorem, 194
tensor product, 478
topological base, 125
topological embedding, 111
topological invariant, 111, 148, 152, 184, 185
topological isomorphism, 232, 241, 289, 291, 293, 438
topological linear space, 270
topological property, 111
topological space, 106
topological sum, 214, 332, 340
topological vector space, 270
topologically isomorphic spaces, 232, 237, 291, 293, 438
topology, 106, 107
total set, 213
totally bounded, 153–157, 159–164
totally cyclic linear manifold, 283
totally disconnected, 185, 188, 192
totally ordered set, 12
trace, 436
trace-class operator, 434–437
transfinite induction, 36
transformation, 5
transitive relation, 7
triangle inequality, 87, 200
trichotomy law, 12
two-sided ideal, 83, 255, 434, 435

ultrametric, 187
ultrametric inequality, 187
unbounded linear transformation, 237, 288, 289, 364
unbounded set, 89
unconditionally convergent, 345, 359
unconditionally summable, 345, 346
uncountable set, 18
uncountably infinite set, 18
undecidable statement, 24
underlying set, 40, 88
Uniform Boundedness Principle, 244
uniform convergence, 97, 246–248, 295, 299, 300, 379, 418, 433
uniform homeomorphism, 111, 135, 138
uniform limit, 246
uniform stability, 248, 381, 463, 506, 516, 517
uniformly bounded, 244, 407
uniformly closed, 250, 382

uniformly continuous composition, 177
uniformly continuous function, 98, 99, 135–138, 143, 152, 156, 218
uniformly equicontinuous, 162
uniformly equivalent metrics, 112, 178
uniformly homeomorphic spaces, 111, 112, 135, 156, 178
uniformly stable operator, 248, 296, 463, 464
unilateral shift, 292, 419, 423, 450, 470
unilateral weighted shift, 472
unit vector, 349
unital algebra, 82
unital algebra $\mathcal{L}[\mathcal{X}]$, 56, 83, 224
unital Banach algebra, 224, 393
unital homomorphism, 84
unital normed algebra, 224, 284
unitarily equivalent operators, 409, 418, 420, 422, 496, 497, 509
unitarily equivalent spaces, 337, 340, 362, 363, 438
unitary equivalence, 336, 363, 410, 442
unitary operator, 389, 422, 431, 439, 440, 444, 457, 500, 502, 508, 509
unitary space, 89, 204
unitary transformation, 337, 339, 340, 388, 404, 409, 419–421, 509
upper bound, 9, 10
upper limit, 171
upper semicontinuity, 176
usual metrics, 88, 90, 91, 95
usual norms, 204, 205, 207, 210, 220

value of a function, 5
vector, 40
vector addition, 40
vector space, 40
von Neumann expansion, 296, 464

wavelet, 361
wavelet expansion, 361
wavelet functions, 361
wavelet vectors, 361
weak convergence, 305–307, 378–383, 398, 415–418
weak limit, 305, 378
weak stability, 306, 381, 431, 506
weaker topology, 108
weakly bounded, 407
weakly closed, 381, 382
weakly closed convex cone, 397, 429
weakly stable operator, 306, 381, 420, 422, 506
weak* convergence, 307
Weierstrass Theorems, 125, 161
weighted shift, 472, 473
weighted sum of projections, 372, 413, 484–489, 491, 492, 508
well-ordered set, 12

Zermelo Well-Ordering Principle, 24
ZF axiom system, 23
ZFC axiom system, 23
zero linear manifold, 43
Zorn's Lemma, 17